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In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. [201]--206.

Persistent URL: http://dml.cz/dmlcz/700561

NEW CARDINAL INVARIANTS FOR TOPOLOGICAL SPACES[1])

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The cardinal invariants mentioned in the title arise from the following question: for a given (arbitrary) topological space X, what is the smallest infinite cardinal $\mathfrak{m}$ such that one can recover the topology of X from its convergent $\mathfrak{m}$-nets? (An $\mathfrak{m}$-net is a net whose directed set has cardinality $\leqq \mathfrak{m}$.) There are two ways of giving this question a precise formulation (see Section 1). Because they yield the Fréchet spaces and the sequential spaces (as defined in [4]) for the case $\mathfrak{m} = \aleph_0$, the new invariants are called the Fréchet character and the sequential character. Section 2 is devoted to the behaviour of these invariants with respect to the formation of subspaces, products, and quotients. Sections 3 and 4 contain results which are new for the case $\mathfrak{m} = \aleph_0$. In Section 3: for ordered spaces and their products, the sequential and Fréchet characters coincide. In Section 4 new estimates are given for the cardinality of sequential spaces in terms of their density character.

This work arose from a function space formulation of the Aleksandrov-Urysohn problem about the cardinality of first countable compact Hausdorff spaces. In spaces of real-valued functions, much is known about when the pointwise topology (= relative product topology) is sequential or Fréchet. Similar results about when the pointwise topology is $\mathfrak{c}$-sequential or $\mathfrak{c}$-Fréchet might shed light on the Aleksandrov-Urysohn problem. See [11] for details.

Over the years, sequential and Fréchet topologies have been studied under various names by many authors. For recent work see [1], [2], [4], [5], [9], [15], and their bibliographies. For other work on generalizations from sequences to nets of higher cardinality, see [1] and [16]. Proofs of some of the results in this paper are in [12] and [13].

1. The Sequential and Fréchet Character of a Topological Space. We now need a precise formulation [12] of the idea of recovering a topology from its convergent $\mathfrak{m}$-nets. (For nets and other topological concepts we follow the terminology of [8], unless noted otherwise.) In a topological space (X, t), we form the $\mathfrak{m}$-*closure* of a subset A by adding to A all limits of t-convergent $\mathfrak{m}$-nets in A; we denote this set by

[1]) The author gratefully acknowledges the support of the National Science Foundation, under Grant GP 6411.

$\mathfrak{m}$-cl A, but observe that this is not a Kuratowski closure operator in general because it need not be idempotent. However, if $\mathfrak{m}$-cl $A = t$-cl A for all subsets A of X, we say that (X, t) is an $\mathfrak{m}$-*Fréchet space.* More generally, if we can obtain the t-closure operator by iteration of the $\mathfrak{m}$-closure operator, we say that (X, t) is an $\mathfrak{m}$-*sequential space.* For the case $\mathfrak{m} = \aleph_0$, these definitions are equivalent[2]) to the usual ones [4], so that we may write sequential (Fréchet) instead of $\aleph_0$-sequential ($\aleph_0$-Fréchet).

If (X, t) is an $\mathfrak{m}$-sequential topology, for each p in t-cl A there is a smallest ordinal η such that p is in the η-th iterate of the $\mathfrak{m}$-closure of A. The ordinal η is called the $\mathfrak{m}$-*Baire order* of p with respect to A and is denoted by $\mathfrak{m}$-ord p.

Although the $\mathfrak{m}$-closure operator is the most general closure operator to be considered in this paper, the construction carried out thus far can be done more generally; see [12]. This is useful for applications in which one is interested in some, but not necessarily all, of the convergent $\mathfrak{m}$-nets. Of course using a smaller collection of nets could increase Baire order.

