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The algebraic formulation of the axioms of quantum field theory

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Quantum field theory may be considered as an attempt to combine quantum mechanics with the special theory of relativity. The basic principles of both theories are incorporated in the Wightman-Gårding axioms. There exist other general frameworks for quantum field theory, and the so called gauge theories of elementary particle physics may not fit into this scheme as it stands now. (These theories have yet to be formulated in a rigorous way.) The Wightman-Gårding axioms are, however, probably the simplest and most clear-cut of the various proposals. A study of the consequences of these axioms and a search for physically interesting models has been a major occupation of mathematical physicists in the last 10-20 years and has advanced developments in other brances of physics and mathematics as well.

I give here a brief account of a reformulation of these axioms in terms of linear functionals on a topological algebra due to Borchers [6] , cf. also Uhlmann [7]. I shall only treat the simplest case, that of a single, hermitean, neutral, scalar Bose field.

Let $\mathcal{S}_n$ denote the space of complex, rapidly decreasing Schwartz test functions on $\mathbb{R}^{4n}$ and define $\underline{\mathcal{S}}$ as the locally convex direct sum $\bigoplus_{n=0}^{\infty} \mathcal{S}_n$ $(\mathcal{S}_0 = \mathbb{C})$. Thus the elements of $\underline{\mathcal{S}}$ are sequences $\underline{f} = (f_0, f_1, \ldots f_m, \ldots)$ with $f_\nu \in \mathcal{S}_\nu$ and all but finitely many f_ν are 0. $\underline{\mathcal{S}}$ becomes a $*$-algebra if one defines a product by $(\underline{f} \times \underline{g})_n = \sum_{\mu+\nu=n} f_\mu \otimes g_\nu$ with the usual tensor product for functions, and a $*$-operation by $(\underline{f}^*)_n (x_1, \ldots x_n) = \overline{f_n (x_n, \ldots x_1)}$ (complex

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conjugation and inverted order of the arguments). The product is continuous in each factor separately, and the $*$-operation is a continuous involution. The Poincaré group, i.e. the group of Lorentz-transformations on $\mathbb{R}^4$ followed by translations, acts as a group of automorphisms on $\underline{\mathcal{S}}$ in the following way: If $a \in \mathbb{R}^4$ is a translation and Λ a Lorentz-transformation, then

$$(\alpha_{(a,\Lambda)} \underline{f})_n (x_1,\dots,x_n) := f_n(\Lambda^{-1}(x_1-a),\dots,\Lambda^{-1}(x_n-a)).$$

With the aid of $\alpha_{(a,1)}$ one defines a left-ideal $I_{sp} \subset \underline{\mathcal{S}}$ as follows: If $h \in \mathcal{S}_1$, let $\tilde{h}$ denote its Fourier-transform and let $\overline{V}^+ \subset \mathbb{R}^4$ be the forward light-cone, $\overline{V}^+ = \{(p^0,p^1,p^2,p^3) \mid p^0 \geq 0,\ (p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2 > 0\}$. Then I_{sp} is defined as the closed left-ideal generated by elements of the form

$$\int (\alpha_{(a,1)} \underline{f})\ h(a)\, da$$

with $\underline{f} \in \underline{\mathcal{S}}$ and $h \in \mathcal{S}_1$ such that $\tilde{h}(p)=0$ if $p \notin \overline{V}^+$. Furthermore, one considers the set of all commutators $\underline{f} \times \underline{g} - \underline{g} \times \underline{f}$ such that $\underline{f} = (0,f,0,\dots)$, $\underline{g} = (0,g,0,\dots)$ and such that the supports of the test functions f and g are spacelike separated, i.e. $f(x) \cdot g(y) = 0$ if $(x-y) \in \overline{V} := \overline{V}^+ \cup -\overline{V}^+$. The smallest, closed two-sided ideal containing this set is denoted by I_c.

The dual space $\underline{\mathcal{S}}'$ of $\underline{\mathcal{S}}$ consists of all sequences $T = (T_0, T_1, \dots$ where $T_0 \in \mathbb{C}$ and T_n is a tempered distribution on $\mathbb{R}^{4n}$. Then $T(\underline{f}) := \sum_\nu T_\nu(f_\nu)$ for $\underline{f} \in \underline{\mathcal{S}}$. A linear functional $T \in \underline{\mathcal{S}}'$ is positive, if $T(\underline{f}^* \times \underline{f}) \geq 0$ for all $\underline{f} \in \underline{\mathcal{S}}$.

