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ON CLOSED LATTICES

by

Jan PAVELKA

Let T be a set. Consider the contravariant hom-functor

$$T^{(-)}: \text{Set} \longrightarrow \text{Set}.$$

For an ordinal number α there is the natural isomorphism

$$\varkappa_X (T^X)^\alpha \approx (T^\alpha)^X$$

defined

$$[\varkappa_X(f_\xi \mid \xi < \alpha)]\, x = (f_\xi\, x \mid \xi < \alpha)$$

If we are now given an α-ary operation $\sigma: T^\alpha \longrightarrow T$ on T there is a natural way to define operations σ_X on the sets T^X; we put

$$\sigma_X = \sigma^X \,.\, \varkappa_X \,.$$

In this context the word "natural" has a precise meaning, namely: for any map $f: x \longrightarrow Y$ the map $T^f: T^Y \longrightarrow T^X$ carries a homomorphism between the algebras $\langle T^Y, \sigma_Y \rangle$, $\langle T^X, \sigma_X \rangle$; in other words, the maps σ_X form a natural transformation

$$\sigma: (T^{(-)})^\alpha \longrightarrow T^{(-)}. \qquad (0.1)$$

Moreover, by Yoneda lemma any transformation (0.1) is induced in this way by the operation

$$\sigma_{T^\alpha}(p_\xi \mid \xi < \alpha)$$

on T, where $p_\xi : T^\alpha \longrightarrow T$ are projections from the cartesian power of T.

It is in the above sense that the power set $PX \approx 2^X$

of a set X owes its natural structure of a complete boolean algebra to the two-point boolean algebra $\mathbf{2}$. Thus, if we want to follow the idea of Zadeh and replace 2 by another set T of membership degrees to obtain " T-fuzzy sets", then the crucial problem is to devise an algebraic structure on which would yield the most fruitful generalization of the set-theoretical calculus.

As the present state of affairs indicates, the structure of a closed lattice, which is the topic of this introductory article, is one of the most likely candidates.

1. Closedness structures on lattices

The general theory of closed categories as developed in [2] deals with the following situation: on a category $\mathcal{K}$ bifunctors

$$\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}, \quad [-,-] : \mathcal{K}^* \times \mathcal{K} \longrightarrow \mathcal{K} \tag{1.1}$$

are given, together with an adjunction in two variables

$$\sigma_{XYZ} : \mathcal{K}(X \otimes Y, Z) \approx \mathcal{K}(X, [YZ]). \tag{1.2}$$

Here we shall be concerned with the special case that arises when $\mathcal{K}$ is small, thin, and skeletal; i.e. a partially ordered set.

For a partially ordered set P, a couple of adjoint bifunctors (1.1) reduces to a couple of binary operations

such that

(1.3) $\otimes : P \times P \longrightarrow P$ is isotone in both variables,

(1.4) $[-,-] : P \times P \longrightarrow P$ is isotone in the second and antitone in the first variable, (1.5) the adjointness condition

$$x \otimes y \leq z \text{ iff } x \leq [yz]$$

holds for all $x,y,z \in P$.

Alternatively, (1.5) can be expressed as

(1.6) for all $x,y \in P$, $x \leq [y, x \otimes y]$ and $[xy] \otimes x \leq$ $\leq y$.

The strength that (1.5) possesses as a link between $\otimes$ and $[-,-]$ is perhaps best illustrated by the following

1.1. Proposition (cf.[6]). Let $\otimes, [-,-]$ be as above and let $e \in P$. Then the following are couples of equivalent conditions on P:

(T1)	$(x \otimes y) \otimes z \leq x \otimes (y \otimes z)$	$[yz] \leq [[xy][xz]]$	(H1)
(T2)	$x \otimes e = x$	$x = [ex]$	(H2)
(T3)	$e \otimes x = x$	$x \leq y$ iff $e \leq [xy]$	(H3)
(T4)	$x \otimes y = y \otimes x$	$x \leq [yz]$ iff $y \leq [xz]$	(H4)
(T5)	$(x \otimes y) \otimes z = x \otimes (y \otimes z)$	$[x \otimes y, z] = [x[yz]]$	(H5)

In accordance with [2] we shall use the term "monoidal closed (shortly, MC) structure on P " to designate a triple $\langle \otimes, [-,-], e \rangle$ where $\otimes, [-,-]$ satisfy (1.3) - (1.5) and $\otimes$, e satisfy (T2),(T3), and (T5). If, moreover, $\otimes$ satisfies (T4), we shall speak about a symmetric monoidal closed (shortly SMC) structure on P.

