TOPOLOGICAL METHODS IN MEASURE THEORY AND THE THEORY OF MEASURABLE SPACES

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There has been a growing interest in topological problems which have applications in measure theory and the theory of measurable spaces. The most important material seems to be related to the existence of projective limits in various categories of measure spaces; for a short expository account we refer to [F3] (the basic technique goes back to Kolmogorov, Bochner and Marczewski, generalized by Choksi and Metivier).

I do not intend to give a systematic survey, but merely a short selection of results and trends which interest me personally. In other words I will talk about the material discussed in my seminars (leaving out projective limits).

In the first two paragraphs some background material is presented. Paragraphs 3 and 4 concern the spaces of measures. The paragraphs 5 and 6 serve as an introduction to § 8 on uniform methods in the theory of measurable spaces. § 7 gives an important example on the construction introduced in § 6, and at the same time provides an exposition of some surprising recent results in non-separable descriptive theory. In conclusion we discuss quite mysterious Borel complete and Baire-Borel complete spaces.

1. Baire and Borel sets

The category of measurable spaces has measurable spaces (pairs $\mathscr{X} = \langle X, \mathscr{B} \rangle$ where $\mathscr{B}$ is a σ-algebra of subsets of X) for the objects, and measurable mappings for the morphisms. Denote by **Bo** the functor which assigns to each topological space X the Borel space of X, i.e. **Bo** X is the set X endowed with the smallest σ-algebra containing the open sets in X. The symbol **Bo** X will be used for the Borel space of X as well as for the structure of the Borel space, i.e. for the σ-algebra of Borel sets. In measure theory the functor **Bo** is commonly used; in many questions another functor is needed, namely the functor **Ba** which assigns to each X the Baire space of X which is defined to be the set X endowed with the smallest σ-algebra such that each continuous (real-valued) function on X is measurable. Again **Ba** X stands for both the Baire space of X and the structure of the Baire space, namely the σ-algebra of all Baire sets in X. Recall that **Ba** X is the smallest σ-algebra which contains all exact open (=co-zero) sets in X. If X is perfectly normal (in particular, if X is metrizable) then $\mathbf{Ba}\,X = \mathbf{Bo}\,X$; it is not known if every normal X such that $\mathbf{Ba}\,X = \mathbf{Bo}\,X$ is necessarily perfectly normal (this is a problem of M. Katětov).

It seems that $\mathbf{Ba}\,X$ more closely reflects the properties of X than $\mathbf{Bo}\,X$ does. For example, notice that

if K is compact Hausdorff, and if $\mathbf{Ba}\,K$ is countably generated then K is metrizable.

It is not known whether **Ba** may be replaced by **Bo** in this statement; the answer in affirmative would be an important theorem. I guess that the problem will be answered in negative.

The second example on close attachment of $\mathbf{Ba}\,X$ to X is the following non-trivial theorem [F 5] which says, roughly speaking, that a Baire measurable mapping of an analytic space into a metrizable space can be regarded continuous, and the image is analytic.

Theorem. *If A is analytic, M is metrizable and if $f : \mathbf{Ba}\,A \to \mathbf{Ba}\,M$ is measurable then there exists an analytic space A' such that $\mathbf{Ba}\,A' = \mathbf{Ba}\,A$ and $f : A' \to M$ is continuous. Hence $f[A]$ is analytic.*

It should be remarked that one can take the graph of f in $A \times M$ for A'.

I don't know of any survey of **Bo** and **Ba**. On the other hand an exposition of many classical results on abstract measurable spaces (mostly countably generated) is provided by B. V. Rao's thesis.

