T. Soundararajan

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WEAKLY HAUSDORFF SPACES AND THE CARDINALITY OF TOPOLOGICAL SPACES

T. SOUNDARARAJAN

Madurai

Introduction. A topological space $(X, \mathscr{T})$ is called a weakly Hausdorff space if each element of X is an intersection of regularly closed sets. A weakly Hausdorff space can also be defined as one in which each element is an intersection of regularly open sets[1]). The class of weakly Hausdorff spaces includes properly the Hausdorff spaces, the T_1 semi-regular spaces and the T_1 π_2 spaces of [8]. First we prove that if $(X, \mathscr{T})$ is a weakly Hausdorff space and has a dense subset of cardinal θ then $|X| \leqq 2^{2^\theta}$. We next consider products of weakly Hausdorff spaces. We then derive that a locally compact Hausdorff group G has a dense subset of cardinal $\leqq \mathfrak{m}$ $(\mathfrak{m} \geqq \aleph_0)$ if and only if $|G| \leqq 2^{2^{\mathfrak{m}}}$ and G is a set union of $\aleph$ compact sets where $\aleph \leqq \mathfrak{m}$. We also make some comparisons of this separation axiom with some of the other separation axioms. We conclude with a conjecture. The author expresses his best thanks to Professor M. Venkataraman for his encouragement and guidance in the preparation of this paper.

§ 1. In this section we proceed to the results on the cardinality of topological spaces.

Proposition 1.1. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. Then the following are equivalent:*

1. *A is regularly closed* $(A = \overline{\operatorname{Int} A} = \overline{A^0})$.
2. $A = \operatorname{cl}(A \cap D)$ *for each dense subset D of X.*
3. $A = \operatorname{cl}(A \cap D)$ *for each dense open subset D of X.*
4. $A = \operatorname{cl}(A \cap D)$ *for each dense set D such that $|D| = d$ where d is the density character of X.*
5. $A = \operatorname{cl}(A^0 \cap D)$ *for each dense subset D of X.*

Proof. Left to the reader.

Theorem 1.2. *Let $(X, \mathscr{T})$ be a weakly Hausdorff space having a dense subset D of cardinal θ. Then $|X| \leqq 2^{2^\theta}$.*

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[1]) Regularly closed sets are also called closed domains and regularly open sets are called open domains, cf. [8].

Proof. By Proposition 1.1 if A and B are two regularly closed sets and $A \neq B$ then we have $A \cap D \neq B \cap D$. Hence the cardinality of the set of regularly closed sets cannot exceed 2^{θ}. Since X is weakly Hausdorff, each element x of X is the intersection of all the regularly closed sets containing x. Hence to each element x of X we can associate the subfamily V_x of all regularly closed sets containing x. If $x \neq y$ we have $V_x \neq V_y$. The family $\{V_x\}_{x \in X}$ has cardinality $\leqq 2^{2^{\theta}}$ and hence it follows that $|X| \leqq 2^{2^{\theta}}$.

Remark 1.3. Theorem 1.2 generalizes a result of Pospíšil[2]). The proof above is similar to showing that the weight of a semi-regular space does not exceed 2^d where d is the density character.

Proposition 1.4. *Let X_α, $\alpha \in I$ be a family of non-void topological spaces. Then the product space $\prod X_\alpha$ is weakly Hausdorff if and only if each X_α is a weakly Hausdorff space.*

Proof. Suppose first that each X_α is weakly Hausdorff. Let $x = (x_\alpha)$ be any element of $\prod X_\alpha$. Let $y = (y_\alpha)$ be any element of $\prod X_\alpha$ such that $y \neq x$. Then there is an $\alpha \in I$ such that $x_\alpha \neq y_\alpha$. Let P_α be the projection map of $\prod X_\alpha$ onto X_α.