Definition: A linear functional $W \in \underline{\mathcal{S}}'$ is called a Wightman-state if the following conditions are satisfied:

(1) W is positive and normalized, i.e. $W_0 = 1$

(2) W is Poincaré-invariant, i.e. $W \circ \alpha_{(a,\Lambda)} = W$ for all a and Λ.

(3) $W(I_{sp}) = \{0\}$

(4) $W(I_c) = \{0\}$

The connection with the usual Hilbert space formulation of quantum mechanics is as follows: The state W gives rise to a representation of the algebra $\underline{\mathcal{S}}$ by linear operators on a Hilbert space. This is achieved by a standard method (GNS-construction): One starts with the (left) regular representation of $\underline{\mathcal{S}}$ and defines a scalar product on $\underline{\mathcal{S}}$ by $\langle \underline{f}, \underline{g} \rangle := W(\underline{f}^* \times \underline{g})$. The null space $N = \{\underline{f} \mid \langle \underline{f}, \underline{f} \rangle = 0\}$ is invariant under the representation and by completing the quotient $\underline{\mathcal{S}}/N$ one obtains a Hilbert space H.

The algebra $\underline{\mathcal{S}}$ acts on a dense domain in H by linear operators, which are in general unbounded. From (2) it follows that H also carries a representation of the Poincaré group. The generators of space-time translations in this representaton are interpreted as the operators corresponding to energy and momentum and (3) implies that their joint spectrum belongs to the forward light-cone. (This is equivalent to the existence of a lower bound for the energy.) Finally, (4) means that operators corresponding to the field amplitude in different domains in space-time commute, provided the domains cannot be joined by any signal travelling no faster than light.

It is evident from the definition that the Wightman states form a closed, convex set $\mathcal{W}$ in the dual space $\underline{\mathcal{S}}'$. The extremal points of $\mathcal{W}$ play a special role. A certain subset of them belongs to quantum fields having the property that the eigenvector of lowest eigenvalue for the energy operator is unique up to a scalar factor. These Wightman states can be characterized in the following way, which resembles the mixing condition in ergodic theory:

(5) $\lim\limits_{\lambda\to\infty} W(\alpha_{(\lambda a,1)}\underline{f}\times\underline{g}) = W(\underline{f})\cdot W(\underline{g})$ for all $\underline{f},\underline{g}\in\underline{\mathcal{S}}$, $a\in\mathbb{R}^4$, $a\neq 0$.

Integral decomposition theory on the convex set $\mathcal{W}$ is not quite standard, for $\mathcal{W}$ is neither compact nor metrizable in any reasonable topology. The fact that $\underline{\mathcal{S}}$ is a nuclear space allows one nevertheless to write any Wightman state as an integral over extremal states. Criteria for the existence of a decomposition into states satisfying (5) can also be given. The decomposition is in general not unique. That (5) is not equivalent to the state being extremal is a mathematical curiosity, without much significance for physics.

Conditions (1)-(5) are the foundation of the so called axiomatic field theory. By now classical results of this theory are e.g.: The definition of the concept of a particle and the description of particle scattering in a field theory; a theorem linking the internal angular momentum (spin) of particles with their "statistics" (Bose- or Fermi-); a theorem stating that the combined effect of interchanging left and right, reversing the direction of time and replacing particles by anti-particles is always a symmetry operation; a definition of natural equivalence classes for fields and a proof that the fields within one class give rise to the same scattering matrix; various characterizations of physically trivial fields. An account of these things and many others is given in references [1] - [5] below.

The general results mentioned above depend to a large extend only on conditions (2), (3) and (4), i.e. on the linear part of the axioms. The main advantage of the algebraic formulation is that the non-linear condition (1) appears in a natural setting. Among recent applications of this formalism giving rise to many interesting problems I would like to mention the work of Hegerfeldt [17] on so-called prime fields and infinitely divisible fields. A systematic study of the algebra $\underline{\mathcal{S}}$, its ideals, automorphisms and positive linear functionals was initiated in [8], [9] and [10], cf. also [11], [12], [13] for further references. Most of the hard problems, however, are still unsolved. The highly developed theory of Banach algebras and of certain commutative topological algebras is here only of limited use. A better mathematical understanding of more general topological algebras would undoubtedly be of considerable advantage to the theory of Wightman fields.

I list some monographs and review articles, together with a few other articles where further references can be found.

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