REMARKS ON TRANSFINITE DIAMETERS[1])

E. HILLE

New Haven

1. Introduction. The *transfinite diameter* is a set function introduced by MIHÁLY FEKETE in 1923. It was originally defined for bounded closed sets in the complex plane. It coincides with the *Čebyšev constant* and with the *logarithmic capacity* of the set. If the set is a continuum and its complement is connected, then the transfinite diameter also coincides with the *exterior conformal mapping radius*. It is possible to define similar notions in any Euclidean space and even in arbitrary metric spaces.

In this direction G. PÓLYA and G. SZEGÖ took up the study of the three dimensional case and various other extensions in 1931. The important investigations of F. LEJA started in 1933; he examined in particular the conformal mapping aspects of the problem, various associated sequences of polynomials, extensions to the space of two complex variables, aed to general metric spaces. Elliptic and hyperbolic metrics for complex numbers were considered by M. TSUJI (1947). The notion of capacity has been much extended by G. CHOQUET, but he does not seem to have considered possible connections with transfinite diameters.

There is an extensive abstract theory of mean values going back to A. N. KOLMOGOROFF and M. NAGUMO (1930) that has an important bearing on our problem.

2. Transfinite diameters. The general notion of a transfinite diameter involves four essential elements:

[1] *A metric space* X.
[2] *An averaging* (mean value) *process* $\mathscr{A}$.
[3] *An extremal problem.*
[4] *A limiting process.*

Let E be a *compact* set in X. Take any n, $n > 1$, points $P_1, P_2, \ldots, P_n$ in E and form the distances

$$\delta_{jk} = d(P_j, P_k)\,, \quad 1 \leqq j < k \leqq n\,, \tag{1}$$

which are

$$N = \tfrac{1}{2}n(n-1)$$

in number. This is the first step.

[1]) This research was supported in part by a grant (DA-ORD-12) from the U. S. Army Research Office (Durham).

The second step involves the averaging process $\mathscr{A}$. Here it is convenient to impose the conditions of Kolmogoroff:[2])

(i) *$\mathscr{A}$ assigns a positive average to every finite set of positive numbers $\{x_j\}$.*

(ii) *$A(x_1, x_2, \ldots, x_m)$ is a continuous symmetric function of its arguments and A is strictly increasing as a function of each of them.*

(iii) $A(x, x, \ldots, x) = x$.

(iv) $A(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m) = A(y, y, \ldots, y, x_{k+1}, \ldots, x_m)$ *if* $y = A(x_1, \ldots, x_k)$.

As a consequence of (ii) and (iii) we have the important inequality

$$\min x_j \leqq A(x_1, x_2, \ldots, x_m) \leqq \max x_j \,, \tag{2}$$

where equality holds if and only if all x_j are equal.

It should be observed that $A(x_1, \ldots, x_m)$ decreases to a nonnegative limit if one or more of the variables decreases to zero, so we can define $A(0, x_2, \ldots, x_m)$ by continuity. It is clear that $A(0, 0, \ldots, 0) = 0$, but it may happen that A is zero if one of its arguments is zero. In particular this happens for what in Section 4 below is called the *natural averaging process in E_n* as soon as $n > 1$. It is clear that in (ii) we must restrict ourselves to strictly positive values of the argument.

A. N. Kolmogoroff and M. Nagumo proved that conditions (i)–(iv) imply that A has a particular form: there exists a continuous strictly monotone function $F(u)$ such that

$$m\, F[A(x_1, x_2, \ldots, x_m)] = \sum_{j=1}^{m} F(x_j)\,. \tag{3}$$

We shall not use this representation. For our purposes it is just as convenient, if not more so, to work merely with the assumptions (i)–(iv). Since mean values defined by (3) have already been used in the theory of transfinite diameters by F. Leja, we cannot expect to produce any new results, but the method of proof based on the abstract postulates offers some advantages.