Observe that

(i) (T1) together with (T4) already imply (T5) so that an SMC-structure is also defined by axioms (T1) - (T4).

(ii) (T4) together with (H5) imply

(H4!) $[x[yz]] = [y[xz]]$,

which therefore holds in any SMC-structure.

(iii) For a general MC-structure the condition (H5), in view of (H3), implies (1.5) and can be understood to express the adjointness "inside P".

(iv) (H4) says that the antitone functions $[-,x]$ are right self-adjoints. A remark analogous to (iii) can be made about (H4!).

For any couple of adjoint operations $\langle \otimes, [-,-] \rangle$ we also have

(TO) the functions $-\otimes x: P \longrightarrow P;\ x \in P$ preserve all suprema that exist in P, and

(HO) the functions $[x,-]: P \longrightarrow P;\ x \in P$ preserve all infima that exist in P.

Moreover, if P is a complete lattice (as we shall assume from now on) then

(i) any isotone binary operation $\otimes$ satisfying (TO) has a right adjoint, uniquely determined by $\otimes$, whose values are given by the formula

$$[yz] = \bigvee\{x \mid x\otimes y \leq z\} \tag{1.7}$$

(ii) Any antiisotone-isotone binary operation $[-,-]$ satisfying (HO) has a left adjoint, uniquely determined by $[-,-]$, whose values are given by the formula

$$x\otimes y = \bigwedge\{z \mid x \leq [yz]\} \tag{1.8}$$

Thus any isotone associative operation $\otimes$ with a two-sided unit e on a complete lattice L is a part of an MC-structure on L iff it satisfies (TO). In [1] a complete lattice endowed with such an operation is called a cl-monoid provided it also satisfies

$(TO)^*$ the functions $x \otimes - : L \longrightarrow L$; $x \in L$ preserve all suprema.

Remark: Unless $\otimes$ is commutative, $(TO)^*$ doest not follow from (TO). What can be proved is that it is equivalent to

$(HO)^*$ the functions $[-,x] : L \longrightarrow L$; $x \in L$ transfer all suprema to infima.

Indeed, assume $(TO)^*$ and let $Y \subset L$, $z \in L$. Since $[-,z]$ is antitone we have $[\bigvee Y,z] \leq \bigwedge[Y,z]$. Hence it suffices to show that the converse inequality also holds in order to prove $(HO)^*$. By (1.5), $\bigwedge[Y,z] \leq [\bigvee Y,z]$ iff $\bigwedge[Y,z] \otimes \bigvee Y \leq z$. By (TO), $\bigwedge[Y,z] \otimes \bigvee Y =$ $= \bigvee\{\bigwedge[Yz] \otimes y \mid y \in Y\}$. Because $\bigwedge[Yz] \otimes y \leq [yz] \otimes y \leq$ $\leq z$ holds for any $y \in Y$ this part is complete. The proof of $(HO)^* \Longrightarrow (TO)^*$ is analogous.

Still, it is useful to assume (TO) for non-commutative $\otimes$ - it guarantees that the opposite multiplication

$$x \otimes^* y = y \otimes x$$

also has an adjoint, say $[-,-]^*$, and allows us to treat both resulting MC-structures simultaneously, as it is done in [4]. Trivially one then has

$$x \leq [yz] \text{ iff } y \leq [xz]^*$$

so that each $[-,z]$ is again a contravariant adjoint on the right. If $\otimes$ is commutative then $[-,-]$ and $[-,-]^*$ coincide and $[-,z]$ is self-adjoint, as was already noted.

1.2. Example: If a complete lattice L is completely distributive then $\langle L, \wedge, 1 \rangle$, where $\wedge$ is the meet in L and 1 is the greatest element of L, is a commutative cl-monoid and therefore a part of an SMC-structure on L: the so called Heyting algebra on L. The right adjoint to $\wedge$ is denoted by $y \longrightarrow z$ and called the Heyting operation on L. (Concerning terminology: a category $\mathfrak{K}$ is called cartesian closed if it has finite products and the product functor is a left adjoint. Since $x \wedge y$ is the categorial product of the objects $x, y \in L$, we have completely distributive = cartesian closed.)

