Gordon Thomas Whyburn
Cohesive spaces and fixed points

In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. 311--317.

Persistent URL: http://dml.cz/dmlcz/700569

# COHESIVE SPACES AND FIXED POINTS

G. T. WHYBURN

Charlottesville

**1. Partially Closed Sets.** In working over material for my book on Analytic Topology I became interested in the type of set which has closed components and which comes near to being closed in other ways but is not so restricted as to actually make it closed. In particular, it was desired that the collection of its components be upper semi-continuous and, even more, that its natural extension to the rest of the space obtained by adding in individual points of its complement also be upper semi-continuous.

The solution to this problem, in the case of compact metric spaces, was found in the notion of a set or collection being semi-closed. A collection of disjoint closed sets is *semi-closed* provided that any sequence of its elements which converges to a limit set meeting the complement of the union of the elements of the collection actually converges to a single point of this complement. A set $K$ is *semi-closed* provided its components form a semi-closed collection. As indicated, it turned out that a collection of disjoint closed sets in a compact metric space is semi-closed if and only if its single-point extension to the whole space is upper semi-continuous.

To illustrate, consider the Sierpiński universal plane curve $S$ taken for convenience on the 2-sphere $S^2$. The collection of boundaries of complementary regions is semiclosed. The same is true if we take as an element each such boundary together with the region of which it is the boundary. In each case the single point extension to $S^2$ yields an usc decomposition of $S^2$ with a monotone natural mapping of $S^2$ onto the decomposition space $\Sigma$. In the latter case this map is also non-separating so that, by the well known theorem of R. L. Moore, $\Sigma$ also is an $S^2$. In the former case the image of $S$ alone is an $S^2$, a true cyclic element of $\Sigma$, although $\Sigma$ itself is a cactoid.

Interest in this type of partly closed set has been revived and heightened recently by the attention attracted by consideration of several types of partially continuous functions and their fixed point and related properties. I refer in particular to connectivities and peripherally continuous functions to be discussed later.

A set $K$ in a topological space $X$ is *quasi-closed* [6] provided it is of *external dimension* 0, that is, for any $x \in X - K$ any open set $U$ about $x$ contains an open set $V$ about $x$ whose boundary $\partial V$ does not meet $K$. In other words, $K$ is quasi-closed provided $\dim_x (K + x) = 0$ for each $x \in X - K$.

Similarly a set $G$ is *quasi-open* provided that $X - G$ is quasi-closed or, equivalently, for each $p \in G$ and each open set $U$ about $p$ there exists an open set $V$ with $p \in V \subset U$ and with boundary $\partial V$ contained in $G$.

It is readily seen that every quasi-closed set is semi-closed but the reverse implication does not hold even in compact metric spaces. (Note that *every* totally disconnected set is semi-closed.) Also, the intersection of an arbitrary collection of quasi-closed sets is quasi-closed. However the union of two such sets may fail to be quasi-closed. Correspondingly, arbitrary unions of quasi-open sets are quasi-open, but the finite intersection axiom does not necessarily hold for such sets.

**2. Connectivities and Peripherally Continuous Functions.** A function $f: X \to Y$ is a *connectivity* provided the graph of each restriction $f \mid C$ of $f$ to a connected subset $C$ of $X$ is a connected set. Thus $f$ is a connectivity provided its graph function $g: X \to X \times Y$ preserves connectedness where $g$ is defined by

$$g(x) = [x, f(x)] \in X \times Y.$$

This is a stronger property than that of preserving connectedness, even for real functions on an interval as was shown in an example by Kuratowski [4]. For real functions of Baire class 1 the two properties are equivalent.

A function $f: X \to Y$ is *peripherally continuous* [3] at $x \in X$ provided that if $U$ and $V$ are open sets about $x$ and $f(x)$ respectively, there exists an open set $W$ with $x \in W \subset U$ and $f(\partial W) \subset V$. Clearly any continuous function on a regular space is peripherally continuous. However the function $f: I \to I$ which is 0 for $x$ irrational and $(\sqrt{2})/2$ for $x$ rational is also peripherally continuous. Note also that this function has no fixed point.