We return to the set of N numbers δ_{jk} and apply $\mathscr{A}$ to this set. We write

$$A(\delta_{11}, \delta_{12}, \ldots, \delta_{n-1,n}) = \mathscr{A}(\delta_{jk})\,. \tag{4}$$

It follows from (2) that

$$0 < \mathscr{A}(\delta_{jk}) \leqq \delta(E)\,, \tag{5}$$

where $\delta(E)$ is the point set theoretical diameter of the set E and equality holds if and only if $\delta_{jk} = \delta(E)$ for all j and k. It follows that the set $\{\mathscr{A}(\delta_{jk})\}$ is bounded when the points P_j range over E. The set has a supremum and, since E is compact, there is at least one choice of the points for which

$$\mathscr{A}(\delta_{jk}) = \sup \mathscr{A}(\delta_{pq}) \equiv \delta_n(E)\,. \tag{6}$$

[2]) I am indebted to Professors C. T. Ionescu Tulcea and Shizuo Kakutani for reminding me of the literature on mean values. I had rediscovered some of the results. [I am also indebted to Professors Z. Frolík and V. Jarník whose valuable observations led to a revision of the manuscript in November 1961.]

We have $\delta_2(E) = \delta(E)$ and for all $n > 1$

$$0 < \delta_n(E) \leqq \delta(E)\,. \tag{7}$$

M. Fekete worked with the geometric mean, G. Pólya and G. Szegö mostly with the harmonic mean. Other cases have been considered by them and by F. Leja.

Next we show that the sequence $\{\delta_n(E)\}$ is never increasing. To prove this, let us choose $n + 1$ points P_j such that $\delta_{n+1}(E)$ is the $\mathscr{A}$-average of the distances $d(P_j, P_k)$. It is immaterial if we take the $\frac{1}{2}n(n + 1)$ distances $d(P_j, P_k)$ with $j < k$ or the $n(n + 1)$ distances with $j \neq k$. This follows from (iv). We choose the second alternative and proceed to separate the elements into groups in two different ways. First we separate the δ_{jk} into $n + 1$ groups of n elements each, the elements in the j-th group being the distances from P_j to the other points. Let the average of the distances from P_j to the other points be denoted by η_j so that

$$A(\delta_{j,1}, \ldots, \delta_{j,j-1}, \delta_{j,j+1}, \ldots, \delta_{j,n+1}) = \eta_j\,. \tag{8}$$

By property (iv) we can replace each element in the j-th group by η_j. In other words, $\delta_{n+1}(E)$ is the average of $\eta_1, \eta_2, \ldots, \eta_{n+1}$, each repeated n times. Using (iv) again, we see we can contract this average to the average of $\eta_1, \eta_2, \ldots, \eta_{n+1}$ taken singly so that

$$\delta_{n+1}(E) = A(\eta_1, \eta_2, \ldots, \eta_{n+1})\,. \tag{9}$$

We can now apply (2) which says that either all the η's are equal to each other and hence to δ_{n+1} or else

$$\min \eta_j < \delta_{n+1}(E) < \max \eta_j\,. \tag{10}$$

Next we group the $n(n + 1)$ quantities δ_{jk} in a different manner. We separate the $2n$ elements involving a distance from P_j in one group and lump the remaining $n(n - 1)$ distance into the other. The average of the first $2n$ elements is η_j by (iv) and (8), while the average of the remaining elements is at most δ_n by the definition of the latter. It follows that $\delta_{n+1}(E)$ cannot exceed the $\mathscr{A}$-average of $n(n - 1)$ quantities δ_n and $2n$ quantities η_j, or again contracting with the aid of (iv)

$$\delta_{n+1} \leqq A(\delta_n, \delta_n, \ldots, \delta_n, \eta_j, \eta_j)\,, \tag{11}$$

where δ_n is repeated $n - 1$ times. Here we want to apply (2). Either all the η's are equal to δ_{n+1} in which case the inequality gives

$$\delta_{n+1} \leqq A(\delta_n, \delta_n, \ldots, \delta_n, \delta_{n+1}, \delta_{n+1})\,, \tag{12}$$

or we can find an $\eta_j < \delta_{n+1}$. For this value of j, formula (11) again gives (12), but now as a strict inequality. Next, either $\delta_n = \delta_{n+1}$ or

$$A(\delta_n, \delta_n, \ldots, \delta_n, \delta_{n+1}, \delta_{n+1}) < \max(\delta_n, \delta_{n+1})\,.$$

In the latter case

$$\delta_{n+1} < \max(\delta_n, \delta_{n+1})$$

which implies that

$$\delta_{n+1} < \delta_n\,.$$

Thus, in any case

$$\delta_{n+1}(E) \leqq \delta_n(E) \tag{13}$$

as asserted.