2. Baire and Borel measures

All topological spaces we consider are assumed to be completely regular. Let $\mathbf{C}_b(X)$ denote the Banach space of all bounded continuous functions on X. Everybody knows that the dual of $\mathbf{C}_b(X)$ can be represented as the space of all finitely additive regular signed measures on $\mathbf{Ba}\,X$; regular means that for each B in $\mathbf{Ba}\,X$ and each $r > 0$ there is a zero set $Z \subset B$ such that

$$B' \subset B - Z\,, \quad B' \in \mathbf{Ba}\,X \quad \text{implies} \quad |\mu B'| < r\,.$$

This representation of the dual of $\mathbf{C}_b(X)$ is usually denoted by $\mathbf{M}(X)$; here we shall use this symbol just for the positive cone, i.e. for the set of all non-negative measures (this will be essential in § 3 and § 4!). There are two distinguished topologies on $\mathbf{M}(X)$, the norm topology (the norm of μ is μX), and the weak topology ($\{\mu_a\}$ converges to μ if and only if $\mu_a(f)$ converges to $\mu(f)$ for each f in $\mathbf{C}_b(X)$).

Consider the following subspaces of $\mathbf{M}(X)$:

$\mathbf{P}(X)$: the probabilities (i.e. $\mu X = 1$).

$\mathbf{P}^2(X)$: the probabilities which assume two values only.

$\mathbf{M}_\sigma(X)$: σ-additive measures.

$\mathbf{M}_\downarrow(X)$: strongly continuous measures (i.e., if $\mu(f_\alpha)$ converges to 0 whenever $\{f_\alpha\}$ is a net in $\mathbf{C}_b(X)$, which pointwise decreases to zero).

$\mathbf{M}_t(X)$: tight or Radon measures (the functionals continuous w.r.t. the compact-open topology on $\mathbf{C}_b(X)$; another description: for each B in $\mathbf{Ba}\,X$ and each $r > 0$ there exists a compact set $K \subset B$ such that $B' \subset B - K$ implies $\mu B' < r$).

The meaning of $\mathbf{P}_\sigma(X)$ and $\mathbf{P}_\downarrow(X)$ is obvious. It is easy to see that

$$\mathbf{M} \supset \mathbf{M}_\sigma \supset \mathbf{M}_\downarrow \supset \mathbf{M}_t\,,$$
$$\mathbf{P}^2 \supset \mathbf{P}^2_\sigma \supset \mathbf{P}^2_\downarrow = \mathbf{P}^2_t\,.$$

The following two results are due to E. Hewitt [1]:

$\mathbf{P}(X) = \mathbf{P}_\sigma(X) \Leftrightarrow \mathbf{M}(X) = \mathbf{M}_\sigma(X) \Leftrightarrow X$ is pseudocompact.

$\mathbf{P}^2_\sigma(X) = \mathbf{P}^2_t(X) \Leftrightarrow X$ is realcompact.

Definition. A space X is called *measure-compact* if $\mathbf{M}_\sigma(X) = \mathbf{M}_\downarrow(X)$, a *Radon space* if $\mathbf{M}_\sigma(X) = \mathbf{M}_t(X)$.

It has been proved by W. Moran that the product of two measure-compact spaces need not be measure-compact ($S \times S$ where S is the Sorgenfrey line), and that the class of Radon spaces is finitely productive. There is no known topological characterization of measure-compact nor Radon topological spaces.

Since $\mathbf{C}_b(X)$ and $\mathbf{C}_b(\beta X)$ are isomorphic, $\mathbf{M}(X)$ and $\mathbf{M}(\beta X)$ are also isomorphic, and if $\tilde{\mu} \in \mathbf{M}(\beta X)$ corresponds to $\mu \in \mathbf{M}(X)$, then

$$\tilde{\mu}B = \mu(B \cap X)$$

for each $B \in \mathbf{Ba}(\beta X)$. Since βX is compact,

$$\mathbf{M}(\beta X) = \mathbf{M}_t(\beta X)\,.$$

Now clearly $\mu \in \mathbf{M}_t(X)$ if and only if X is Caratheodory measurable in $\langle \beta X, \mathbf{Ba}\,(\beta X), \tilde{\mu} \rangle$. Thus we can say that X is a Radon space if and only if X is Caratheodory measurable for every $\mu \in \mathbf{M}_\sigma(X)$. For example, every analytic space is a Radon space [F 3], [S 1], a locally compact space need not be a Radon space (it is if and only if it is measure-compact).