The two types of functions just defined are markedly different in the case of real valued functions of a real variable, since we have noted that whereas connectivities preserve connectedness in particular, a peripherally continuous function on the interval may have a discrete set of two or more points as its image set. However, remarkably enough, it turns out that on a large class of domain spaces, including in particular all Euclidean manifolds of dimension $\geqq 2$, the two properties are entirely equivalent so that the two classes of functions coincide. [1, 8]. These domains are the cohesive spaces to be discussed shortly.

Peripherally continuous functions may be characterized in a simple and useful way exhibiting clearly their relation to mappings (= continuous functions). For it turns out that whereas a function $f: X \to Y$ is continuous if and only if the inverse of every closed set in $Y$ is a closed set in $X$, to characterize peripherally continuous functions we need only change the second "closed set" to "quasi-closed set". Thus $f$ is peripherally continuous if and only if the inverse of every closed set is quasi-closed [6]. Note that in the example above of such a function from $I$ into $I$ the inverse image of 0 was the irrationals and of $(\sqrt{2})/2$ was the rationals, both being quasi-closed sets which definitely are not closed. Despite the simple character of this characterization of peripheral continuity we note a certain lack of symmetry. Starting with a *closed* set in $Y$ we get a *quasi-closed* set as its inverse in $X$. Starting with a *quasi-closed* set in $Y$ we know little about its inverse. This indicates, what actually is the case, that peripherally continuous functions do not compose. In other words the composition of two such functions is not necessarily peripherally continuous [5].

**3. Cohesive Spaces.** A connected space or set $M$ is *cohesive* [6] between two of its closed subsets $A$ and $B$ provided $H_a \cdot H_b$ is connected for every representation $M = H_a + H_b$ where $H_a$ and $H_b$ are closed and connected and contain $A$ and $B$ respectively. Thus cohesion of a space between $A$ and $B$ is a sort of unicoherence between these two sets.

A connected regular $T_1$-space $X$ is *locally cohesive* provided that each open set $U$ about a point $x$ of $X$ contains the closure of a *canonical region* about $x$, i.e., a connected open set $R$ having a connected boundary $\partial R$ and such that $X$ is cohesive between $x$ and $X - R$ (equivalently, $\bar{R}$ is cohesive between $x$ and $\partial R$). Note that a locally cohesive space is always locally connected and locally peripherally connected. Also it can have no local cut point.

It is apparent that all manifolds of dimension $\geqq 2$ are locally cohesive, as are all polyhedra without local cut points. Indeed it is readily shown *that any locally unicoherent connected locally connected and locally compact Hausdorff space is locally cohesive at each of its non local cut points.* (Local unicoherence means that each point is interior to some unicoherent connected subset.) Thus if there are no local cut points we have local cohesion at all points of the space.

This brings us to the first of two key properties of a locally cohesive space $X$ entering into the equivalence of connectivities and peripherally continuous functions on such spaces as domains. For if $W$ is any canonical region in $X$ about $a \in X$, then any set $K$ separating $a$ and $\partial W$ in $\overline{W}$ contains the boundary of a canonical region $R$ about $a$ lying in $W$. This is readily shown using the cohesion property of $X$ between $a$ and $\partial W$. Accordingly, *if $E$ is any quasi-closed set in $X$, any open set $U$ in $X$ about $a \in X - E$ contains a canonical region $R$ about $a$ whose boundary does not meet $E$.* Equivalently, *if $G$ is any quasi-open set in $X$, any open set in $U$ about $a \in G$ contains a canonical region $R$ about $a$ whose boundary lies in $G$.*

The second key property is concerned with preservation of connectedness under peripherally continuous functions on a locally cohesive domain space $X$. If $f: X \to Y$ is such a function where $X$ is locally cohesive and $Y$ is completely normal, then not only is connectedness preserved under $f$ but also the induced graph function $g: X \to X \times Y$ of $f$ is peripherally continuous [1, 8]. Thus in case $X \times Y$ is completely normal, connectedness is also preserved under $g$ so that $f$ is a connectivity.

The connectedness preservation of $f$ here follows from the interesting result that *in a locally cohesive space $X$, any connected set in $X$ lying in the union of two* *disjoint quasi-open sets lies entirely in one of them* [7]. As might be expected, the first basic property concerned with canonical regions enters strongly into the proof of this one.

