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CAUCHY-KOWALEWSKI THEOREMS IN CLIFFORD ANALYSIS : A SURVEY

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1. Introduction

Along with the theory of holomorphic functions of several complex variables, other directions have been followed to construct function theories in higher dimension. In particular we think of quaternionic analysis set up by Moisil-Theodorescu [11] and Fueter [6] . It was also Fueter who introduced the generalized Cauchy-Riemann operator in the framework of Clifford algebra (see [7]), an idea which was taken up again by Iftimie [10] , Hestenes [9] and Delanghe [4,5] in the late sixties.

Meanwhile this theory has grown out to a rather vast domain called Clifford analysis. The progress made by the Ghent school in the last four years was brought together into book-form (see [1]).

Within the scope of this paper it is of course impossible to give an overview of all results obtained, but nevertheless the Cauchy-Kowalewski Theorems dealt with, will give a rather nice impression of what Clifford-analysis looks like, how it has indeed fulfilled the initial aim of generalizing holomorphic function theory in one variable and of refining the theory of harmonic functions, and how it creates an opening towards differential geometry and mathematical physics, as recently elaborated by Sommen in [15, 16] .

2. Clifford algebra

Constructed in 1878 by Clifford in an attempt to combine the properties of the Grasmann algebra and Hamilton's quaternion algebra into a single geometric algebra, the so-called Clifford algebra was not used in physics until just before 1930.

The way Clifford algebra is briefly introduced here is rather constructive, as contrasted with other possible approaches [2,3,12]

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which are mostly formal algebra ; moreover we confine ourselves to the special Clifford algebra which is used in the sequel.

Let V be an n-dimensional real (resp.complex) vector space with a bilinear form $(v|w)$, v, w $\in$ V and an associated orthonormal basis $(e_1, e_2, \ldots, e_n)$ such that

$$(e_i|e_j)=0 \quad \text{if } i\neq j$$

$$(e_i|e_i)=-1,\ i=1,\ldots,n.$$

Then consider the 2^n-dimensional real (resp. complex) vector space A (resp. A^c) with basis

$$\{e_A=e_{h_1\ldots h_r} : A=\{h_1,\ldots, h_r\} \in P\{1,\ldots,n\}:1\leq h_1<\ldots<h_r\leq n\},$$

e_ϕ being written as e_o or 1. An arbitrary element of A(resp.A^c) is then written as

$$a = \sum_A e_A a_A,\quad a_A \in \mathbb{R} \quad (\text{resp.} a_A \in \mathbb{C}).$$

Now a product may be defined on A by the rule

$$e_A e_B=(-1)^{\#(A\cap B)}(-1)^{p(A,B)}e_{A\Delta B}$$

where $p(A,B)=\sum_{j\in B} p(A,j)$, $p(A,j)=\#\{i\in A,\ i>j\}$, the sets A,B and $A\Delta B$ being ordered in the prescribed way.

It follows at once from this multiplication rule that

(i) e_o is the identity element;

(ii) $e_ie_j+e_je_i=-2\delta_{ij}$

(iii) if $h_1<h_2<\ldots<h_r$ then $e_{h_1}.e_{h_2}\ldots e_{h_r}=e_{h_1\ldots h_r}$.

It is an easy matter now to check that in this way A(resp.A^c) is turned into a linear, associative but non-commutative algebra, called the real(resp.complex) universal Clifford algebra over V.

As A(resp.A^c) is isomorphic to $\mathbb{R}^{2^n}$ (resp. $\mathbb{C}^{2^n}$) we may provide it with the Euclidean norm

$$|a| = \left(\sum_A |a_A|^2\right)^{1/2}$$

and it is easy to show that

$$|a.b| \leq 2^{n/2}|a|.|b|, \quad a,b \in A(\text{resp.}A^c).$$

Hence $A(\text{resp.}A^c)$ is a Banach algebra for the Clifford norm

$$|a|=2^{n/2}|a|.$$

3. Monogenic functions

Clifford analysis is developed within the following framework.