3. Prochorov spaces

In this section we assume that $\mathbf{M}(X)$ is endowed with the weak topology. By a Prochorov space we mean a space X with the following property:

If C is compact subset of $\mathbf{P}_t(X)$, and if $r > 0$ then there exists a compact set $K \subset X$ such that $\mu(X - K) < r$ for each μ in C.

Certainly every compact space is a Prochorov space. About ten years ago Varadarajan asserted that every Radon space is a Prochorov space (a Prochorov space need not be Radon), but his proof gives just that every locally compact space is a Prochorov space (the mapping $\mathbf{C}_b(X) \times \mathbf{M}_t(X) \to \mathbf{R}$ is continuous where $\mathbf{C}_b(X)$ is given the compact-open topology). Varadarajan noticed that each G_δ subspace of a Prochorov space is a Prochorov space.

This year D. Preiss proved that the space Q of rational numbers is not a Prochorov space, and that an absolute Borel separable metrizable space is completely metrizable if and only if it does not contain a copy of Q as a G_δ subspace. Hence

Theorem (Preiss [1]). *The following three properties on a separable metrizable absolute Borel space X (more generally, coanalytic space) are equivalent:*

1. *X is completely metrizable.*
2. *X contains no G_δ copy of* Q.
3. *X is a Prochorov space.*

It should be remarked that V. Prochorov proved in 1956 that every separable completely metrizable space is a Prochorov space. For a connection of Prochorov spaces with null sets of vector measures we refer to G. Choquet.

In conclusion we should note that a separable metrizable Prochorov space need not be completely metrizable, and that it would be useful to complete the study of Prochorov spaces.

4. BC-spaces

Call a space X a BC-space if the following condition is fulfilled:

If T is a 0-dimensional compact space then for every continuous mapping $f : T \to \mathbf{P}_t(X)$ there exists a Radon probability μ on $\mathbf{C}(T, X)$ endowed with the compact-open topology such that

$$f = f_\mu$$

where $f_\mu t$ is the image of μ under the evaluation mapping e_t at t; by definition

$$f_\mu\, t(B) = \mu(e_t^{-1}[B]) .$$

It is easy to see that if T is compact and X is any space, then each $f_\mu : T \to \mathbf{P}_t(X)$ is continuous. R. M. Blumental and H. H. Corson [1] proved:

Theorem. *Every completely metrizable space is a* BC-*space.*

Every BC-space is a Prochorov space. Indeed if C is a compact subset of $\mathbf{P}_t(X)$ then we can take a continuous mapping f of a 0-dimensional compact space T onto C.

Let μ be a Radon probability on $\mathbf{C}(T, X)$ such that $f = f_\mu$. Given $r > 0$ take a compact subset K' in $\mathbf{C}(T, X)$ such that $\mu K' > 1 - r$. Consider the image K of $T \times K'$ in X under the mapping $T \times \mathbf{C}(T, X) \to X$. Clearly $ftK > 1 - r$ for each $t \in T$.

It follows that in the theorem of § 3 we can add one more equivalent condition: X is a BC space.

The statement "X is a BC-space" is equivalent to: if f is a continuous mapping of a 0-dimensional compact space into $\mathbf{P}(X)$ then there exists a Radon probability μ on $\mathbf{C}(T, X)$ such that the random function

$$\{e_t : \langle \mathbf{C}(T, X), \mu \rangle \to X\}$$

has the 1-dimensional distributions prescribed by f. With this motivation in mind, one is interested in the properties of the set of all probabilites representing a given mapping, and also in the existence theorems for T as general as possible. We mention another theorem by Blumenthal and Corson [2]:

Theorem. *Let $\mathbf{P}_a(X)$ be the space of all $\mu \in \mathbf{P}(X)$ such that X is the support of μ. If X is a compact metrizable space then X is connected and locally connected if and only if each continuous mapping f of a compact metrizable space T into $\mathbf{P}_a(X)$ can be represented by a Radon measure on $\mathbf{C}(T, X)$ (i.e. $f = f_\mu$ for some μ).*

5. Separable measurable spaces

In this paragraph let $\mathscr{X} = \langle X, \mathscr{B} \rangle$ be a measurable space.

