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ISOMORPHISMS OF PRODUCTS

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Problems of isomorphisms of products have been studied for various structures, namely algebraic, relational and topological ones. In 1933, S. Ulam put a problem (see [6]) whether there exist two non-homeomorphic topological spaces X, Y such that X^2 and Y^2 are homeomorphic. Ulam's problem was solved positively by R. H. Fox in 1947 (see [1]). In 1957, W. Hanf (see [2]) constructed a Boolean algebra B isomorphic to B^3 but not to B^2. (Obviously, putting $C = B$, $D = B^2$ one obtains non-isomorphic Boolean algebras with isomorphic squares.) By [3], the similar assertion is true also for locally compact metrizable spaces.

The problems mentioned can be generalized as problems of representations of commutative semigroups by products in a following way : Let $(S,+)$ be a commutative semigroup, $\underline{C}$ a category with finite products. A collection $\{X(s);\ s \in S\}$ of objects of $\underline{C}$

is called a representation of $(S,+)$ by products in $\underline{C}$ if the following two conditions are satisfied :

(1) $X(s+s')$ is isomorphic to $X(s)\times X(s')$ for all $s, s'\in S$;

(2) $X(s)$ is isomorphic to $X(s')$ iff $s=s'$.

The representation of commutative semigroups by products in various structures has been investigated at the Seminar on General Mathematical Structures in Prague, under the leading of V. Trnková.

A survey on representations of commutative semigroups is given in [4]. Let us recall Trnková's general method for constructions of productive representations :

According to [4], any commutative semigroup is isomorphic to a subsemigroup of $(\exp N^{\aleph_0\cdot \mathrm{card}\, S},+)$ (where the additive operation + on the power-set $\exp N^{\aleph_0\cdot \mathrm{card}\, S}$ is defined by

$$A+B=\{h\in N^{\aleph_0\cdot \mathrm{card}\, S} ; (\exists f\in A,\ g\in B)(\forall a\in \aleph_0\cdot \mathrm{card}\, S)(h(a)= f(a)+g(a))\})$$

Thus, it suffices to construct for any subset A of

$N^{\aleph_0 \cdot \mathrm{card}\, S}$ an object $X(A)$ of a given category such that for every $A, B \in \exp N^{\aleph_0 \cdot \mathrm{card}\, S}$ the following two conditions hold :

(i) $X(A+B)$ is isomorphic to $X(A) \times X(B)$,

(ii) $X(A)$ is isomorphic to $X(B)$ iff $A = B$.

If a given category has arbitrary products and coproducts and if the distributivity of finite products and arbitrary coproducts is satisfied, it suffices to find a collection $\{X_a \,;\, a \in \gamma\}$ (where γ is the first ordinal with $\mathrm{card}\, \gamma = \aleph_0 \cdot \mathrm{card}\, S$) such that for every $A, B \in \exp N^{\gamma}$ the following condition holds :

(*) $\coprod_{2^{\gamma}} \coprod_{h \in A} \prod_{a \in \gamma} X_a^{h(a)}$ is isomorphic to $\coprod_{2^{\gamma}} \coprod_{k \in B} \prod_{a \in \gamma} X_a^{k(a)}$ iff $A = B$.

Representations of semigroups by products of topological spaces have been investigated with respect to special properties, namely the connectedness, the 0-dimensionality and the metrizability. While V. Trnková constructed in [5] a connected metric space X homeomorphic to

X^3 but not to X^2 (and more generally, she proved that every finitely generated Abelian group can be represented by products of connected metric spaces), the similar problem for metric 0-dimensional spaces was still open. Moreover, V. Trnková proved that if a compact metric 0-dimensional space Y is homeomorphic to Y^3 then it is also homeomorphic to Y^2.

In the present note, there is given a sketch of a construction of a metric 0-dimensional space which is isometric to its cube but which is not homeomorphic to its square (moreover, every commutative semigroup has a representation by products of metric 0-dimensional spaces).

Denote by $\underline{C}$ the category of metric spaces with a diameter ≤ 1 and Lipschitz mappings with a constant ≤ 1. Obviously, $\underline{C}$ has arbitrary products and coproducts. (If I is a set and $\{(X_i, \varrho_i);\ i \in I\}$ is a collection of objects of $\underline{C}$ then $\prod_{i \in I}(X_i, \varrho_i) = (\prod_{i \in I} X_i, \varrho)$ where $\varrho((x_i)_{i \in I}, (y_i)_{i \in I}) = \sup_{i \in I} \varrho_i(x_i, y_i)$.
One can see easily that the functor assigning to

each metric space (X, ϱ) a topological space with the topology induced by ϱ preserves finite products and arbitrary coproducts.

Now, an application of Trnková's general method is the following : for every $a \in \gamma$ find a 0-dimensional object X_a of $\underline{C}$ such that $(*)$ is satisfied and for every $f \in N^{\gamma}$ the space $\prod_{a \in \gamma} X_a^{f(a)}$ is also 0-dimensional.

Construction. For every $a \in \gamma$ choose a set of cardinal numbers $B_a = \{ \beta_{a,n} ; n \in N \}$ such that the following conditions hold :

$$2^{\gamma} < \beta_{0,0} \ , \ \beta_{a,n} < \beta_{a,\, n+1} \ ,$$

$$\beta_{a,0} > (\sup \{ \beta_b ; b < a \})^{\gamma} \quad \text{where}$$

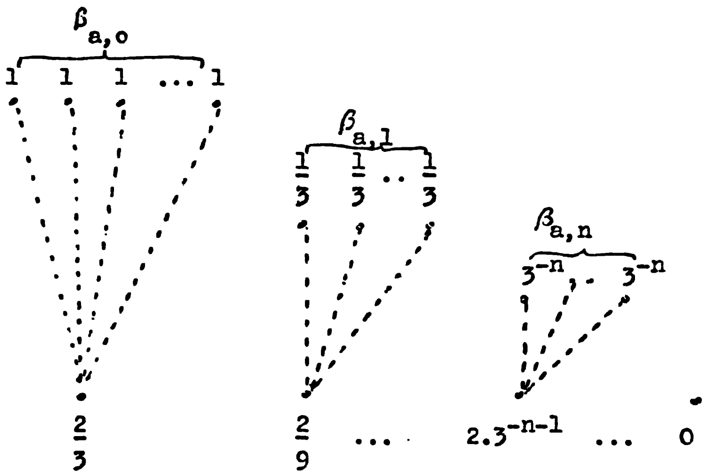
$\beta_b = \sup \{ \beta_{b,n} ; n \in N \}$. Let

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\frac{3^n-1}{2}} \left] \frac{2i-1}{3^n} , \frac{2i}{3^n} \right[$$

be the Cantor set (with the usual metric) ,

$C_n = [2.3^{-n-1}, 3^{-n}] \cap C$, $D = \{ 2.3^{-n} ; n \in N \setminus \{0\} \} \cup \cup \{ 0 \}$ (again with the usual real-line metric).

For every $a \in \gamma$ define a metric space X_a by glueing $\beta_{a,n}$ copies of C_n to the point 2.3^{-n-1} of D as shown in the picture .



The proof of (*) and of the 0-dimensionality of products $\prod_{a \in \gamma} X_a^{f(a)}$ will be published in [7].

R e f e r e n c e s

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