On the one hand we have the Euclidean space $\mathbb{R}^{m+1}$, the points of which are denoted alternatively by

$$x=(x_o,\ x_1,\dots,x_m)=(x_o,\vec{x}),$$

$\vec{x}$ laying in the hyperplane $x_o=0$ which is identified with $\mathbb{R}^m$. By

$\mathbb{R}_o^{m+1}$ we mean $\mathbb{R}^{m+1}\setminus\{0\}$.

On the other hand we have the Clifford algebra A, its space of 1-vectors $A_1=\text{sp}\{e_i:\ i=1,\dots,n\}$ having dimension n; it is assumed that $m\leq n$.

For $x \in \mathbb{R}^{m+1}$, $\vec{x} \in \mathbb{R}^m$ we put

$$x=\sum_{i=0}^{m} e_i x_i=x_o+\vec{x},\quad \vec{x}=\sum_{j=1}^{m} e_j x_j,$$

$$\overline{x}=\sum_{i=0}^{m} \overline{e}_i x_i=x_o-\vec{x}.$$

By $\Omega(\text{resp.}\vec{\Omega})$ we always denote an open set in $\mathbb{R}^{m+1}(\text{resp.}\mathbb{R}^m)$. The functions under consideration are of the form

$$f:\Omega\to A,\ x\mapsto f(x)=\sum_A e_A f_A(x),\ f_A:\Omega\to\mathbb{R}.$$

Introducing the generalized Cauchy-Riemann operator

$$D=\sum_{i=0}^{m} e_i\partial_{x_i}=e_o\partial_{x_o}+D_o,\qquad D_o=\sum_{j=1}^{m} e_j\partial_{x_j}$$

and its conjugate

$$\overline{D}=\sum_{i=0}^{m}\overline{e}_i\partial_{x_i}=e_o\partial_{x_o}-D_o,$$

monogenic functions are defined as follows.

Definition 1. A function $f\in C^1(\Omega;A)$ is left (resp.right) monogenic in Ω if and only if $Df=0$ (resp. $fD=0$) in Ω.

The space of all left monogenic functions in Ω is denoted by $M(\Omega;A)$; it is a right A-module.
Now as $D\overline{D}=\overline{D}D=\Delta_{m+1}e_o$ (linearization of the (m+1)-dimensional Laplacian) we have at once

$$M(\Omega;A)\subset \mathrm{Harm}(\Omega;A)\subset \mathcal{A}(\Omega;A)\subset E(\Omega;A)$$

where $\mathrm{Harm}(\Omega;A)$, $\mathcal{A}(\Omega;A)$ and $E(\Omega;A)$ stand for the respective right A-modules of A-valued harmonic, analytic and indefinitely differentiable functions in Ω.
Endowed with the topology of uniform compact convergence, $M(\Omega;A)$ becomes a right Fréchet A-module.

4. Taylor series expansion

To give some idea of how monogenic functions look like, we pay attention to the local behaviour of a monogenic function. To that end introduce the hypercomplex variables

$$z_\ell=x_\ell e_o-x_o e_\ell,\quad \ell=1,\ldots,m$$

and the homogeneous polynomials

$$V_{\ell_1\ldots\ell_k}(x)=\frac{1}{k!}\sum_{\pi(\ell_1,\ldots,\ell_k)} z_{\ell_1}\ldots z_{\ell_k}$$

where the sum runs over all distinguishable permutations of the sequence $(\ell_1,\ldots,\ell_k)\in\{1,\ldots,m\}^k$, which are clearly both left and right monogenic in the whole of $\mathbb{R}^{m+1}$.