A family $\{B_a \mid a \in A\}$ is called $\mathscr{B}$-preserving if the union of each subfamily $\{B_a \mid a \in A'\}$, $A' \subset A$, belongs to $\mathscr{B}$. A family $\{X_a \mid a \in A\}$ is called $\mathscr{B}$-discrete if there exists a $\mathscr{B}$-preserving disjoint family $\{B_a\}$ such that $B_a \supset X_a$ for each a. Hence, a disjoint $\mathscr{B}$-preserving family is $\mathscr{B}$-discrete, and a disjoint cover of X is $\mathscr{B}$-discrete if and only if it is $\mathscr{B}$-preserving. Observe that a disjoint cover $\{X_a \mid a \in A\}$ of X is $\mathscr{B}$-discrete if and only if the associated quotient mapping $f : \langle X, \mathscr{B} \rangle \to \langle A, \exp A \rangle$ is measurable ($f[X_a] = (a)$). A $\mathscr{B}$-discrete family $\{X_a \mid a \in A\}$ need not be $\mathscr{B}$-preserving even if $X_a \in \mathscr{B}$ for each a. For example, take a family $\{X_a \mid a < \omega_1\}$ such that X_a is a Baire set of class a in the unit interval I, and consider the subset $B' = \sum\{X_a\}$ in the sum space $X = \sum\{I \mid a < \omega_1\}$. Observe that X is metrizable, B' is no Baire set in X, but $\{(a) \times X_a \mid a \in A\}$ is $\mathbf{Ba}(X)$-discrete.

Definition. Call a measurable space $\langle X, \mathscr{B} \rangle$ *separable* if every $\mathscr{B}$-discrete cover of X is countable.

If $2^{\aleph_1} > 2^{\aleph_0}$ then every countably generated measurable space is separable. If there exists an uncountable separable metrizable space X such that $\exp X = \mathbf{Ba}\, X$, then $\langle X, \mathbf{Ba}\, X \rangle$ is countably generated but not separable.

On the other hand, a separable measurable space need not be countably generated. This follows from the following consequence of the theorem in § 1.

Theorem. *If A is analytic then* **Ba** *A is separable. If $\langle X, \mathscr{B}\rangle$ is a pseudoanalytic space then $\mathscr{B}$ is separable.*

In some spaces X there is a close connection between **Ba**-discreteness and topological discreteness. This will be shown in § 7.

6. Hyper-rocks

We have noticed that the union of a $\mathscr{B}$-discrete family in $\mathscr{B}$ need not belong to $\mathscr{B}$. We shall see later on that it is a nice property of $\mathscr{B}$ if the union of every $\mathscr{B}$-discrete family in $\mathscr{B}$ belongs to $\mathscr{B}$. We say that a σ-algebra with this nice property is **h**-closed.

Definition. If $\mathscr{B}$ is a σ-algebra on X, we denote by $\mathbf{h}\mathscr{B}$ the smallest σ-algebra $\mathscr{C} \supset \mathscr{B}$ with the property: the union of each $\mathscr{B}$-discrete family in $\mathscr{C}$ belongs to $\mathscr{C}$. We denote by $\mathbf{H}\mathscr{B}$ the smallest σ-algebra $\mathscr{C} \supset \mathscr{B}$ with $\mathbf{h}\mathscr{C} = \mathscr{C}$.

Clearly $\mathbf{h}\mathscr{B} = \mathscr{B}$ if and only if $\mathscr{B}$ is **h**-closed, and $\mathbf{h}\mathscr{B} = \mathscr{B}$ is equivalent to $\mathbf{H}\mathscr{B} = \mathscr{B}$.

It is obvious that $\mathbf{H}\mathscr{B}$ is obtained by transfinite iteration of **h**, i.e.

$$\mathbf{H}\mathscr{B} = \bigcup\{h_\alpha\mathscr{B}\}$$

where $h_\alpha\mathscr{B} = \mathbf{h}\mathscr{C}$ with $\mathscr{C}$ being the smallest σ-algebra which contains $\bigcup\{h_\beta\mathscr{B} \mid \beta < \alpha\}$.

