COVERING PROPERTIES AND PRODUCT SPACES

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During the last five years General Topology has been noted for a rapid development of concepts and methods involving covering properties. The time is coming to make a coordination and a regulation in the investigations of so many covering axioms (see [12], Vocabulary on p. 74). A central role among them play Compactness and Paracompactness. The behavior of topological properties with respect to various kinds of mappings was investigated before and especially after the lecture of P. S. Aleksandrov at the first Prague Symposium. The Cartesian product $X \times Y$ of spaces X and Y does not inherit, in general, covering properties of X and Y. There are singular examples in this area (see [12], pp. 42–54). The following problem given by H. Tamano [7] in his lecture at the second Prague Symposium, is not yet solved:

"Which space X satisfies the condition that $X \times Y$ is normal for any paracompact space Y".

He proved there the statement: Let X be a completely regular space. Then $X \times Y$ is normal for any paracompact space Y iff $X \times Y$ is paracompact for any paracompact space Y.

We say that a paracompact space X is *productible*, if the product space $X \times Y$ is paracompact for any paracompact space Y.

X is productible, if any of the following conditions holds:

(1) X is compact (J. Dieudonné [1]).

(2) X is paracompact and σ-locally compact (K. Morita [4]).

(3) X is paracompact and it has a linearly locally finite covering by compact sets (H. Tamano [6]).

(4) X is a closed image of a paracompact, locally compact space (T. Ishii [2]).

(5) X is a closed image of a paracompact, perfectly normal, σ-locally compact space (M. Tsuda [14]).

(6) X is a closed image of a paracompact, σ-locally compact space (R. Telgársky [8], [9] and [10]).

(7) X has an order locally finite open cover $\{U_i : i \in I\}$ and a closed cover $\{F_i : i \in I\}$ by compact sets such that $F_i \subseteq U_i$ for each $i \in I$ (Y. Katuta [3]).

(8) X is paracompact and it has an order locally finite open cover $\{U_i : i \in I\}$

and a closed cover $\{F_i : i \in I\}$ by C-scattered sets such that $F_i \subseteq U_i$ for each $i \in I$ (R. Telgársky [8], [10]).

C-scattered space means: each its nonvoid closed subspace F has a point $p \in F$ with a compact neighborhood in F ([8], [9], [10]).

Among the previous conditions the following implications hold:

$$\begin{tikzcd}
 & & & (7) \arrow[dr] & \\
(1) \arrow[r] & (2) \arrow[r] & (3) \arrow[ur] \arrow[dr] & & (8) \\
 & & & (6) \arrow[ur] & \\
 & & (4) \arrow[ur] & & (5) \arrow[ul]
\end{tikzcd}$$

(9) X has a locally finite closed cover by productible spaces ([8], [10]).

(10) X is an F_σ-subset in a productible space ([8], [10]).

(11) X is the perfect image or the perfect preimage of a productible space ([8], [9], [10]).

(12) $X = Y \times Z$, where Y and Z are productible ([8], [10]).

(13) X is paracompact and it has a closed, locally compact subspace Y such that every closed subspace of X contained in $X \setminus Y$ is productible (J. Suzuki [5], R. Telgársky [8] and [10]).

(14) X is paracompact and it has a closed, σ-locally compact, G_δ-subset Y such that each closed subspace of X contained in $X \setminus Y$ is productible (R. Telgársky [8], [10]).

We are convinced that an appropriate generalization of "C-scattered" and "closure-preserving covering by compact sets" shall lead to the solution of the problem of H. Tamano.

For to see how the assumption "C-scattered" works for some stronger covering properties, we present the following results:

(a) If X is a paracompact scattered space and Y is an absolutely paracompact space, then the product space $X \times Y$ is absolutely paracompact (R. Telgársky [11]).

(b) Let $X = \{0\} \cup \{1/m + 1/n : m \in N \text{ and } n \in N\}$ equipped in the relative topology and Y be Cantor's set. Then X is totally hypocompact and scattered, Y is compact, $X \times Y$ is C-scattered, hypocompact, and absolutely paracompact, but $X \times Y$ is not totally hypocompact (R. Telgársky [11]).

(c) If X is a Lindelöf C-scattered space and Y is a Hurewicz space (Lindelöf space), then $X \times Y$ is a Hurewicz space (Lindelöf space, resp.) (R. Telgársky [11]).

(d) If X is a σ-compact regular space and Y is a Hurewicz space (Lindelöf space), then $X \times Y$ is a Hurewicz space (Lindelöf space, resp.) (R. Telgársky [11]).

In [13] we introduced the property: X has a closure-preserving covering by compact sets. We believe that a regular space with a closure-preserving covering by compact sets must be paracompact.[1]) If we consider completely regular spaces with a closure-preserving covering by compact sets, then the matter looks a bit simpler, because it suffices to prove only the normality of the spaces. It follows from the following statements:

(I) If X has the closure-preserving covering $\{C_i : i \in I\}$ by compact sets and Y is compact, then $X \times Y$ has the closure-preserving covering $\{C_i \times Y : i \in I\}$ by compact sets.

(II) If $X \times \beta X$ is normal, then X is paracompact — so reads the famous theorem of H. Tamano.

References

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[1]) Added in proof. Professor E. Michael has kindly communicated me that our conjecture is disproved by H. Potoczny in his paper "A non-paracompact space which admits a closure-preserving cover of compact sets", which will be published in Proc. Amer. Math. Soc.