Theorem 1. Let f be left monogenic in the open ball $B(0,R)$. Then

$$f(x) = \sum_{k=0}^{\infty} \sum_{(\ell_1,\ldots,\ell_k)} V_{\ell_1\ldots\ell_k}(x)\, \partial_{x_{\ell_1}} \cdots \partial_{x_{\ell_k}} f(0), \tag{1}$$

considered as a multiple power series, converges uniformly on compact subsets of $\mathring{B}(0,(\sqrt{2}-1)R)$, while, bracketing terms together

$$f(x) = \sum_{k=0}^{\infty} P_k f(x) \tag{2}$$

with

$$P_k f(x) = \sum_{(\ell_1,\ldots,\ell_k)} V_{\ell_1\ldots\ell_k}(x)\partial_{x_{\ell_1}} \cdots \partial_{x_{\ell_k}} f(0),$$

converges uniformly on compact subsets of $\mathring{B}(0,R)$.

Notice that in the special case where f is left monogenic in the whole of $\mathbb{R}^{m+1}$ then both Taylor series (1) and (2) converge in $\mathbb{R}^{m+1}$.

5. The Cauchy-Kowalewski Theorem in the complex case

If we compare the functions $\sin x$ and $\operatorname{arctg} x$ ($x \in \mathbb{R}$) then, both being analytic in $\mathbb{R}$, the Taylor series of $\sin x$ at the origin converges in $\mathbb{R}$, while for $\operatorname{arctg} x$ the convergence only holds in $]-1,1[$. The very reason for this fact lays of course in the complex plane where $\sin x$ is holomorphic in $\mathbb{C}$ with a Taylor series expansion at the origin converging in $\mathbb{C}$, while the Taylor series of $\operatorname{arctg} x$ at the origin converges only in the unit disk, due to the poles at i and -i. So we see that the behaviour of real-analytic functions on the real axis is in fact governed by their holomorphic extensions in the complex plane. This is the contents of the so-called Cauchy-Kowalewski Theorem for holomorphic functions: if $f(z)$ is holomorphic in $\Omega \subset \mathbb{C}$ with $\vec{\Omega} = \Omega \cap \mathbb{R} \neq \phi$ then the restriction of f to $\vec{\Omega}$ is a complex-valued analytic function in $\vec{\Omega}$; conversely if $\vec{f}$ is analytic in $\vec{\Omega} \subset \mathbb{R}$ open, then there exists an open neighbourhood Ω of $\vec{\Omega}$ in $\mathbb{C}$ and a holomorphic function f in Ω such that $\vec{f}=f|\vec{\Omega}$. We shall see in the next section that the same goes through for monogenic functions in $\mathbb{R}^{m+1}$ versus analytic functions in $\mathbb{R}^m$.

6. The Cauchy-Kowalewski Theorem in Clifford analysis

As an immediate consequence of the Taylor series expansion of a monogenic function (see section 4) we have

Proposition 1. Let $f \in M(\Omega;A)$ and $\Omega\cap\{x_0=0\}=\vec{\Omega}\neq\phi$; then $\vec{f}=f|\vec{\Omega}\in \mathcal{A}(\vec{\Omega};A)$.

As to the converse we first need the following definition.

Definition 2. The open set Ω in $\mathbb{R}^{m+1}$ is called an x_0-normal open neighbourhood of $\vec{\Omega}\subset\mathbb{R}^m$ if for each $x\in\Omega$ the line segment $\{x+te_0\}\cap\Omega$ is connected and contains just one point of $\vec{\Omega}$.

Theorem 2 (Cauchy-Kowalewski). Let $\vec{f} \in \mathcal{A}(\vec{\Omega};A)$; then there exist a maximal x_0-normal open neighbourhood Ω of $\vec{\Omega}$ and a unique function $f \in M(\Omega;A)$ such that $f(0+\vec{x})=\vec{f}(\vec{x})$.