We can now define

$$\delta_0(E) \equiv \lim_{n\to\infty} \delta_n(E) \tag{14}$$

as the *transfinite diameter of E with respect to the average* $\mathscr{A}$.

The set function $\delta_0(E)$ has important properties of monotony and continuity.

I. *If* E_1 *and* E_2 *are compact and if* $E_1 \subset E_2$, *then*

$$\delta_0(E_1) \leqq \delta_0(E_2)\,.$$

This follows from the definition: any average of distances between points of E_1 is also an average of distances between points of E_2 so that the supremum of $\mathscr{A}(\delta_{jk})$ with respect to E_1 for a given n cannot exceed the corresponding supremum with respect to E_2. It may very well happen that $\delta_0(E_1) = \delta_0(E_2)$ even though E_1 is a very small subset of E_2. Thus if ∂E denotes the (outer) boundary of E, we often have the relation

$$\delta_0(\partial E) = \delta_0(E)\,. \tag{15}$$

II. *If* E_ε *is the set of points whose distance from* E *does not exceed* ε, *and if* E *and* E_ε *are compact, then*

$$\lim_{\varepsilon\to 0} \delta_0(E_\varepsilon) = \delta_0(E)\,. \tag{16}$$

This is essentially a consequence of the continuity and monotony properties of A. The argument given by M. Fekete for the plane case carries over with minor modifications.

3. Čebyšev functions. The original problem of Fekete is closely related to the theory of Čebyšev polynomials. Similar structures can be introduced in any metric space and for any averaging process satisfying the above conditions.

Let E be a compact set in a metric space X. Let $P_1, P_2, \ldots, P_n$, and P be arbitrary points of X and form the average of the distances from P to the points P_j. Set

$$f(p) \equiv A[d(P, P_1), d(P, P_2), \ldots, d(P, P_n)]\,. \tag{17}$$

For fixed points $P_1, P_2, \ldots, P_n$, not necessarily distinct, this is a continuous function of P which tends to a nonnegative limit when P approaches any one of the points P_j. The function $f(P)$ has a supremum on the given compact set E and this value is reached for at least one choice of P in E. As the points P_j range independently of each other over X, the corresponding maxima $m[f]$ form a set of nonnegative real numbers. This set has an infimum which may be zero. In any case the infimum is attained, that is, there is at least one choice of base points $Q_1, Q_2, \ldots, Q_n$ such that if

$$\check{C}_n(P) \equiv A[d(P, Q_1), d(P, Q_2), \ldots, d(P, Q_n)]\,, \tag{18}$$

then

$$\max_{P \in E} \check{C}_n(P) = \min \max_{P \in E} f(P) \equiv M_n(E) . \tag{19}$$

We call $\check{C}_n(P)$ a Čebyšev function for E of order n. It is immaterial for our purposes if $\check{C}_n(P)$ is unique. In the case studied by Fekete

$$\check{C}_n(P) = |T_n(z)|^{1/n} ,$$

where $T_n(z)$ is the unique Čebyšev polynomial for E of degree n.

The numbers $M_n(E)$ are nonnegative and they do not exceed $\delta(E)$ for any n. In order to prove that the sequence $\{M_n(E)\}$ tends to a limit, we find it convenient to impose an additional, possibly redundant, condition on the averaging process $\mathscr{A}$.

(v) *The average of n entries a and one entry b tends to a when n becomes infinite.*

Let us denote this average by a_n. Then *the sequence $\{a_n\}$ is monotone.* To fix the ideas, suppose that $0 < a < b$, so that the sequence becomes *strictly decreasing.* We note first that for all n

$$a < a_n < b .$$

We now form

$$a_n = A(a, \ldots, a, b) = A(a, a_{n-1}, a_{n-1}, \ldots, a_{n-1}) .$$

In the first average, a occurs n times, in the second a_{n-1} occurs $n - 1$ times. Equality between the two averages follows from (iv). We now observe that the second average is less than a_{n-1} by inequality (2) since $a < a_{n-1}$. It follows that

$$a_n < a_{n-1} \quad \text{if} \quad a < b . \tag{20}$$

Thus the sequence $\{a_n\}$ is strictly decreasing so that it has a limit $\geqq a$; condition (v) asserts that the limit equals a.