Since x_α, $y_\alpha \in X_\alpha$ and $x_\alpha \neq y_\alpha$ there is a regularly closed set A_α in X_α such that $x_\alpha \in A_\alpha$ and $y_\alpha \notin A_\alpha$. Now $A_\alpha = \operatorname{cl} A_\alpha^0$ where A_α^0 is the interior of A_α in X_α. Now $P_\alpha^{-1}(A_\alpha^0)$ is an open set in $\prod X_\alpha$ and $\operatorname{cl} P_\alpha^{-1}(A_\alpha^0) = P_\alpha^{-1}(A_\alpha)$. (This follows for instance from the well known proposition that if $B_\alpha \subseteq X_\alpha$ then $\operatorname{cl}(\prod B_\alpha) = \prod(\operatorname{cl} B_\alpha)$). Thus $P_\alpha^{-1}(A_\alpha)$ is a regularly closed set in $\prod X_\alpha$. Also $x \in P_\alpha^{-1}(A_\alpha)$ and $y \notin P_\alpha^{-1}(A_\alpha)$. It follows that x is the intersection of all regularly closed sets containing it. Thus $\prod X_\alpha$ is weakly Hausdorff.

Suppose now that $\prod X_\alpha$ is weakly Hausdorff. Consider any X_α. Let $x_\alpha \in X_\alpha$ be any element. Let $y_\alpha \in X_\alpha$ and $y_\alpha \neq x_\alpha$. For each $a \neq \alpha$ choose a $z_a \in X_a$. Now consider the elements x any y in $\prod X_a$ defined as follows: $x = (x_a)$ such that $x_a = z_a$ if $a \neq \alpha$ and the α-th coordinate of x is x_α, $y = (y_a)$ such that $y_a = z_a$ if $a \neq \alpha$ and the α-th coordinate of y is y_α. Now $x \neq y$ and $\prod X_a$ is weakly Hausdorff. So there is a regular open set O such that $x \in O$ and $y \notin O$. Since O is open there is a basic open set V such that $x \in V \subseteq O$. So now $y \notin \operatorname{Int} \operatorname{cl} V$ since $\operatorname{Int} \operatorname{cl} V \subseteq O$. Now $V = \prod W_a$ where each W_a is open in X_a and except for a finite number of indices $W_a = X_a$ itself. Since $x_a = y_a$ for all $a \neq \alpha$ and $y \notin V$, we must have that $W_\alpha \neq X_\alpha$. Let $W'_\alpha = \operatorname{Int} \operatorname{cl} W_\alpha$ in X_α. Then $V' = \prod Z_a$ where for all $a \neq \alpha$, $Z_a = W_a$ and $Z_\alpha = W'_\alpha$ is an open set such that $V' \supseteq V$ and $V' \subseteq O$ and $y \notin V'$. So $y_\alpha \notin W'_\alpha$. But $x_\alpha \in W'_\alpha$. Hence x_α belongs to a regular open set W'_α such that $y_\alpha \notin W'_\alpha$. So our assertion follows.

Theorem 1.5. *Let $(X, \mathcal{T})$ be a space having a dense subset H of cardinal θ $(\geqq \aleph_0)$. Then $X^{2^{\theta}}$ also has a dense subset of cardinal θ.*

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[2]) Cf. B. Pospíšil, Časopis pro pěst. mat. a fys. *67* (1937—8), 89—96.

Proof. The assertion is, in fact, proved in [2].

Theorem 1.6.[3]) *For each $a \in A$ let X_a be a weakly Hausdorff space with at least two elements. Let $X = \prod_{a \in A} X_a$ with the product topology. If X has a dense subset of cardinal $\leqq \theta$ with $\theta \geqq \aleph_0$ then each X_a has a dense subset of cardinal $\leqq \theta$ and $|A| \leqq 2^\theta$.*

Proof. Suppose X has a dense subset of cardinal $\leqq \theta$. Since for each a, X_a is a continuous image of X, X_a also has a dense subset of cardinal $\leqq \theta$. If $|A| > 2^\theta$ then $|X| > 2^{2^\theta}$. But by Theorems 1.4 and 1.2, X is weakly Hausdorff and $|X| \leqq 2^{2^\theta}$. Hence it follows $|A| \leqq 2^\theta$.