This pair (f,Ω) is called the C-K-extension of $\vec{f}$ in $\vec{\Omega}$. It is immediately clear that, given $\vec{\Omega}$ and $\vec{f}$, two problems have to be solved for determining the C-K-extension: both the maximal region Ω and the function f have to be found. And, thinking of the complex case, it is also clear that in general no complete solution to this problem can be given. Indeed, for $\sin x \in \mathcal{A}(\mathbb{R})$ we have as C-K-extension $(\sin x,\mathbb{C})$, while for $\operatorname{arctg} x \in \mathcal{A}(\mathbb{R})$ the C-K-extension is $(\operatorname{arctg} z, \mathbb{C}\setminus\{iy:(y)\geq 1\})$. Moreover if we substitute z for x in

$$\operatorname{arctg} x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\ ,\ -1<x<1$$

then the holomorphic extension

$$\operatorname{arctg} z=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{2k+1}$$

is only obtained in the open unit disk, which is not maximal.

However, for calculating the extension f of $\vec{f}$ in the Clifford case we have the disposal of the following formula.

Proposition 2. If $\vec{f}$ is analytic in $\vec{\Omega}\subset\mathbb{R}^m$ then the function f given by

$$f(x)=\sum_{k=0}^{\infty}(-1)^k\,\bar{D}\left[\frac{x_o^{2k+1}}{(2k+1)!}\,\Delta_m^k\,\vec{f}(\vec{x})\right]$$

is left monogenic in a neighbourhood Ω of $\vec{\Omega}$ in $\mathbb{R}^{m+1}$ and satisfies $f(0+\vec{x})=f(\vec{x})$ in $\vec{\Omega}$.

Further we would expect an analogous result as in the complex case if series expansion is under consideration; that this it not entirely the case, but that in fact the region of convergence is dramatically contracted, is shown in the following theorem.

First we introduce

Definition 3. Call

$$C_m(0,\vec{R})=\{\vec{x}\in\mathbb{R}^m : |x_j|<R_j,\ j=1,\dots,m\}$$

and

$$C_{m+1}(0,\vec{R})=\{x\in\mathbb{R}^{m+1} :\ x_o^2+x_j^2<R_j^2,\ j=1,\dots,\ m\}.$$

Clearly $C_{m+1}(0,\vec{R})\cap\{x_o=0\}=C_m(0,\vec{R})$.

Theorem 3. Let $\vec{f}$ be analytic such that

$$\vec{f}(\vec{x})=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\ell_1=1}^{m}\cdots\sum_{\ell_k=1}^{m}x_{\ell_1}\dots x_{\ell_k}\lambda_{\ell_1\dots\ell_k}$$

converges as a multiple power series in $C_m(0,\vec{R})$. Then

$$\sum_{k=0}^{\infty}\left(\frac{1}{k!}\sum_{\ell_1=1}^{m}\cdots\sum_{\ell_k=1}^{m}z_{\ell_1}\dots z_{\ell_k}\,\lambda_{\ell_1\dots\ell_k}\right)\tag{3}$$

converges uniformly on the compact subsets of $C_{m+1}(0,2^{-m/2}\vec{R})$ to a left monogenic function f such that $f|C_m(0,2^{-m/2}\vec{R})=\vec{f}$.

It should be remarked that in section 8 better results will be obtained as to the region of absolute convergence of the series (3) then considered as a multiple power series. However this Theorem 3 has an interesting corollary.

Definition 4. If $\vec{f}$ is analytic in $\mathbb{R}^m$ such that its Taylor series at the origin converges as a multiple power series in the whole of $\mathbb{R}^m$, then $\vec{f}$ is called entire-analytic, which is denoted by $\vec{f} \in \mathcal{A}_{ent}(\mathbb{R}^m;A)$.

Corollary. If $\vec{f}$ is entire-analytic then its left C-K-extension f is left monogenic in the whole of $\mathbb{R}^{m+1}$.

This corollary enables us to define a product of monogenic functions which is not at all trivial since, due to the non-commutativity of the multiplication in A, the pointwise product of two (left) monogenic functions is not necessarily (left) monogenic anymore.

Let f and g be left monogenic in $\mathbb{R}^{m+1}$; then $\vec{f}=f|\mathbb{R}^m$ and $\vec{g}=g|\mathbb{R}^m$ are entire-analytic in $\mathbb{R}^m$ and so is $\vec{f}.\vec{g}$; the left C-K-extension of $\vec{f}.\vec{g}$, which is left monogenic in $\mathbb{R}^{m+1}$, is by definition the C-K-product of f and g, denoted by $f\odot_L g$.