We now return to the numbers $M_n(E)$. They satisfy two important inequalities, namely

$$M_{n+1} \leqq A[M_n, M_n, \ldots, M_n, \delta(E)] , \tag{21}$$

$$M_{m+n} \leqq \max(M_m, M_n) . \tag{22}$$

Here M_n is repeated n times in (21) and m, n are arbitrary positive integers. For the proof we form functions of type (17) involving the base points of the Čebyšev functions under consideration. In the first case, let

$$F(P) \equiv A[d(P, Q_1), \ldots, d(P, Q_n), d(P, Q)] ,$$

where $Q_1, \ldots, Q_n$ are the base points of $\check{C}_n(P)$ and Q is an arbitrary point in E. Since, by (iv)

$$F(P) = A[\check{C}_n(P), \ldots, \check{C}_n(P), d(P, Q)] ,$$

we see that the maximum of $F(P)$ on E cannot exceed the right member of (21). On the other hand, this maximum is at least equal to M_{n+1}. This proves (21). In the second case we form

$$G(P) \equiv A[d(P, P_1), \ldots, d(P, P_m), d(P, Q_1), \ldots, d(P, Q_n)] = \\ = A[\check{C}_m(P), \ldots, \check{C}_m(P), \check{C}_n(P), \ldots, \check{C}_n(P)]$$

with obvious notation. The maximum of $G(P)$ on E is at least equal to M_{m+n} and at most equal to

$$A(M_m, \ldots, M_m, M_n, \ldots, M_n) \leqq \max(M_m, M_n) .$$

This gives (22).

Suppose now that the positive numbers $\{c_k\}$ satisfy

$$c_{m+n} \leqq \max(c_m, c_n) \tag{23}$$

for all m and n. Such a sequence need not be convergent. Thus if

$$c_k = \alpha_m \text{ for } k = 2^m p, \quad m = 0, 1, 2, \ldots, \; p = 1, 3, 5, \ldots,$$

where $\{\alpha_m\}$ is a never increasing sequence of positive numbers, then $\{c_k\}$ satisfies (23) and there are infinitely many limit points if there are infinitely many distinct α's. In any case we have

$$\lim_{m\to\infty} c_{2m-1} = \limsup c_n \equiv \beta . \tag{24}$$

To prove this we note first that (23) implies that $c_n \leqq c_1$ for all n, so such a sequence is necessarily bounded. Next, if there is a γ, $0 < \gamma$, and an integer j such that

$$c_j < \gamma , \quad c_{j+1} < \gamma ,$$

then $c_n < \gamma$ for all large n. It suffices that $n > j^2$. Now if $\delta > 0$ is given, we can find an N as large as we please such that

$$\beta - \delta < c_N , \quad \beta = \limsup c_n .$$

From

$$\beta - \delta < c_N \leqq \max(c_k, c_{N-k}) ,$$

it follows that either c_k or c_{N-k} exceeds $\beta - \delta$, where $k = 1, 2, \ldots, N - 1$. Since δ and N are arbitrary, we conclude that

$$\beta \leqq c_k$$

holds for at least half the positive integers. Moreover, since $c_j < \beta$, $c_{j+1} < \beta$, implies $c_n < \beta$ for all large n, either

$$\beta \leqq c_{2m} \quad \text{for all } m , \tag{25}$$

or

$$\beta \leqq c_{2m-1} \quad \text{for all } m . \tag{26}$$

In the second case, (24) obviously holds.

Now if c_n has a unique limit, the latter must coincide with β and (24) holds a fortiori. In order to prove that (24) is always true, it is enough to show that (26) always holds. Suppose, then, that there is an odd integer $2k + 1$ such that

$$c_{2k+1} < \beta .$$

By (23) we have then for every positive integer p

$$c_{(2k+1)p} < \beta .$$

This, however, implies that neither (25) nor (26) can hold for all m and we have seen that at least one of them must be true. This contradiction shows that (26) is always true and this proves (24).