These key properties of locally cohesive spaces enable us to prove without too much difficulty that on such a domain space any peripherally continuous function is a connectivity. The reverse implication also holds in case the space is locally compact and metric [3, 5]. The best proof of this result [6] hinges on the facts (1) that if $f: X \to Y$ is a connectivity where $X$ is compact, then for any closed set $C$ in $Y$, $f^{-1}(C)$ is semi-closed so that the single point extension of the collection of its components is upper semi-continuous [2], and (2) the rather surprising theorem that a *Peano continuum $M$ is cohesive between two of its points $a$ and $b$ if and only if the cyclic chain $C(a, b)$ is unicoherent* [6].

An interesting problem concerning locally cohesive spaces is that of determining the natural class of mappings under which the property of being locally cohesive is invariant. For monotone mappings it seems to be *necessary* that no point inverse separate any region in the domain space containing it and *sufficient* that the domain space be "locally cohesive about each point inverse" in a rather obvious way. Whether this latter property is a consequence of the local cohesion at individual points, however, does not seem to be a readily answerable question.

**4. Separation and Intersection Theorems.** We turn now to a consideration of a new type of separation theorem which can be established in locally cohesive spaces. Two sets $A$ and $B$ in a space $X$ are *weakly separated* in $X$ provided no component of $X$ meets them both. Of course, that they are *separated* in $X$ means that there exists a separation

$$X = X_a + X_b$$

of $X$ between them so that $A \subset X_a$, $B \subset X_b$.

We recall the standard separation theorem that in a compact metric space $K$, any two closed sets $A$ and $B$ that are weakly separated in $X$ are actually separated in $X$. The same result holds in a compact Hausdorff space $H$. Worded slightly differently this says that if no component of $H$ meets both of two closed sets $A$ and $B$ then there is a separation of $H$ between $A$ and $B$.

Remarkably enough an entirely analogous theorem holds in locally cohesive spaces where the sets involved in the theorem are neither compact, closed nor open [7]. To facilitate the proof it is stated first in the form involving quasi-open sets:

**Separation Theorem.** *Let $G$ be a quasi-open set in a locally cohesive $T_1$-space $X$. If two disjoint relatively closed subsets $G_a$ and $G_b$ of $G$ are weakly separated in $G$, they are actually separated in $G$. Indeed if $A$ and $B$ are closed sets in $X$ satisfying $A \cdot G = G_a$, $B \cdot G = G_b$, there exist disjoint open sets $U_a$ and $U_b$ in $X$ such that $G \subset U_a + U_b$ and $U_a \cdot B = U_b \cdot A = \emptyset$.*

It can be shown that there is no loss of generality if we assume $A$ and $B$ each to be of positive dimension at each of its points. The proof then goes as follows.

Proof. For each $x \in G$ let $R_x$ be a canonical region about $x$ with boundary $C_x$ lying in $G$ and so chosen that (i) for $x \in G_a (G_b)$, $C_x$ meets $A$ (resp. $B$) but $R_x$ does not meet $B$ (resp. $A$), (ii) for $x \in G - G_a - G_b$, $R_x$ meets neither $A$ nor $B$. The union $U_a$ of all sets $R_x$ which are finitely chainable to $G_a$ by regions $[R_x]$ is open as is also the union $U_b$ of all such regions not so chainable to $G_a$. Further, $G \subset U_a + U_b$ and clearly $U_b \cdot A = \emptyset$. Thus we have left only to show that $U_a \cdot B = \emptyset$. If this is not so, then some $b \in G_b$ is in a set of $[R_x]$ which is finitely chainable to $a \in G_a$; and we have a simple chain $a \in R_1, R_2, \ldots, R_n \ni b$ of regions $[R_x]$ only the first of which contains $a$ and only the last of which contains $b$. However, this is impossible because if $C_i$ is the boundary of $R_i$, $1 \leqq i \leqq n$, $C = \bigcup_1^n C_i$ is a connected subset of $G$ and $G_a \cdot C_1 \neq \emptyset \neq G_b \cdot C_n$ so that $C$ meets both $A$ and $B$.