Evidently $\mathbf{h}\mathscr{B}$ can be obtained by iterating the operations involved. The operation of taking σ-discrete unions is very important, and therefore some details may be in place.

Definition. Let $\mathscr{B}$ be a σ-algebra on X. Denote by $\mathbf{d}_{\mathscr{B}}$ the map from exp (exp X) into itself which assigns to each $\mathscr{C} \subset \exp X$ the set of all unions of $\mathscr{B}$-discrete families in $\mathscr{C}$.

The operation $\mathbf{d}_{\mathscr{B}}$ need not be idempotent; however, obviously, it is if $\mathscr{B}$ is **h**-closed.

The other operation involved is ϱ, which assigns to each $\mathscr{C} \subset \exp X$ the smallest $\mathscr{D} \supset \mathscr{C}$ with $\mathscr{D}_\sigma = \mathscr{D}_\delta = \mathscr{D}$. The elements of $\varrho\mathscr{D}$ are called the *rocks* over $\mathscr{D}$. The operation ϱ is idempotent, and $\varrho\mathscr{D}$ is a σ-algebra if $\mathscr{D}$ is.

Now we may write

$$\mathbf{h}\mathscr{B} = \bigcup\{\mathscr{B}_\alpha\},$$

where $\mathscr{B}_0 = \mathscr{B}$, and $\mathscr{B}_\alpha = \mathbf{d}_{\mathscr{B}}\mathscr{B}'_\alpha$ or $\mathscr{B}_\alpha = \varrho\mathscr{B}'_\alpha$ according to as α is odd or even, with $\mathscr{B}'_\alpha = \bigcup\{\mathscr{B}_\beta \mid \beta < \alpha\}$.

Remark. Sometimes the following situation occurs: We are given a σ-algebra $\mathcal{C} \subset \mathcal{B}$, and we consider the smallest σ-algebra $\mathcal{D} \supset \mathcal{C}$ with $\mathbf{d}_{\mathcal{B}}\mathcal{D} = \mathcal{D}$. This σ-algebra is denoted by $\mathbf{h}_{\mathcal{B}}\mathcal{C}$. This is the situation when one investigates completeness of bi-measurable spaces, see § 10.

There are two more functors of measurable spaces into itself. The first assigns to each σ-algebra the smallest σ-algebra with the property that the meet of two discrete covers is discrete, and the other one assigns to each $\mathcal{B}$ the σ-algebra of all bi-Souslin sets over $\mathcal{B}$; for the definition see § 7.

7. Non-separable Baire sets

The main result says:

Theorem. *If $\mathcal{B}$ is the σ-algebra of all Baire sets in a completely metrizable space X then*

$$\mathcal{C} = \mathbf{H}\mathcal{B} = \mathbf{h}\mathcal{B} = \textbf{bi-Souslin}\ \mathcal{B} = \textbf{bi-Souslin}\ X,$$

and $\mathcal{C}$ is locally determined in X.

Recall that if $\mathcal{M}$ is a cover of a set Y, then **Souslin** $\mathcal{M}$ stands for the collection of all Souslin sets over $\mathcal{M}$, and **bi-Souslin** $\mathcal{M}$ stands for the collection of all $M \subset Y$ such that the two sets M and $Y - M$ belong to **Souslin** $\mathcal{M}$. If X is a space, then

$$\textbf{Souslin}\ X = \textbf{Souslin}\ \mathcal{M}, \quad \textbf{bi-Souslin}\ X = \textbf{bi-Souslin}\ \mathcal{M}$$

where $\mathcal{M}$ is the collection of all closed sets in X. The Souslin sets over $\mathcal{M}$ are the sets of the form

$\mathbf{S}M = \bigcup\{\bigcap\{Ms \mid s \in S, s < \sigma\} \mid \sigma \in \Sigma\}$ where S is the set of the finite sequences in N, Σ is the set of all sequences in N, $<$ indicates restriction, and $Ms \in \mathcal{M}$ for each s.