We can apply this analysis to the case $c_k = M_k(E)$. In view of the inequality (22) we see that

$$\lim_{m\to\infty} M_{2m-1}(E) = \limsup_{n\to\infty} M_n(E) \equiv \beta \,. \tag{27}$$

We shall show that condition (21) together with (v) implies the existence of

$$\lim_{n\to\infty} M_n(E) = \chi(E) \tag{28}$$

and the inequality

$$M_n(E) \geqq \chi(E) \tag{29}$$

for all n. In view of (27) it is sufficient to prove (29) in order to obtain (28).

We know that (29) holds for all odd values of n. Suppose there is an even value, $2k$ say, such that

$$M_{2k}(E) = \gamma < \beta \,.$$

In view of (22) we have then also

$$M_{2^pk}(E) \leqq \gamma \,, \quad p = 1, 2, 3, \ldots \tag{30}$$

We now use (21) with $n = 2^pk$ and we replace $\delta(E)$ by a larger number η. We have then certainly $\gamma < \eta$ and (21) gives

$$M_{2^pk+1}(E) < A(\gamma, \gamma, \ldots, \gamma, \eta) \,,$$

where γ is repeated 2^pk times. By (27) the left member tends to β when $p \to \infty$ and by condition (v) the limit of the right member is $\gamma < \beta$. This contradiction shows that (30) cannot hold for any k and p. It follows that (29) holds and this implies the uniqueness of the limit of $M_n(E)$.

We call this limit $\chi(E)$ the *Čebyšev constant* of E.

If A satisfies (i)–(iv) we have also

$$\chi(E) \leqq \delta_0(E) \,. \tag{31}$$

This is proved as follows. We choose $P_1, P_2, \ldots, P_n$ in E so that

$$\delta_n = \delta_n(E) = \mathscr{A}(\delta_{jk})$$

and then P_{n+1} also in E such that

$$\begin{aligned} &A[d(P_{n+1}, P_1), \ldots, d(P_{n+1}, P_n)] = \\ &= \max_{P\in E} A[d(P, P_1), \ldots, d(P, P_n)] \geqq M_n(E) \,. \end{aligned}$$

By the definition of δ_{n+1} and property (iv) we have

$$\begin{aligned} \delta_{n+1} &\geqq A(\delta_{11}, \ldots, \delta_{n-1,n}, \delta_{1,n+1}, \ldots, \delta_{n,n+1}) = \\ &= A(\delta_n, \ldots, \delta_n, M_n, \ldots, M_n) \,, \end{aligned}$$

where δ_n is repeated $\frac{1}{2}n(n-1)$ times and M_n occurs n times. By (2)

$$\delta_{n+1} \geqq \min(\delta_n, M_n) = M_n$$

since $\delta_n \geqq \delta_{n+1}$. This gives (31) when $n \to \infty$.

For most of the known cases equality holds in (31), but it is not always true.

4. Euclidean spaces. Let $\mathscr{X} = \mathscr{E}_n$ be the Euclidean space of n dimensions, $n \geqq 1$, with its natural metric. We may ask if there is also a *natural averaging* process in such a space. This question is too vague to admit of an answer, but we could possibly accept a process $\mathscr{A}$ as natural in $\mathscr{E}_n$, if it gives $\delta_0(S) = R$ for a sphere S of radius R. Such an average exists and it is defined by formula (3) where the function $F(u) = F_n(u)$ is defined as follows:

$$F_n(u) = \begin{cases} u\,, & n = 1\,, \\ \log \dfrac{1}{u}\,, & n = 2\,, \\ \dfrac{1}{u^{n-2}}\,, & n > 2\,. \end{cases} \tag{32}$$

Even without using the results of A. N. Kolmogoroff and M. Nagumo, we can prove that such an averaging process $\mathscr{A}_n$ satisfies conditions (i)–(iv). If $n = 2$, $\mathscr{A}_2$ is the geometric mean used by M. Fekete and for $n = 3$, $\mathscr{A}_3$ is the harmonic mean considered by G. Pólya and G. Szegö. In both cases it is known that $\delta_0(S) = R$ for a "sphere" of radius R.