Stated in terms of quasi-closed sets we have

**Separation Theorem** (alternate form). *Let $L$ be a quasi-closed set in a locally cohesive $T_1$-space $X$. If two closed sets $A$ and $B$ in $X$ are weakly separated in $X$ by $L$, there exists a closed set $K$ in $L$ which separates $A - K$ and $B - K$ in $X$.*

To get this form of the theorem we take $G = X - L$, $G_a = A \cdot G$, $G_b = B \cdot G$, $K = X - (U_a + U_b)$, where $U_a$ and $U_b$ are given by the previous theorem. Then $A - K \subset U_a$ since $U_b \cdot A = \emptyset$, $B - K \subset U_b$ since $U_a \cdot B = \emptyset$, so that $X - K = U_a + U_b$ provides the required separation.

Using this form of the Separation Theorem we now obtain a basic extension of the Hurewicz-Wallman intersection result for closed sets in the unit interval $I^n$ of Euclidean space $E^n$. For each $i$, $1 \leqq i \leqq n$, let $A_i$ and $B_i$ be the faces of $I^n$ on which $x_i = 0$ and $x_i = 1$ respectively.

**Intersection Theorem** [7]. *Given quasi-closed sets $C_1, C_2, \ldots, C_n$ in $I^n$ such that for each $i$, $1 \leqq i \leqq n$, $C_i$ weakly separates $A_i$ and $B_i$ in $I^n$. Then $\bigcap_1^n C_i \neq \emptyset$.*

Proof. For each $i$, by the above, $C_i$ contains a closed set $K_i$ which separates $A_i - K_i$ and $B_i - K_i$ in $I^n$, so that $I^n - K_i = U_i + V_i$, where $U_i$ and $V_i$ are disjoint and open and contain $A_i - K_i$ and $B_i - K_i$, respectively. Now define a function $f(x)$, $x \in I^n$, by letting $f(x)$ be the terminal end of the position vector $x + d(x)$ in $E^n$, where the $i^{\text{th}}$ component $d_i$ of the vector $d(x)$ is $\pm\varrho(x, K_i)$, the sign being $+$ for $x \in U_i$ and $-$ for $x \in V_i$. Then for $x \in U_i$ and each $i$, we have $d_i = \varrho(x, K_i) \leqq 1 - x_i$ so that $0 \leqq x_i + d_i \leqq 1$; and for $x \in V_i$, $d_i = -\varrho(x, K_i) \geqq -x_i$ and again $0 \leqq x_i + d_i \leqq x_i \leqq 1$. Thus in any case $f(x) \in I^n$. Since $f$ clearly is continuous and $f: I^n \to I^n$, we have $f(x_0) = x_0$ for some $x_0 \in I^n$ by the Brouwer fixed-point theorem. Thus $d(x_0) = 0$ and $x_0 \in \bigcap_1^n K_i \subset \bigcap_1^n C_i$.

**5. Fixed Point Theorem.** As an application of the separation and intersection theorems for locally cohesive spaces we give next a simple direct proof for the Hamilton-Stallings extension of the Brouwer Fixed Point Theorem to connectivities and peripherally continuous functions [3, 5, 7]. This famous theorem has been extended in a number of different directions, including the following: (i) Relaxation of the single valuedness condition on the mapping. Here there are important results due to Kakutani, A. D. Wallace, Eilenberg-Montgomery, Strother, Smithson and others. (ii) Relaxation of the structure requirements on the domain space. Included here are results of Lefschetz, Eilenberg-Montgomery, Ayres, Borsuk, Cartwright, Hamilton and others. (iii) Relaxation of the continuity condition on the function. The extension we give now belongs to this last group of results and stems from the simple observation that, in the case of a real function $f: I^1 \to I^1$, the graph will cross the diagonal in $I^1 \times I^1$ (and thus $f$ will have a fixed point) so long as it is connected.

Using the results developed above, a proof basically like this and almost as simple is now available to us for the

**Fixed-Point Theorem** (Hamilton-Stallings). *Any peripherally continuous function of $I^n$ into itself, $n \geqq 2$, has at least one fixed point. The same is true of any connectivity function of $I^n$ into itself for $n \geqq 1$.*