This C-K-product enjoys the following properties:

(i) It is associative.

(ii) If $(f|\mathbb{R}^m).(g|\mathbb{R}^m)=(g|\mathbb{R}^m).(f|\mathbb{R}^m)$ then $f\odot_L g=g\odot_L f$.

(iii) $1\odot f=f\odot_L 1=f$.

(iv) $M(\mathbb{R}^{m+1};A)_{,+,\odot_L}$ is a real algebra.

In illustration of this C-K-product let us give a few examples. The C-K-extension of $x_i (i=1,\dots,m)$ is $z_i=x_ie_o-x_oe_i$; the C-K-extension of x_ix_j is $\frac{1}{2!}(z_iz_j+z_jz_i)$ if $i\neq j$ or z_i^2 if $i=j$; this implies that

$$z_i\odot_L z_j=z_i\odot_R z_j=V_{ij}(x) \qquad \text{if } i\neq j$$

The left and right C-K-product of z_i and z_j coincide and moreover it is commutative ; so we arrive at

$$z_{\ell_1}\odot\ldots z_{\ell_k}=z_1^{n_1}\odot\ldots z_m^{n_m}=n_1!\ldots n_m!V_{\ell_1\ldots\ell_k}(x)$$

where n_i stands for the number of times that i appears in $(\ell_1,\ldots\ell_k)$.
With an analogous meaning for n_i' we also have

$$V_{\ell_1\ldots\ell_k}\odot V_{s_1\ldots s_t}=\left[\prod_{i=1}^{m}\binom{n_i+n'_i}{n_i}\right]V_{\ell_1\ldots\ell_k\, s_1\ldots s_t}.$$

A second application of the Cauchy-Kowalewski extension theorem is the construction of elementary functions, especially the exponential functions which are used in defining Fourier and generalized Laplace transforms in a Clifford setting (see [1, 13, 14]).

First for $\vec{t}\in\mathbb{R}^m$ and $\vec{x}\in\mathbb{R}^m$ define for $j=1,\ldots,m$

$$\begin{aligned}\exp(t_jx_je_j)&=e_o\cos(t_jx_j)+e_j\sin(t_jx_j)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\,t_j^kx_j^ke_j^k\end{aligned}$$

which are entire-analytic in $\mathbb{R}^m$. Their left C-K-extensions are then the functions $\exp(t_jz_je_j)$ given by

$$\exp(t_jz_je_j)=\sum_{k=0}^{\infty}\frac{1}{k!}\,t_j^kz_j^ke_j^k\qquad (j=1,\ldots,m),$$

which are left monogenic in $\mathbb{R}^{m+1}$.
Next we define the entire-analytic function in $\mathbb{R}^m$

$$\begin{aligned}E(\vec{t},\vec{x})&=\prod_{j=1}^{m}\exp(t_jx_je_j)\\&=\sum_{k=0}^{\infty}\ \sum_{k_1+\ldots+k_m=k}\frac{t_1^{k_1}\ldots t_m^{k_m}}{k_1!\ldots k_m!}\,x_1^{k_1}\ldots x_m^{k_m}e_1^{k_1}\ldots e_m^{k_m}\end{aligned}$$

yielding as left C-K-extension the so-called exponential function

$$E(\vec{t},x)=\prod_{j=1}^{m}{\odot_L}\exp(t_jz_je_j)$$

$$= \sum_{k=0}^{\infty} \sum_{k_1+\ldots+k_m=k} t_1^{k_1} \ldots t_m^{k_m} V_{\underbrace{1\ldots1}_{k_1}\ldots\underbrace{m\ldots m}_{k_m}}(x)\, e_1^{k_1} \ldots e_m^{k_m}$$

which is left monogenic in $\mathbb{R}^{m+1}$.
Observe that in the particular case where m=1 the functions

$$E(\vec{t},\vec{x})=\exp(t_1x_1e_1)=e_o\cos(t_1x_1)+e_1\sin(t_1x_1)$$

and

$$E(\vec{t},x)=\exp(t_1z_1e_1)=\exp(t_1(x_1e_o-x_oe_1)e_1)$$

may be identified with e^{-itx} and e^{-itz}, $t\in\mathbb{R}, x\in\mathbb{R}, z\in\mathbb{C}$ respectively.