For an arbitrary topological space X there is a smallest $\mathfrak{m}$ such that X is $\mathfrak{m}$-sequential; this $\mathfrak{m}$ is called the *sequential character* of X and denoted by $\sigma(X)$. The *Fréchet character* of X is defined similarly and denoted by $\Phi(X)$. These numbers always exist and are topological invariants. They are related to a familiar topological invariant, the *local character* or *point character* $\chi(X)$ (= the least cardinal $\mathfrak{m}$ such that each point in the space has a neighborhood base of cardinality $\leqq \mathfrak{m}$). The relation is seen in the following

Proposition 1.1. *For an arbitrary topological space X*

(1) $\sigma(X) \leqq \Phi(X) \leqq \chi(X)$,

(2) $\Phi(X) \leqq \exp \sigma(X)$.

Proof. The first inequality in (1) is trivial and the second is clear. We now prove (2), i.e., prove that every $\mathfrak{m}$-sequential space is an exp $\mathfrak{m}$-Fréchet space. Suppose that X is $\mathfrak{m}$-sequential, A is a subset of X and $p \in \mathrm{cl}\, A$; we must show that there is an exp $\mathfrak{m}$-net in A converging to p. This follows readily if we show that there is a subset B of A with $p \in \mathrm{cl}\, B$ and card $B \leqq \mathfrak{m}$. We proceed by induction on λ, the $\mathfrak{m}$-Baire order of p with respect to A. If $\lambda > 0$ there is an $\mathfrak{m}$-net $\{x_\nu, \nu \in D\}$ converging to p with $x_\nu \in \mathrm{cl}\, A$ and $\mathfrak{m}$-ord $x_\nu < \lambda$. By the inductive hypothesis there exists $B_\nu \subset A$ with card $B_\nu \leqq \mathfrak{m}$ and $x_\nu \in \mathrm{cl}\, B_\nu$. Let $B = \bigcup\{B_\nu \colon \nu \in D\}$. Then B is the desired set, because card $B \leqq \mathfrak{m}$ card $D \leqq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$, and the proof is complete. (This argument is patterned after a similar one for Baire functions [10, page 491]; there the set B is called an "ancestral set" for p.)

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[2]) The equivalence follows from the fact that every countable directed set with no last element has a cofinal sequence.

We shall see later (Section 3) that, for a given infinite cardinal $\mathfrak{m}$, there are many spaces for which $\sigma = \Phi = \chi = \mathfrak{m}$. On the other hand, Example 1.2 shows that $\chi > \exp \sigma$ can also occur, so that the upper bounds for Φ in (1) and (2) are not comparable in general.

Under the assumption of the generalized continuum hypothesis (2) cannot be improved; i.e., for any space X either $\Phi(X) = \sigma(X)$ or $\Phi(X) = \exp \sigma(X)$, and both cases are known to occur. There are other inequalities which are also best possible in this sense: the upper bounds for σ in 2.1 and for Φ in 2.2.

Example 1.2. Assume that $\mathfrak{m}$ is a regular cardinal and X is a set of cardinality $\geqq \mathfrak{m}$ with a distinguished point p. We define a topology on X by specifying that every point except p is isolated and a set containing p is open iff its complement has cardinality $< \mathfrak{m}$. In this case, $\sigma(X) = \Phi(X) = \mathfrak{m}$ and by increasing the cardinality of X we can make $\chi(X)$ as large as we please.

2. Permanence Properties of Sequential and Fréchet Characters. This section summarizes what is known on the extent to which the sequential and Fréchet characters are preserved (not increased) in passing to subspaces, quotients and products. The behaviour of the local character is also described for comparison purposes. It is interesting to note that in each situation there is a well behaved invariant, but no single invariant works for all three cases.

2.1. Subspaces. If Y is a subspace of X then $\Phi(Y) \leqq \Phi(X)$ [12] and of course $\chi(Y) \leqq \chi(X)$. If Y is closed or open in X then $\sigma(Y) \leqq \sigma(X)$ [12], but a similar inequality for σ does not hold in general. (In fact, in any space X for which $\sigma(X) < \Phi(X)$, there exist subspaces Y for which $\sigma(Y) > \sigma(X)$ [12].) Clearly, however, there is an upper bound for $\sigma(Y)$: for any subspace Y of X we have $\sigma(Y) \leqq \Phi(X)$. Thus every subspace of an $\mathfrak{m}$-sequential space is $\exp \mathfrak{m}$-Fréchet.