1.3. Example: Let R be an associative ring with unit. Let L denote the complete lattice of all two-sided ideals in R. The operation

$$\mathfrak{X} \otimes \mathcal{Y} = \{ \sum_{i=1}^{n} a_i b_i \mid n \in \mathbb{N}, a_i \in \mathfrak{X}, b_i \in \mathcal{Y} \}$$

makes L into a cl-monoid. Again, the unit coincides with the greatest element 1 = R of L. If R is commutative, so is $\otimes$.

In the sequel we shall restrict our attention to those MC-structures on a lattice L whose unit coincides with the unit of L (for the reason see [7]) and call them, in accordance with [1], integral MC-structures.

The latter example, though rather esoteric from our point of view, motivated an investigation of integral MC-structures on lattices carried out - long before the concept of a closed category was proposed - by R.P. Dilworth and M. Ward (see [3],[4]). Concerning terminology:

in [3], the operation [-,-] is called residuation and $[x,y]$ is denoted by $y:x$.

In [4] where non-symmetric structures are considered,

$$\left.\begin{matrix}[xy]\\ [xy]^*\end{matrix}\right\}\text{ is denoted by } \left\{\begin{matrix} y.x^{-1} \\ x^{-1}.y\end{matrix}\right\} \quad \text{and called}$$

the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ residual of x with respect to y.

The above mentioned papers concentrate on the following questions.

Q_1: Do lattices with nice properties admit integral MC-structures?

Q_2: How far does an integral MC-structure on L affect its lattice structure?

We shall not go into much detail here, but let us at least touch Q_1.

As to completely distributive lattices Q_1 is settled in the affirmative (Example 1.2). Anyway, complete distributivity is not a necessary condition because we have

1.4. Example: Let L be a complete lattice containing an element a with the properties

(i) $a<1$,

(ii) for any $x \in L$ either $x \leq a$ or $x \geq a$,

(iii) the sublattice $\{x \mid x \geq a\}$ is a chain.

Put

$$x \otimes y = \begin{cases} 0 & \text{if} \\ x \wedge y & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\otimes$ makes L into a commutative integral cl-monoid.

The above example suggests that the answer to Q_1 depends primarily on the behavior of L "in the vicinity of 1". Indeed, let L admit an integral MC-structure and let $\otimes$ be its multiplication. Then by monotony of $\otimes$

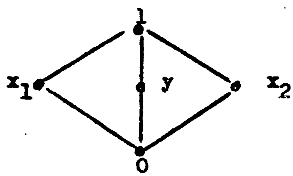
(1.9) $\quad x \otimes y \leq (1 \otimes y) \wedge (x \otimes 1) = x \wedge y$

and by (TO)

(1.10) for any $X \subseteq L,\ y \in L$: $\bigvee X = 1 \Longrightarrow$

$$\Longrightarrow y = (\bigvee X) \otimes y = \bigvee \{ x \otimes y \mid x \in X \} \leq \bigvee \{ x \wedge y \mid x \in X \}.$$

Now observe that this already disqualifies the modular lattice



($X = \{x_1, x_2\}$ and y fail to satisfy the necessary condition (1.10)).

2. Special lattices: chains and boolean algebras

We have already noted that if the lattice L in question is completely distributive, questions Q_1 and Q_2 cease to be interesting. On the other hand, new problems arise such as:

Q_3: How many integral MC (or, in particular, SMC) structures does L admit? Can they be described and classified in a reasonable way?

Q_4: For two integral MC-structures $\mathcal{S}_1 = \langle \otimes_1, [-,-]_1 \rangle$,

$\mathcal{S}_2 = \langle \otimes_2, [-,-]_2 \rangle$ on L put

$\mathcal{S}_1 \le \mathcal{S}_2 \equiv_{df} x \otimes_1 y \le x \otimes_2 y$ for all $x,y \in L$

(this condition is equivalent to $[xy]_1 \ge [xy]_2$, all $x,y \in L$). (1.9) says that the Heyting algebra on L is the greatest element of the resulting partially ordered set $\mathfrak{S}(L)$. What else can be proved about $\mathfrak{S}(L)$?

Q_5: Given an SMC-structure on L, the function $\varphi = [-,0]$ has the following properties:

(i) φ sends all suprema to infima,

(ii) $\varphi x \ge x \longrightarrow 0$ for all $x \in L$,

(iii) $x \le \varphi\varphi x$ for all $x \in L$.

Given a function $\varphi : L \longrightarrow L$ satisfying (i) - (iii), does there exist an SMC-structure on L with $\varphi = [-,0]$?

We shall end this article by several results in this direction.