7. The Cauchy-Kowalewski Theorem on an anlytic surface

Let Σ be an analytic m-dimensional surface in $\mathbb{R}^{m+1}$ and let g be an A-valued analytic function in Σ. The problem now is to find a left monogenic function f in a certain neighbourhood of Σ such that $f|\Sigma=g$.

This problem is solved by a Cauchy-Kowalewski type theorem, but first we introduce some notions and notations.

For $p \in \Sigma$ we denote by $(N_o(p),\ldots, N_m(p))$ the components of the unit normal vector to Σ at the point p ; we also put

$$N(p)=\sum_{i=0}^{m} N_i(p)e_i .$$

<u>Definition 5.</u> An open neighbourhood ω of Σ is called normal with respect to Σ if the following conditions are satisfied :

(i) for every $x \in \omega$ there exist unique $p_x \in \Sigma$ and $n_x \in \mathbb{R}$ such that $x = p_x+n_xN(p_x)$;

(ii) p_x and n_x are both C_∞-functions in ω;

(iii) for every $x \in \omega$, the line segment

$$\{p_x+\lambda N(p_x):0<\lambda \text{ sgn } n_x \leqslant |n_x|\}$$

is contained in ω.

Definition 6. An A-valued C_∞-function f in ω is called projective if it satisfies $f(x)=f(p_x)$ for all $x \in \omega$.

The unit normal vector field being $N=\sum_{i=0}^{m} N_i e_i$, we put

$\partial_N=\langle N,\nabla\rangle=\sum_{i=0}^{m} N_i\partial_{x_i}$. Furthermore in ω we introduce the Cauchy-Riemann operators associated with Σ :

$$\Gamma_o=\overline{N}D-\partial_N$$

$$\Gamma_k=\frac{1}{k}[\partial_N,\Gamma_{k-1}]=\frac{1}{k}(\partial_N\Gamma_{k-1}-\Gamma_{k-1}\partial_N),\ k=1,2,\ldots\ .$$

Theorem 4. Let g be analytic in Σ. Then there exists a unique left monogenic function f in a maximal normal open neighbourhood ω of Σ such that $f|\Sigma=g$.

This extension f is called the Cauchy-Kowalewski extension of g.

Now it is possible to give a structure formula for the C-K-extension f of g, which is made explicit in the special cases where $\Sigma=\mathbb{R}^m$ and $\Sigma=S^m$.

Indeed, if f is analytic in ω then f can be developed into a unique series of the form

$$f(x)=\sum_{k=0}^{\infty} n_x^k\, f_k(x),$$

the functions f_k, $k \in \mathbb{N}$, being projective, converging absolutely in a maximal open neighbourhood of Σ depending upon f and the analyticity of Σ.
Introducing a local coordinate system (n,θ) in ω, given by

$$(n,\theta)(x)=(n_x,\theta(p_x)),$$

θ being a local coordinate system in Σ, the above expansion then reads

$$f(x)=\sum_{k=0}^{\infty} n^k f_k(\theta).$$

Now if moreover f is assumed to be left monogenic in ω, then it may be proved that

$$f_k(\theta) = \Theta_k f_o(\theta), \quad k=1,2,\ldots$$

where the differential operators Θ_k, $k \in N_o$, are polynomials in $\Gamma_o, \Gamma_1, \ldots, \Gamma_k$.