Michael [7, page 44] has raised the question of characterizing the subspaces of sequential spaces. The above argument shows that it is necessary that such spaces be $\mathfrak{c}$-Fréchet. It is an open question whether or not the condition is sufficient[3]); however, Michael (ibid.) has also given an example of a space X which is regular and $\mathfrak{c}$-Fréchet

[3]) The above condition is not sufficient, because it is now known that Michael's example, $N \cup \{p\}$, is not a subspace of any sequential space, regular or not (Fleischer and Franklin, On compactness and projections, Proc. Symp. on extension theory of topological structures, Berlin 1967).

Lynn Imler has proved the following stronger result: every subspace of a sequential space is a c-space. (A c-space can be described as a space in which the closure of any set is the union of the closures of its countable subsets). This is stronger because it follows from the proof of (2) in Proposition 1.1 that every c-space is $\mathfrak{c}$-Fréchet. The above counterexample also shows that the converse of Imler's result is false.

but is not a subspace of any regular sequential space. (His example: $X = N \cup \{p\}$, with $p \in \beta N - N$.)

2.2. Quotients. The sequential character is the only invariant that is preserved with respect to quotients. If Z is a quotient of X, then

(1) $\sigma(Z) \leqq \sigma(X)$,

(2) $\Phi(Z) \leqq \exp \sigma(X)$,

(3) $\chi(Z) \leqq \exp \operatorname{card} X$.

The inequality (1) was proved in [12] as a step in the characterization of $\mathfrak{m}$-sequential spaces as quotients of spaces of local character $\leqq \mathfrak{m}$. (2) follows from (1).

2.3. Products. In general χ is well behaved with respect to products, σ and Φ are not. However for some products σ and Φ are well behaved because they coincide with χ (Section 3).

Let X be the product of a family of topological spaces $\{X_i: i \in I\}$, where each X_i is non-trivial (has at least one non-void proper open set). Then

(1) $\sigma(X) \geqq \max\{\operatorname{card} I, \sup\{\sigma(X_i): i \in I\}\}$,

(2) $\Phi(X) \geqq \max\{\operatorname{card} I, \sup\{\Phi(X_i): i \in I\}\}$,

(3) $\chi(X) = \max\{\operatorname{card} I, \sup\{\chi(X_i): i \in I\}\}$.

The fact that $\sigma(X) \geqq \operatorname{card} I$ is proved in [13, Lemma 2]; the rest of (1) and (2) follows from the fact that X_i is both a subspace and a quotient of X.

A number of counterexamples have been published which show that equality does not hold in (1) or (2); see [2], [3], [6], [14].

3. Spaces for Which $\sigma = \Phi = \chi$. The main result of this section (from [13]) is that for products of ordered spaces $\sigma = \Phi = \chi$. (Note however that the behavior of their subspaces is much more complicated; look at spaces of real-valued functions.) Greater generality is obtained here by generalizing the notion of an order topology (from that found for example in [8]) and by treating the extension to products separately.

Definition. *A locally ordered topological space is one in which each point has an open neighborhood which can be (totally) ordered so that the order topology agrees with the given topology in the neighborhood.*

A circle provides an example of locally ordered topological space which cannot be globally ordered (i.e., there is no total ordering of the entire space such that the order topology is the given topology).

Theorem 3.1. *If X is a locally ordered space, then* $\sigma(X) = \Phi(X) = \chi(X)$.

The proof is essentially that given in [13] for ordered spaces.

The next result shows that product spaces have the property $\sigma = \Phi = \chi$ if either there is a sufficiently large number of coordinate spaces or enough of the coordinate spaces themselves have the property.