Proof. Let $f: I^n \to I^n$, $n \geqq 2$, be any peripherally continuous function. We visualize $I^n \times I^n$ in the form $I^n \times I'^n$ where $I^n = I_1 \times I_2 \times \ldots \times I_n$ and $I'^n = I'_1 \times I'_2 \times \ldots \times I'_n$; and for $x = (x_1, x_2, \ldots, x_n) \in I^n$ let $f(x) = x' = (x'_1, x'_2, \ldots, x'_n) \in I'^n$. Let $g: I^n \to I^n \times I'^n$ be the graph function for $f$, i.e., $g(x) = [x, f(x)]$; and for $1 \leqq i \leqq n$, let $p_i = \pi_i g: I^n \to I_i \times I'_i$ be the projection of the graph of $f$ into the plane $I_i \times I'_i$ given by $p_i(x) = \pi_i(x, x') = (x_i, x'_i) \in I_i \times I'_i$, where $f(x) = x' = (x'_1, x'_2, \ldots, x'_n)$. Let $\Delta_i$ be the diagonal $x_i = x'_i$ in the plane $I_i \times I'_i$. That $f$ has a fixed point now follows from three assertions:

(i) For each $i$, $1 \leqq i \leqq n$, $p_i$ is peripherally continuous.

(ii) For each $i$, $1 \leqq i \leqq n$, the set $C_i = p_i^{-1}(\Delta_i)$ is quasi-closed and it weakly separates in $I^n$ the faces $A_i$ and $B_i$ of $I^n$ on which $x_i = 0$ and $x_i = 1$, respectively.

(iii) $\bigcap_1^n C_i \neq \emptyset$ and $f(x) = x$ for each $x \in \bigcap_1^n C_i$.

To verify (i), let $E$ be any closed set in $I_i \times I'_i$. Then $\pi_i^{-1}(E)$ is closed since $\pi_i$ is continuous. Thus the set $g^{-1} \pi_i^{-1}(E) = p_i^{-1}(E)$ is quasi-closed, since $f$ is peripherally continuous and thus so also is its graph function $g$. Whence, $p_i$ is peripherally continuous.

To prove (ii) note that since $\Delta_i$ is closed, $C_i$ is quasi-closed by (i). Now if some component $Q$ of $I^n - C_i$ intersected both $A_i$ and $B_i$, $p_i(Q)$ would be connected since connectedness is preserved under $p_i$ by (i). However, $p_i(Q)$ would then meet $\Delta_i$ because it contains a point $p_i(a)$, $a \in A_i$, where $x_i' \geqq x_i$ and also a point $p_i(b)$, $b \in B_i$, where $x_i' \leqq x_i$.

Assertion (iii) now follows from (ii) and the Intersection Theorem, and this concludes the proof of the first statement in our theorem.

The second statement is verified by the simple argument given above in the first paragraph of this section in case $n = 1$. For $n \geqq 2$ it results from the fact that $I^n$ is locally cohesive along with the equivalence of connectivities and peripherally continuous functions on such spaces as discussed above in § 3.

In conclusion it may be noted that the above proof of the Intersection Theorem leans heavily on the Brouwer Fixed Point Theorem. It would be highly desirable to have an elementary and independent proof of this basic result. This would then make the above proof of the Hamilton-Stallings theorem complete and free of dependence on the Brouwer Theorem. The latter would then be a direct and true corollary to the former.

**Bibliography**


[1] *M. R. Hagan:* Equivalence of connectivity maps and peripherally continuous transformations, Proc. Amer. Math. Soc. vol. *17* (1966), 175—177.

[2] *M. R. Hagan:* Upper semi-continuous decompositions and factorizations of certain non-continuous transformations, Duke Math. Jour. vol. *32* (1965), 679—687.

[3] *O. H. Hamilton:* Fixed points for certain non-continuous functions, Proc. Amer. Math. Soc. vol. *8* (1957), 750—756.

[4] *C. Kuratowski:* Topologie, vol. II, 3rd edition (1961), 80—81.

[5] *J. Stallings:* Fixed point theorems for connectivity maps, Fund. Math. vol. *47* (1959), 249 to 263.

[6] *Gordon T. Whyburn:* Loosely closed sets and partially continuous functions, Mich. Math. Jour. vol. *14* (1957), 193—205.

[7] *Gordon T. Whyburn:* Quasi-closed sets and fixed points, Proc. Natl. Acad. Sci. vol. *57* (1967), 201—205.

[8] *Gordon T. Whyburn:* Connectivity of peripherally continuous functions, ibid., vol. *55* (1966), 1040—1041.


UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, U.S.A.