This means that if f is the C-K-extension in ω of the analytic function g in Σ, then

$$f(x) = \sum_{k=0}^{\infty} n^k \, \Theta_k g(\theta).$$

In the case where $\Sigma = \mathbb{R}^m$ we obtain that

$$\Theta_k = \frac{(-1)^k}{k!} (D - \partial_{x_o})^k$$

leading to the formula already obtained in Proposition 2.

In the case of the unit sphere $\Sigma = S^m$ we have that

$$\Theta_k = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k-1} S_{k,\ell} \Gamma^{k-\ell}$$

where

$$S_{k,\ell} = \sum_{j_i < \ldots < j_\ell} j_1 \ldots j_\ell, \quad j_i = 1, \ldots, k-1$$

and Γ is the spherical Cauchy-Riemann operator given by

$$\Gamma = \bar{\omega} \partial_\omega, \quad \partial_\omega = \sum_{j=1}^{m} \frac{1}{\left| \frac{\partial \omega}{\partial \theta_j} \right|^2} \frac{\partial \omega}{\partial \theta_j} \theta_j$$

with $\omega = \sum_{i=0}^{m} \omega_i e_i = \sum_{i=0}^{m} e_i \frac{x_i}{|x|} \in S^m$

and θ_j, $j=1,\ldots,m$ the angular spherical coordinates.

8. The convergence problem optimally solved

Now let us come back to the convergence problem for the expansion of the C-K-extension of a given analytic function in $\vec{\Omega} \subset \mathbb{R}^m$.

In [8] Hayman proved that a harmonic function, and a fortiori a holomorphic function in the open unit disk, which of course has an analytic restriction to $]-1,1[\ \subset \mathbb{R}$, admits a multiple Taylor series expansion about the origin converging absolutely in the square $\{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\}$.

This result is now generalized to monogenic functions in the following sense: taking an analytic function $\vec{f}$ the multiple Taylor series of which converges absolutely in a domain of $\mathbb{R}^m$, we look for the optimal domain in $\mathbb{R}^{m+1}$ where the multiple Taylor series of the monogenic extension of $\vec{f}$ converges absolutely. To this end we define a hypercomplex version of the Radon transform

$$P : \mathcal{O}'(\overline{B}(0,1) \to \mathcal{O}(B(0,1))$$

given by

$$P(T)(u) = \langle T_\zeta, \frac{1}{1-u\zeta} \rangle,$$

where $\mathcal{O}(B(0,1))$ stands for the space of holomorphic functions in the open unit disk and $\mathcal{O}'(\overline{B}(0,1))$ for the space of continuous linear functionals on $\mathcal{O}(\overline{B}(0,1) = \lim\limits_{\varepsilon>0} \operatorname{ind} \mathcal{O}(B(0,1+\varepsilon))$.

The kernel function for this generalized Radon transform is

$$P(u,\zeta) = \frac{1-\langle\vec{u},\vec{\zeta}\rangle - u_o\vec{\zeta}}{(1-\langle\vec{u},\vec{\zeta}\rangle)^2 + u_o^2 \sum\limits_{j=1}^{m} \zeta_j^2} = \sum_{k=0}^{\infty} (\langle\vec{u},\vec{\zeta}\rangle e_o - u_o\vec{\zeta})^k$$

where for $u = u_o + \vec{u} \in \mathbb{R}^{m+1}$ and $(\zeta_1,\ldots,\zeta_m) \in \mathbb{C}^m$, $\vec{\zeta} = \sum\limits_{j=1}^{m} e_j\zeta_j$ and $\langle\vec{u},\vec{\zeta}\rangle = \sum\limits_{j=1}^{m} u_j\zeta_j$.

This kernel is holomorphic in the variable $\vec{\zeta}$ and monogenic in the variable u in the region $(\mathbb{R}^{m+1}\times\mathbb{C}^m)\setminus S$, where S is the set of zeros of the denominator.