For $n = 1$ the following result is obtained. *If S is a closed set, contained in the interval $[a, b]$ and containing the points a and b, then the transfinite diameter with respect to $\mathscr{A}_1$, that is, the arithmetic mean, is given by*

$$\delta_0(E) = \tfrac{1}{2}(b - a)\,. \tag{33}$$

A "sphere" in this case is an interval, so we have $\delta_0(S) = R$ as desired. In this case the boundary of E consists of the two points a and b and $\delta_0(\partial E)$ is also given by (33) so that formula (15) holds. The proof of (33) is elementary. We have

$$\delta_n(E) = \max \sum_{k=1}^{n} (n - 2k + 1)\, x_k\,, \tag{34}$$

where $b \geqq x_1 \geqq x_2 \geqq \ldots \geqq x_n \geqq a$ and each x_k belongs to E. The maximum is reached for $x_k = a$ or b according as $n - 2k + 1 < 0$ or > 0. In the limit we obtain (33).

For $n > 3$ we have to delve into the theory of hyperspherical harmonic functions and related theories of capacity. The Čebyšev function for the sphere S in $\mathscr{E}_n$ with radius R and center at the origin is

$$\check{C}_n(P) = d(P, 0) \tag{35}$$

for all n so that $\chi(S) = R$. This is with respect to the average $\mathscr{A}_n$. On the other hand, the integral

$$(36) \qquad \omega_n^{-1} \int\limits_{\partial S} [d(P, Q)]^{2-n} \, \mathrm{d}Q \,,$$

extended over the surface of the sphere whose area is ω_n, represents a harmonic function of P (with respect to the n dimensional Laplace operator) provided $d(P, Q) > > R$ and the value of this function is

$$[d(P, Q)]^{2-n} \,.$$

It follows that the double integral

$$(37) \qquad \omega_n^{-2} \int\limits_{\partial S} \int\limits_{\partial S} [d(P, Q)]^{2-n} \, \mathrm{d}P \, \mathrm{d}Q = R^{2-n} \,.$$

But this is the *energy integral* corresponding to the *equilibrium distribution* on the sphere and it has been proved by OTTO FROSTMAN [4] that the the minimum value of the energy, corresponding to the equilibrium distribution, equals the transfinite diameter with respect to the average $\mathscr{A}_n$. Frostman gives this only for $n = 3$ but the argument extends to higher dimensions.

We have no assurance that the average $\mathscr{A}_n$ is the only choice for which $\delta_0(S) = R$ even though this seems plausible. There are some results of Frostman's for averages $\mathscr{A}_\alpha$, where α need not be an integer, which enable us to compute $\delta_0(S)$ for S in E_n even if $\alpha \neq n$. It seems likely that Frostman's assumption $n = 3$ can be generalized.

References

[1] *G. Choquet*: Theory of Capacities. Annales de l'Institut Fourier, *5* (1953–54), 131–295.

[2] *M. Fekete*: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math. Zeitschrift, *17* (1923), 228–249.

[3] *M. Fekete*: Über den transfiniten Durchmesser ebener Punktmengen, I, II. Math. Zeitschrift, *32* (1930), 108–114, 215–221. III. Ibid., *33* (1933), 635–646.

[4] *O. Frostman*: Potentiel d'Équilibre et Capacité des Ensembles avec quelques Applications à la Théorie des Fonctions. Thèse, Lund, 1935.

[5] *E. Hille*: Analytic Function Theory. Vol. II, Chapter 16, Ginn, Boston 1961.

[6] *A. N. Kolmogoroff*: Sur la notion de la moyenne. Atti della Reale Accademia Nazionale dei Lincei, (*6*) *12*, (1930), 388–391.

[7] *F. Leja*: Sur la définition du diamètre et de l'écart transfini d'un ensemble. Annales de la Société Polonaise de Mathématique, *12* (1934), 29–34.

[8] *F. Leja*: Généralisation de certaines fonctions d'ensemble. Annales de la Société Polonaise de Mathématique, *16* (1937), 41–52.

[9] *F. Leja*: Teoria Funkcji Analitycznych. Chapter XI, Bibliography. Biblioteka Matematyczna, Vol. 14, Warszawa 1957.

[10] *M. Nagumo*: Über eine Klasse der Mittelwerte. Japanese Journal of Mathematics, *7* (1930), 71–79.

[11] *G. Pólya* and *G. Szegö*: Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen. Journal für die reine und angewandte Mathematik, *165* (1931), 4–39.

[12] *M. Tsuji*: Potential Theory in Modern Function Theory. Chapter III, Bibliography, Maruzen, Tokyo 1959.