Theorem 1.7.[3]) [4]) *Let G be an infinite locally compact Hausdorff topological group. Then G has a dense subset of cardinal $\leqq \mathfrak{m}$ $(\mathfrak{m} \geqq \aleph_0)$ if and only if (1) $|G| \leqq 2^{2^{\mathfrak{m}}}$ and (2) G is a set union of $\aleph$ compact sets where $\aleph \leqq \mathfrak{m}$.*

Proof. Suppose G has a dense subset of cardinal $\leqq \mathfrak{m}$, then by Theorem 1.2 $|G| \leqq 2^{2^{\mathfrak{m}}}$. Since G is a locally compact group, G contains an open subgroup H such that $H \supset G_0$, the connected component at the identity of G, and H/G_0 is compact. Since G/H will also have a dense subset of cardinal $\leqq \mathfrak{m}$, we get that $|G/H| \leqq \mathfrak{m}$. Now since H/G_0 is compact we have that there exists a compact normal subgroup H_1 of H such that H/H_1 is separable metric [4] and locally compact. Now H/H_1 is σ-compact, for if V is a compact symmetric neighbourhood of the identity in H/H_1 then $\bigcup V^n$ is an open subgroup in H/H_1 which is σ-compact and which has a countable index in H/H_1. Hence H/H_1 is σ-compact. If $H/H_1 = \bigcup B_n$ each B_n compact in H/H_1 then, since H_1 is compact, the preimage of B_n is compact in H [7] and so we get H is σ-compact. Now G is a union of at most $\mathfrak{m}$ cosets of H each of which is σ-compact. Hence (2) follows.

Suppose (1) and (2) are satisfied. As above G has an open subgroup H such that $H \supset G_0$, the connected component at the identity e of G and H/G_0 is compact. Then H has a compact invariant subgroup H_1 such that H/H_1 is separable metric [4]. Now consider the compact group H_1, $|H_1| \leqq 2^{2^{\mathfrak{m}}}$. If the identity element is of character θ then $|H_1| = 2^\theta$ [3]. Hence we have that the character at the identity is $\leqq 2^{\mathfrak{m}}$, but any compact group is dyadic and we can assume that it is a continuous image of $D^{\mathfrak{n}}$, $D = \{0, 1\}$, D discrete, and $\mathfrak{n}$ is the weight of G. Also the weight of a dyadic compact space is the upper bound of all its point characters taken over any dense set (Efimov). Hence $\mathfrak{n} \leqq 2^{\mathfrak{m}}$. By Theorem 1.6, $D^{2^{\mathfrak{m}}}$ and hence D^n has a dense subset of cardinal $\leqq \mathfrak{m}$.

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[3]) We are proving this with the assumption $\mathfrak{m} > 2^{\mathfrak{n}}$ implies $2^{\mathfrak{m}} > 2^{2^{\mathfrak{n}}}$. This appears to be independent of the other axioms of set theory c.f. W. B. Easton: Powers of regular cardinals, dissertation, Princeton (1964).

[4]) This result was derived in a conversation with M. Rajagopalan.

Hence H_1 has a dense subset of cardinal $\leqq \mathfrak{m}$. Since H/H_1 is separable we get that H itself has a dense subset of cardinal $\leqq \mathfrak{m}$. Since H is open and G/H is a union of at most $\mathfrak{m}$ compact sets we get that the discrete space G/H is also a union of at most $\mathfrak{m}$ compact sets. But any compact set of G/H is finite. Hence $|G/H| \leqq \mathfrak{m}$. Since H has a dense set of cardinal $\leqq \mathfrak{m}$, we get that G, which is a union of at most $\mathfrak{m}$ cosets of H, also has a dense subset of cardinal $\leqq \mathfrak{m} \,.\, \mathfrak{m} = \mathfrak{m}$. This completes the proof.

§ 2. In this section we consider properties of weakly Hausdorff spaces and their relation with some of the other separation axioms. As most of the proofs are straightforward we leave them to the reader.

Proposition 2.1. *Any Hausdorff space is weakly Hausdorff but not conversely.*

Proof. Example for the converse[5]):

Let $X = [0, 1] \cup [2, 3] \cup A$ where A is a countable set disjoint from $[0, 1]$ and $[2, 3]$.

