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ON MINIMAL POINTS WITH RESPECT TO A SET IN BANACH SPACES

by

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Let Y be a linear subspace of the (real) normed linear space X and $\lambda \geq 1$. We assign to each set $M \subset Y$ a set $M^{\lambda}_{Y,X} \subset X$ in the following way: $x \in M^{\lambda}_{Y,X}$ if there exists no $y \in Y$, $y \neq x$, such that:

$$\|y-m\| \leq \lambda \|x-m\| \qquad \text{for each } m \in M.$$

When Y = X and $\lambda = 1$ then $M^{1}_{X,X}$ is the set of minimal points with respect to M studied by B. Beauzamy and B. Maurey in [1] and denoted there by min M. When $Y \subset X$ and $\lambda = 1$ then $M^{1}_{Y,X}$ was introduced and studied in [2].

For each $M \subset Y$ and $1 \leq \lambda \leq \mu$ we have $M^{\lambda}_{Y,Y} = M^{\lambda}_{Y,X} \cap Y$, $M^{\lambda}_{X,X} \subset M^{\lambda}_{Y,X}$ and $M^{\mu}_{Y,X} \subset M^{\lambda}_{Y,X}$, the inclusions above (as well as $M^{\lambda}_{Y,Y} \subset M^{\lambda}_{Y,X}$) being strictly in general, as examples show.

B. Beauzamy and B. Maurey [1] proved the following result: Let X be a reflexive, strictly convex and smooth Banach space and Y a closed linear subspace of X. If min Y = Y (in our notation $Y^{1}_{X,X} = Y$) then there exists a (unique) norm one linear projection of X onto Y . They also remarked that the existence of a norm one linear projection of X onto Y implies min Y = Y . In [2] we gave also a necessary and sufficient condition for the existence of a norm one linear projection of X onto Y , weakening the conditions on X (requiering only the smoothness of X) but strengthening the condition min Y = Y.

When there are no restrictions on X we have the following result. Let $\tilde{\mathcal{P}}(X,Y,\lambda)$ be the set of all mappings $\tilde{P}: X \to Y$ with the following properties:

(1)	$\tilde{P}^2(x) = \tilde{P}(x)$	$(x \in X)$
(2)	$\tilde{P}(\alpha x) = \alpha \tilde{P}(x)$	$(x \in X,\ \alpha \in R)$
(3)	$\tilde{P}(x+y) = \tilde{P}(x)+\tilde{P}(y)$	$(x \in X,\ y \in Y)$
(4)	$\Vert \tilde{P}(x) \Vert \leq \lambda \Vert x \Vert$	$(x \in X)$

Theorem 1. Let Y be a closed linear subspace of the norme linear space X and $\lambda \geq 1$. We have $\tilde{\mathcal{P}}(X,Y,\lambda) \neq \emptyset$ if and only if $Y^{\lambda}_{Y,X} = Y$.

Let us denote $S_X = \{ x \in X : \Vert x \Vert = 1 \}$., sm S_X the set of all $x \in S_X$ such that there exists a unique $x^*_X \in S_{X^*}$ with $x^*_X(x) = 1$, and $\mathcal{P}(X,Y,\lambda)$ the set of all linear projections P of X onto Y with $\Vert P \Vert \leq \lambda$.

Theorem 2.([2]). Let X be a normed linear space and Y a closed linear subspace of X. A necessary, and if $S_Y \subset$ sm S_X also sufficient condition for $\mathcal{P}(X,Y,1) \neq \emptyset$ is that $Y^1_{Y,X} = Y$. If $S_Y \subset$ sm S_X then $\mathcal{P}(X,Y,1)$ contains at most one element.

When $\lambda \geq 2$ we can exhibit subspaces $Y \subset X$ with $\tilde{\mathcal{P}}(X,Y,\lambda) \neq \emptyset$. Indeed, for $Y \subset X$ and $x \in X$ let $P_Y(x)$ be the set of all best approximations of x out of Y , i.e., $P_Y(x) = \{ y_0 \in Y : \Vert x-y_0 \Vert = \text{dist}(x,Y) \}$. Y is called a proximinal subspace of X if $P_Y(x) \neq \emptyset$ for each $x \in X$ and a Čebyšev subspace of X if $P_Y(x)$ contains exactly one element for each $x \in X$. We shall denote the elements of $P_Y(x)$ by $\pi_Y(x)$. When Y is a proximinal subspace of X we can choose a selection $\pi_Y(x) \in P_Y(x)$ satisfying the conditions (1)-(3), and since $\Vert \pi_Y(x) \Vert \leq 2 \Vert x \Vert$ for each $x \in X$, it follows that $\widetilde{\mathcal{P}}(X,Y,\lambda) \neq \emptyset$ for each $\lambda \geq 2$, and so $Y^{\lambda}_{Y,X} = Y$ for each $\lambda \geq 2$. Using this remark and a now classical result of J. Lindenstrauss and L. Tzafriri [3] it follows that for $\lambda \geq 2$ we have not a result similar with Theorem 2.

Some immediate consequences of the above results are:

<u>Corollary</u> 1. If $S_Y \subset \text{sm}\ S_X$ then $\widetilde{\mathcal{P}}(X,Y,1) = \mathcal{P}(X,Y,1)$ and $\widetilde{\mathcal{P}}(X,Y,1)$ contains at most one element.

<u>Corollary</u> 2. Let Y be a proximinal subspace of the normed linear space X such that $S_Y \subset \text{sm}\ S_X$ and $\|\pi_Y(x)\| \leq \|x\|$ for each $\pi_Y(x) \in P_Y(x)$ and each $x \in X$. Then Y is a Čebyšev subspace of X and π_Y is linear. Moreover $\mathcal{P}(X,Y,1) = \{\pi_Y\}$.

We conclude this note with the following particular case. Let E be a normed linear space and we regard E as a subspace of its second dual E^{**}. If $X = E^{**}$ and $Y = E$, then for each $M \subset E$ we denote $\text{MIN}\ M = M^{1}_{E,E^{**}}$. F. Sullivan [4] called a Banach space E to be very smooth if $S_E \subset \text{sm}\ S_{E^{**}}$. Examples of very smooth nonreflexive spaces as well as some properties of very smooth spaces are given in [4]. The last part of Theorem 2 is a generalization of the following result of [4]. If E is a very smooth Banach space then $\mathcal{P}(E^{**},E,1)$ contains at most one element. An immediate consequence of Theorem 2 is:

<u>Corollary</u> 3. ([2]). If E is a very smooth Banach space, then $\mathcal{P}(E^{**},E,1) \neq \emptyset$ if and only if $\text{MIN}\ E = E$.

The proofs of our results which are not contained in [2] will be given elsewhere in a more general setting.

References

[1] B. Beauzamy et B. Maurey : Points Minimaux et Ensembles Optimaux dans les Espaces de Banach, J.Functional Analysis, 24(1977), 107-139.

[2] G. Godini : On minimal points, (to appear).

[3] J. Lindenstrauss and L. Tzafriri : On complemented subspace problem, Israel J.Math., 9(1971), 263-269.

[4] F. Sullivan : Geometrical properties determined by the higher duals of a Banach space, Illinois J. of Math., 21(1977), 315-331.