The crucial step in the proof of Theorem is the following important and very new result by R. W. Hansell [1]:

Lemma. *Assume that $\{X_a\}$ is a disjoint family in a completely metrizable space X such that the union of each subfamily is a Souslin set in X. Then we can write $X_a = \bigcup\{X_{an} \mid n \in \mathsf{N}\}$, $a \in A$, where $\{X_{an} \mid a \in A\}$ is topologically discrete in X for each n.*

We may assume that each X_{an} is closed in X_a, and then $\{X_{an}\}$ is a σ-discrete refinement of $\{X_a\}$ by Souslin sets. It is easy to show that each σ-discrete family of Souslin sets has the property that the union of each subfamily is Souslin. Since

$$\mathcal{B} \subset \textbf{Souslin}\ \mathcal{B} \subset \textbf{Souslin}\ X$$

in any space X (with $\mathscr{B} = \mathbf{Ba}\, X$), in our setting we get

$$\mathbf{H}\mathscr{B} = \mathbf{h}\mathscr{B} \subset \mathbf{Souslin}\, X\,.$$

By another theorem of Hansell [3] (First principle type theorem)

$$\mathbf{bi\text{-}Souslin}\, X \subset \mathbf{h}\mathscr{B}\,.$$

In the next section we shall need another deep property of Baire-discrete covers in completely metrizable spaces.

Proposition. *Every Baire-discrete cover of a completely metrizable space is bounded, i.e. the supremum of the Baire classes of the members is countable.*

The proof depends on Hansell's Lemma.

Corollary. *Let X be a completely metrizable space, and let $\{X_a\}$, $\{Y_b\}$ be two Baire-discrete covers of X. Then $\{X_a \cap Y_b\}$ is Baire-discrete.*

Proof. Every σ-discrete (topologically) family of Baire sets of a bounded class is Baire-preserving.

Remark. The meet of two $\mathscr{B}$-discrete covers need not be $\mathscr{B}$-discrete. E.g. consider the measurable product $\langle Y, \mathscr{B}\rangle$ of $\langle X, \exp X\rangle$ by itself, and assume that $\mathscr{B} \neq \exp Y$. Clearly $\{(x) \times X \mid x \in X\}$ and $\{X \times (x) \mid x \in X\}$ are $\mathscr{B}$-discrete, but the meet $\{(\langle x, y\rangle) \mid x \in X, y \in X\}$ is not $\mathscr{B}$-discrete.

8. Basic functors into uniform spaces

It seems to me that an appropriate application of uniform methods to measurable spaces may be useful for the theory of measurable spaces as well as for the theory of uniform spaces. At least there are some highly non-trivial results the discovery of which has been stimulated by the viewpoint of uniform spaces. Note that it has been known for a long time that all non-trivial examples in the general theory of uniform spaces were "measurable".

Throughout this paragraph let $\mathscr{X} = \langle X, \mathscr{B}\rangle$ be a measurable space; it is convenient to assume that $\mathscr{B}$ is just an algebra, not necessarily a σ-algebra.

The proximity induced by $\mathscr{B}$ is designated by $\mathbf{p}_{\mathscr{B}}$, or simply $\mathbf{p}$, and is defined by setting

$$M_1 \mathbf{p} M_2 \Leftrightarrow M_1 \subset B\,, \quad M_2 \subset X - B \quad \text{for no} \quad B \in \mathscr{B}\,.$$

Evidently $\mathbf{p}$ has the following property: if M_1 and M_2 are distant then there exists a set B such that $M_1 \subset B$, $M_2 \subset X - B$, and B is distant to $X - B$. Such a proximity is called *measurable*. It is easy to see that $\{\mathscr{B} \to \mathbf{p}_{\mathscr{B}}\}$ is a bijection of algebras onto measurable proximities; if p is measurable then $p = \mathbf{p}_{\mathscr{B}}$ where $\mathscr{B} = \{Y \mid Y \text{ is distant to } X - Y\}$.

The proximities induced by σ-algebras are described as follows:

Theorem 1 [F 9]. *A proximity p on a set X is induced by a σ-algebra if and only if the following condition is fulfilled:*

If the union of a sequence $\{A_n\}$ is proximal to B then some A_n is proximal to B.