Now let Ω^* be one of the following three domains of holomorphy in $\mathbb{C}^m$:

$$\Lambda(\vec{R}) = \{\vec{\zeta} \in \mathbb{C}^m : |\zeta_j| < R_j\rangle$$

$$\widetilde{\Lambda}(\vec{R}) = \{\vec{\zeta} \in \mathbb{C}^m : \sum_{j=1}^{m} R_j|\zeta_j| < 1\}$$

$$B^*(0,R)=\{\vec{\zeta}\in\mathbb{C}^m:|\vec{\zeta}|^2=\sum_{j=1}^m|\zeta_j|^2<R^2\};$$

with each of those domains we make correspond an open region Ω in $\mathbb{R}^{m+1}$ which is given respectively by

$$\Pi(\vec{R})=\{x\in\mathbb{R}^{m+1}:|x_o||\vec{R}|+\sum_{j=1}^m R_j|x_j|<1\}$$

$$\tilde{\Pi}(\vec{R})=\bigcap_{j=1}^m\{x\in\mathbb{R}^{m+1}:|x_o|+|x_j|<R_j\}$$

$$\Pi^*(R^{-1})=\{x\in\mathbb{R}^{m+1}:|x_o|+|\vec{x}|<R\}.$$

Then, if $\mathcal{O}^*_{(\ell)}(\overline{\Omega}^*;A^c)$ denotes the right A^c-module of all continuous left A^c-linear functionals on

$$\mathcal{O}_{(\ell)}(\overline{\Omega}^*;A^c)=\lim_{\varepsilon>0}\text{ind}\ \mathcal{O}_{(\ell)}(\Omega^*_\varepsilon;A^c),$$

Ω^*_ε being an open ε-neighbourhood of $\overline{\Omega}^*$, the following isomorphism may be proved.

<u>Theorem 5.</u> Let $T\in\mathcal{O}^*_{(\ell)}(\overline{\Omega}^*;A^c)$, then

$$P(T)(u)=\langle T_{\vec{\zeta}},P(u,\vec{\zeta})\rangle$$

is left monogenic in Ω and its multiple Taylor series expansion about the origin converges absolutely in Ω.
Conversely, if f is left monogenic in Ω such that its multiple Taylor series converges absolutely in Ω, then there exists $T\in\mathcal{O}^*_{(\ell)}(\overline{\Omega}^*;A^c)$ such that $P(T)=f$.

This characterization of $P(\mathcal{O}^*(\overline{\Omega}^*;A^c))$ will now lead to an optimal version of the Cauchy-Kowalewski extension theorem. Hereby notice that if $\vec{f}(\vec{\zeta})$ is an A^c-valued holomorphic function in Ω^*, then its restriction $\vec{f}(\vec{x})$ is an A^c-valued analytic function in Ω^*_{rest}, given respectively by

$$\Lambda(\vec{R})_{rest}=\{\vec{x}\in R^m:|x_j|<R_j\}=C_m(0,\vec{R})$$

$$\tilde{\Lambda}(\vec{R})_{rest}=\{\vec{x}\in R^m:\sum_{j=1}^m R_j|x_j|<1\}$$

$$B^*(0,R)_{rest}=\{\vec{x}\in R^m:|\vec{x}|<R\}.$$

Next we have that the above cited regions Ω in $\mathbb{R}^{m+1}$ are optimal with respect to absolute convergence, meaning that for every $u \in \mathbb{R}^{m+1} \setminus \Omega$ there exists a left monogenic function f in Ω such that the multiple Taylor series expansion of f about the origin converges absolutely in Ω but not in u.

Finally we obtain

Theorem 6. Let $\vec{f}(\vec{\zeta})$ be an A^c-valued holomorphic function in Ω^* and let $\vec{f}(\vec{x})$ be its A^c-valued analytic restriction to Ω^*_{rest}. Then $\vec{f}(\vec{x})$ admits a unique left monogenic extension f(x) to Ω, given by

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \, \overline{D} \left[\frac{x_0^{2k+1}}{(2k+1)!} \, \Delta_m^k f(x) \right].$$

The multiple Taylor series expansion of f about the origin converges absolutely in Ω, which is optimal with respect to absolute convergence.

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