For each element of $[0, 1) \cup (2, 3]$, the usual neighbourhoods constitute a basis at that point. For each $x \in A$, $\{x\}$ itself is a neighbourhood of x. A set B containing 1 is a neighbourhood of 1 provided it contains some $(\varepsilon, 1]$, $0 < \varepsilon < 1$ and all but a finite number of elements of A. Similarly a set C containing 2 is a neighbourhood of 2 provided C contains some $[2, \varepsilon')$, $2 < \varepsilon' < 3$ and all but a finite number of elements of A. This gives rise to a topology $\mathscr{T}$ on X. $\mathscr{T}$ is not Hausdorff since every neighbourhood of 1 intersects every neighbourhood of 2. But $\mathscr{T}$ is weakly Hausdorff.

Remark 2.2. The example in the proposition 2.1 has the following further properties.

1. $(X, \mathscr{T})$ is compact.
2. $(X, \mathscr{T})$ is semi-regular (i.e. has a base of regular open sets) and is a T_1-space.
3. Sequences need not have unique limits.
4. Compact sets are not necessarily closed.
5. $(X, \mathscr{T})$ satisfies the first axiom of countability.
6. The set of limit points of a net need not be regularly closed.

Remark 2.3. Let $X = [0, 1] \cup [2, 3] \cup A$ where now A is an uncountable set; we define a topology $\mathscr{J}$ on X as follows: whenever $x \neq 1, 2$ a basis of neighbourhoods of x is defined as in Proposition 2.1.

If $x = 1$ then a base of neighbourhoods consists of all sets of the form $(\varepsilon, 1] \cup B$ where $0 < \varepsilon < 1$ and $B \subset A$ such that $A - B$ is countable. Similarly when $x = 2$,

[5]) This example is due to S. P. Franklin.

a base of neighbourhoods consists of all sets of the form $[2, \varepsilon') \cup C$, where $2 < \varepsilon' < 3$ and $C \subset A$ such that $A - C$ is countable. This $(X, \mathscr{J})$ has the following properties.

1. $(X, \mathscr{J})$ is weakly Hausdorff.
2. Compact sets are closed in X.
3. Any sequence converges to at most one element.
4. $(X, \mathscr{J})$ is semi-regular and T_1.
5. $(X, \mathscr{J})$ is not Hausdorff.

Remark 2.4. This shows that even under the properties (2) and (3), a weakly Hausdorff space need not be Hausdorff.

Proposition 2.5. *Any semi-regular T_1 space is weakly Hausdorff. But a weakly Hausdorff space need not be semi-regular.*

Proof. Since there are Hausdorff non semi-regular spaces [8], p. 99, the second part of the proposition follows.

Definition 2.6. *A space is called a π_2 space [8] if each open set is a union of regularly closed sets.*

Proposition 2.7. *Any π_2 and T_1 space is weakly Hausdorff but the converse is not true.*

Proof. Since there are Hausdorff spaces which are not π_2 [8], p. 99, the second part of the assertion follows.

Proposition 2.8. *There is a T_1 space in which compact sets are closed and any sequence converges to at most one point but which is not weakly Hausdorff.*

Proof. Let X be an uncountable set and let $\mathscr{T} = \emptyset \cup \{O \subset X \mid O' \text{ is countable}\}$.

Proposition 2.9. *Let $(X, \mathscr{T})$ be a weakly Hausdorff space. Then any open subset is again weakly Hausdorff. A regularly closed subset need not be weakly Hausdorff.*

Proof. For the second part consider the subset $A \cup \{1, 2\}$ in the example in Proposition 2.1.

Open Questions. 1. Is a compact weakly Hausdorff space semi-regular or π_2?

2. Characterize weakly Hausdorff spaces by convergence of nets.

3. The class of weakly Hausdorff spaces is a class of spaces between the classes of T_1 and T_2 spaces and for which Theorem 1.2, Proposition 1.4 and Theorem 1.6 hold. Here we have two questions:

Question 3.1. Is this a maximal class with the above properties?

Question 3.2. Is this the largest class with the above properties?

We conjecture that the answer is affirmative for both the questions 3.1 and 3.2 mentioned here.

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MADURAI UNIVERSITY, MADURAI, INDIA