Thus one might say that p is induced by a σ-algebra if and only if p is σ-additive. Observe that σ-additivity implies measurability. The next result describes the compactifications associated with σ-additive proximities.

Theorem 2 [F 9], [Hager 1]. *Let $\langle X, p\rangle$ be a dense proximal subspace of a compact space K. Then p is σ-additive if and only if K is basically disconnected, X is G_δ-dense in K, and X is a P-space.*

It would be of some interest to find how to study measurable spaces by means of the corresponding compact spaces.

The precompact uniformity associated with $\mathbf{p}_{\mathscr{B}}$ is denoted by $\mathbf{u}_{\aleph_0}\mathscr{B}$, and the corresponding uniform space is denoted by $\mathbf{u}_{\aleph_0}\mathscr{X}$. Clearly $\{\mathscr{B} \to \mathbf{p}_{\mathscr{B}}\}$ defines a functor of measurable spaces into proximity spaces, and $\{\mathscr{B} \to \mathbf{u}_{\aleph_0}\mathscr{B}\}$ defines a functor into uniform spaces. It is easy to see that $\mathbf{u}_{\aleph_0}\mathscr{B}$ is projectively generated by all bounded $\mathscr{B}$-measurable functions. Let $\mathbf{u}_{\aleph_1}\mathscr{B}$ be the uniformity projectively generated by all measurable functions (not necessarily bounded).

Proposition. *The uniform partitions in $\mathbf{u}_{\aleph_1}\mathscr{X}$ form a basis for the uniform covers. If $\mathscr{U}$ is a separable uniformity with the precompact part $\mathbf{u}_{\aleph_0}\mathscr{X}$, then $\mathscr{U}$ is contained in $\mathbf{u}_{\aleph_1}\mathscr{B}$.*

Thus $\mathbf{u}_{\aleph_1}\mathscr{B}$ is proximally fine among separable uniformities. The property that the uniform partitions form a basis will be needed throughout, and we must agree on a short name for the uniformities with this property.

Definition. An *ultra-uniformity* is a uniformity with the property that the uniform partitions form a basis for the uniform covers.

An expository account of results and methods in separable uniform spaces with particular attention to measurable spaces is provided by A. Hager's papers listed in References, which are primarily concerned with the rings of functions.

Here we want to discuss the non-separable uniformities. There are some old problems connected with general uniformities, and therefore we restrict our attention to ultra-uniformities.

Definition. Let m be an infinite cardinal. Denote by $\mathbf{u}_m\mathscr{B}$ the uniformity on X which has all $\mathscr{B}$-discrete covers of cardinal less than m for a sub-basis of uniform covers. Put $\mathbf{u}\mathscr{B} = \bigcup\{\mathbf{u}_m\mathscr{B}\}$.

It is easy to see that this definition for $m = \aleph_0$, and $\aleph_1$ gives the previously defined uniformities.

One can say that $\mathbf{u}_m\mathscr{B}$ is projectively generated by all mappings

$$f : X \to M ,$$

where M is a uniformly discrete space of cardinal less than m, and $f : \langle X, \mathscr{B} \rangle \to \to \langle M, \exp M \rangle$ is measurable.

We noticed in § 7 that the meet of two $\mathscr{B}$-discrete covers need not be $\mathscr{B}$-discrete, and hence the uniformity $\mathbf{u}_m\mathscr{B}$ need not be proximally equivalent to $\mathbf{u}_{\aleph_0}\mathscr{B}$.

Definition. An algebra $\mathscr{B}$ is called *proximally fine* if $\mathbf{u}\mathscr{B}$ is proximally equivalent to $\mathbf{u}_{\aleph_0}\mathscr{B}$.

In other words, $\mathscr{B}$ is proximally fine if and only if the meet of any two $\mathscr{B}$-discrete covers is a $\mathscr{B}$-discrete cover.

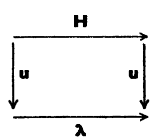
Example. The Baire σ-algebra of a completely metrizable space is proximally fine. This is proved in the conclusion of § 7.