2.1. Proposition ([3]). Every boolean algebra admits exactly one MC-structure.

Proof. First we realize that a completely distributive lattice is a boolean algebra iff $x \vee (x \longrightarrow 0) = 1$ holds for all $x \in L$. Now suppose $\langle \otimes, [-,-] \rangle$ is an integral MC-structure on L. For any $x \in L$ we have $x = (x \vee (x \longrightarrow 0)) \otimes x =$
$= (x \otimes x) \vee ((x \longrightarrow 0) \otimes x) \le (x \otimes x) \vee ([x,0] \otimes x) = x \otimes x$
whence $x \wedge y \le (x \wedge y) \otimes (x \wedge y) \le x \otimes y$ holds for all $x,y \in L$. By (1.9), the proof is complete.

On the other hand, there are $2^{2^{\aleph_0}}$ non-isomorphic SMC-structures on the compact interval of reals (see [5]) with continuous multiplication. To these, a theorem of Mostert and Shields can be applied to obtain a fairly simple description of all possibilities. The classification of general SCM-structures on $I = [0,1]$ seems to be an open problem.

2.2. Proposition. Denote by $\mathfrak{S}_0(L)$ the subposet of $\mathfrak{S}(L)$ consisting of all symmetric elements. Every nonempty chain in $\mathfrak{S}_0(L)$ has a supremum.

Proof. Let $\emptyset \neq \mathcal{U}$ be a chain in $\mathfrak{S}_0(L)$. We show that

$$x \overline{\otimes} y = \bigvee\{x \otimes y \mid \langle \otimes, [-,-] \rangle \in \mathcal{U}\}$$

defines the supremum of $\mathcal{U}$ in $\mathfrak{S}_0(L)$. Since $\overline{\otimes}$ is l.u.b. of the multiplicative parts in $L^{L\times L}$ it suffices to prove that it is a part of some SMC-structure on L. The equalities $x = x \overline{\otimes} 1 = 1 \overline{\otimes} x$, $x \overline{\otimes} y = y \overline{\otimes} x$, and preservation of suprema are evident. That leaves associativity to be proved.

We have $(x \overline{\otimes} y) \overline{\otimes} z = \bigvee_{\otimes \in \mathcal{U}}((\bigvee_{\otimes' \in \mathcal{U}}(x \otimes' y)) \otimes z) = \bigvee_{\otimes \in \mathcal{U}} \bigvee_{\otimes' \in \mathcal{U}}((x \otimes' y) \otimes z)$ by (TO). Since $\mathcal{U}$ is a chain this equals $\bigvee_{\otimes \in \mathcal{U}}((x \otimes y) \otimes z) = \bigvee_{\otimes \in \mathcal{U}}(x \otimes y \otimes z)$. By the same argument (we need (TO)*, but that is taken care of by the commutativity of $\otimes$'s), $\bigvee_{\otimes \in \mathcal{U}}(x \otimes y \otimes z) = x \overline{\otimes} (y \overline{\otimes} z)$.

2.3. Proposition. If L is a chain then any $\varphi : L \to L$ with properties (i) - (iii) is $[-,0]$ for some SMC-structure on L.

Proof: Put

$$x \otimes y = \begin{cases} 0 & \text{if } x \leq \varphi y \\ x \wedge y & \text{otherwise} \end{cases}$$

and check that it works.

In [5] it is shown that in case of $L = [0,1]$ and φ an antiisomorphism of L there is an SMC-structure isomorphic with $\langle \mathrm{Max}\ (0, x + y - 1), \mathrm{Min}\ (1, 1 - y + z) \rangle$ such that $\varphi = [-,0]$.

References

[1] Birkhoff, G.: Lattice theory, AMS, Providence, Rhode Island 1967

[2] Eilenberg S. and Kelly G.M.: Closed Categories, in Proceedings of the Conference on Categorial Algebra, La Jolla 1965, Springer-Verlag 1966, 421-562

[3] Dilworth R.P. and Ward M.: Residuated lattices, Trans. Amer. Math. Soc. 45(1939), 335-354

[4] Dilworth R.P.: Non-commutative residuated lattices, Trans. Amer.Math. Soc. 46(1939), 426-444

[5] Menu J. and Pavelka J.: A note on tensor products on the unit interval, Comment. Math. Univ. Carolinae 17(1976), 71-83

[6] Pavelka J.: A note on closed categories, submitted to Comment. Math. Univ. Carolinae

[7] Pultr A.: On categories over the closed categories of fuzzy sets, these Proceedings, pp.