Theorem 3.2. *Let X be the product of a family of non-trivial topological spaces $\{X_i\colon i \in I\}$. Then $\sigma(X) = \Phi(X) = \chi(X)$ if one of the following conditions holds*:

(a) $\operatorname{card} I \geqq \chi(X_i)$ *for each i in I*,

(b) *For each $i \in I$ there is a $j \in I$ such that $\chi(X_j) \geqq \chi(X_i)$ and $\sigma(X_j) = \chi(X_j)$*.

Proof. It suffices to show $\sigma(X) \geqq \chi(X)$. In case (a) we have, by 2.3, $\sigma(X) \geqq \geqq \operatorname{card} I \geqq \chi(X)$. In case (b) let $\sigma(X) = \mathfrak{m}$. Then by 2.3 and (b) it follows that, for each i, $\mathfrak{m} \geqq \sigma(X_j) = \chi(X_j) \geqq \chi(X_i)$. Since $\mathfrak{m} \geqq \operatorname{card} I$, we can conclude that $\mathfrak{m} \geqq \geqq \chi(X)$.

4. On the Cardinality of Sequential Spaces. It is well known that, for any Hausdorff space X, $\operatorname{card} X \leqq \exp \exp \delta(X)$, where $\delta(X)$ is the density character of X (= the least cardinal of a dense subset of X). This estimate cannot be improved in general (consider for example βN). However, for sequential spaces, the estimate can be improved while at the same time weakening the separation hypothesis to: sequential limits are unique. (For results on the precise extent to which sequential spaces with unique sequential limits need not be Hausdorff, see [5].)

Theorem 4.1. *If X is a sequential space in which sequential limits are unique, then*

(1) $\operatorname{card} X \leqq \exp \delta(X)$,

(2) $\operatorname{card} X \leqq (\delta(X))^{\aleph_0}$.

Corollary 4.2[4].) *A separable sequential space with unique sequential limits has cardinality* $\leqq \mathfrak{c}$.

Corollary 4.3. *If X is a sequential space with unique sequential limits and $\delta(X)$ is an infinite cardinal of the form* $\exp \mathfrak{n}$, *for some* $\mathfrak{n}$, *then* $\operatorname{card} X = \delta(X)$.

Proof. The proof is patterned after a standard inductive argument for counting Baire functions. Both parts of the theorem are proved similarly; we prove (1) here. Let X be a sequential space with unique sequential limits and let Y be a dense subset

[4]) First proved by Lynn Imler (see [7. page 24]).

with card $Y = \delta(X)$. Let $Y_\eta = \{x \in X\colon \text{ord}\, x = \eta\}$, where ord x denotes the $\aleph_0$-Baire order of x with respect to Y (for sequences).

Let $T_\eta = \bigcup\{Y_\alpha\colon \alpha < \eta\}$. We show that

$$(*) \qquad \text{card}\, Y_\eta \leqq \exp \delta(X)$$

by induction on η. It is clear for $\eta = 0$, since $Y_0 = Y$. Now assume that $\eta > 0$ and card $Y_\alpha \leqq \exp \delta(X)$ for all $\alpha < \eta$. Then, since $\eta < \omega_1$, we have card $T_\eta \leqq \leqq \aleph_1 \exp \delta(X) = \exp \delta(X)$. Since every element of Y_η is the limit of a sequence in T_η and no sequence has more than one limit, we have card $Y_\eta \leqq (\text{card}\, T_\eta)^{\aleph_0} \leqq \leqq (\exp \delta(X))^{\aleph_0} = \exp \delta(X)$. This proves (*). From the fact that $X = \bigcup\{Y_\eta\colon \eta < \omega_1\}$ we get card $X \leqq \exp \delta(X)\, \aleph_1 = \exp \delta(X)$.

The first corollary is clear. To prove the second corollary, merely note that $(2^{\text{n}})^{\aleph_0} = 2^{\text{n}\aleph_0} = 2^{\text{n}}$.

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