Theorem 3. *A σ-algebra $\mathscr{B}$ is* **h**-*closed if and only if it is proximally fine and* $\mathbf{u}\mathscr{B}$ *is locally fine.*

Recall that a uniform space is called locally fine if every uniformly locally uniform cover is uniform, see Isbell [1]. The proof of Theorem 3 is routine. Observe that in Theorem 3 one may replace locally fine by: every uniformly locally "finite uniform" cover is uniform.

It is shown in § 7 that the σ-algebra of Baire sets in a completely metrizable space X need not be **h**-closed, and it is a result of Hansell [3] that it is not if X is a Baire space, i.e. if $X = M^{\aleph_0}$ for some uncountable discrete space M.

It is easy to see that $\mathbf{uH}\mathscr{B}$ is the locally fine coreflection of $\mathbf{u}\mathscr{B}$. Thus the following diagram is commutative:



where **H** is a coreflection in measurable spaces defined in § 6, and λ is the coreflection of uniform spaces onto locally fine uniform spaces.

For the details and for the consequences in general theory of uniform spaces we refer to [F 8].

9. Another functors into uniform spaces

We get a scale of functors depending on an infinite cardinal m if we consider all partitions of cardinal less than m by measurable sets. This uniformity seems to be too fine, and I don't see anything interesting to be said about these functors.

There is another possibility, something between $\mathbf{u}_m$ and the latter one.

Definition. Let $\mathcal{X} = \langle X, \mathcal{B} \rangle$ be a measurable space. A *free* family in $\mathcal{B}$ is a family $\{B_a \mid a \in A\}$ such that there is a well ordering $\leqq$ on A such that

$$\bigcup\{B_a \mid a < b\} \in \mathcal{B}$$

for each b in A.

It is easy to check that the meet of two free partitions is a free partition. Thus we can define the functors $\mathbf{v}_m$ into ultra-uniform spaces, where the free partitions of cardinal less than m form a basis for uniform covers in $\langle X, \mathbf{v}_m\mathcal{B} \rangle$.

These functors may be quite interesting. I don't know of any deep theorem, however some examples I know are of certain interest.

10. Complete measurable spaces

A measurable space $\mathcal{X} = \langle X, \mathcal{B} \rangle$ is called complete if the uniform space $\mathbf{u}_{\aleph_1}\mathcal{X}$ is complete. It is immediate that $\mathcal{B}$ is complete if and only if every $\mathcal{B}$-ultrafilter with the countable intersection property (abb. CIP) is fixed.

It is proved in [F 7] that $\mathcal{B}$ is complete if and only if $\mathbf{h}\mathcal{B}$ is complete, and the cardinal of every $\mathcal{B}$-discrete partition is non-measurable. One may replace $\mathbf{h}$ by $\mathbf{H}$.

Examples. A topological space X is said to be *Baire complete* or *Borel complete* if, respectively, $\mathbf{Ba}\,X$ or $\mathbf{Bo}\,X$ is complete. By the Hewitt's theorem quoted in § 2 a space is Baire complete if and only if it is realcompact. Almost nothing is known about Borel complete spaces, see Hager [4] and [F 8].

Definition. Let $\mathcal{C}$ and $\mathcal{B}$ be σ-algebras on X, and let $\mathcal{C} \supset \mathcal{B}$. We say that the bi-measurable space $\langle X, \mathcal{B}, \mathcal{C} \rangle$ is *complete* if every $\mathcal{B}$-ultrafilter which extends to a $\mathcal{C}$-ultrafilter with CIP is fixed.

Clearly if $\mathcal{B}$ or $\mathcal{C}$ is complete then $\langle \mathcal{B}, \mathcal{C} \rangle$ is complete. A topological space X is called *Baire-Borel complete* if $\langle \mathbf{Ba}\,X, \mathbf{Bo}\,X \rangle$ is complete. The category of all Baire complete spaces is a simple epireflective category, the product of an uncountable number of at least two point spaces is not Borel complete. The category of Baire-Borel complete spaces is epireflective; is it the epireflective hull of the Borel complete spaces? For the details and other results we refer to [F 7, 8].

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