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ON DOMAINS OF MONOGENICITY  
IN CLIFFORD ANALYSIS

Richard Delanghe, Freddy Brackx and Willy Pincket

1. Introduction

One of the basic problems encountered when passing from holomorphic functions of a single complex variable to holomorphic functions of several complex variables is caused by the fact that not any domain in  $C^n$  ( $n > 1$ ) is a domain of holomorphy, as it is the case in the complex plane. As is well known a classic counterexample is provided by the Hartogs Extension Theorem stating that any function which is holomorphic in  $\Omega \setminus K$ , where  $\Omega \subset C^n$  is open,  $K$  is compact and  $\Omega \setminus K$  is connected, may be extended to a holomorphic function in  $\Omega$ .

Let us recall a classical characterization of domains of holomorphy in  $C^n$ .

Theorem. If  $\Omega$  is a domain in  $C^n$  then the following conditions are equivalent :

- (i)  $\Omega$  is a domain of holomorphy;
- (ii)  $\Omega$  is holomorphically convex, i.e. for each compact subset  $K \subset \Omega$  its holomorphic hull  $\hat{K}_\Omega = \{z \in \Omega : |f(z)| \leq \sup_{u \in K} |f(u)|, \text{ for all } f \in O(\Omega)\}$  is again compact;
- (iii) there exists a function  $f \in O(\Omega)$  which cannot be continued holomorphically beyond  $\Omega$ , i.e.  $\Omega$  is a holomorphic existence domain.

The aim of this paper is to investigate if, such as in the complex plane, any domain in  $R^{m+1}$  ( $m > 1$ ) satisfies all of the three conditions mentioned above, with respect to the monogenic functions. It can be shown in a straightforward manner that any domain  $\Omega$  in  $R^{m+1}$  is a domain of monogenicity (§2) and that it is moreover mono-

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"This paper is in final form and no version of it will be submitted for publication elsewhere".

genically convex (§3); but it is not known yet if for each domain  $\Omega \subset \mathbb{R}^{m+1}$  there exists a monogenic function in  $\Omega$  which cannot be extended monogenically beyond  $\Omega$ . Nevertheless in section 4 some sufficient conditions are given for a domain  $\Omega \subset \mathbb{R}^{m+1}$  to be a monogenic existence domain.

For the definitions and properties concerning the monogenic functions the reader is referred to [3].

## 2. Domains of monogenicity.

DEFINITIONS 2.1. Let  $\Omega$  be a domain in  $\mathbb{R}^{m+1}$ .

- (i)  $\Omega$  is called a weak domain of monogenicity if for each domain  $\Omega' \supset \Omega$  there exists a monogenic function in  $\Omega'$  which is not the restriction to  $\Omega$  of a monogenic function in  $\Omega'$ ;
- (ii)  $\Omega$  is called a domain of monogenicity if it is impossible to find two domains  $U_1$  and  $U_2$  satisfying the following two conditions :
  - (a)  $\emptyset \neq U_2 \subset \Omega \cap U_1 \subset U_1$ ;
  - (b) for each monogenic function  $f$  in  $\Omega$  there exists a monogenic function  $\hat{f}$  in  $U_1$  such that  $f = \hat{f}$  on  $U_2$ .

REMARK. Clearly any domain of monogenicity is also a weak domain of monogenicity. The fact that both notions coincide is a consequence of the following theorem.

THEOREM 2.2. Every domain in  $\mathbb{R}^{m+1}$  is a domain of monogenicity.

Proof. Let  $\Omega \subset \mathbb{R}^{m+1}$  be a domain which is not a domain of monogenicity. Then there exist two domains  $U_1$  and  $U_2$  satisfying the conditions (a) and (b) of Definition 2.1.(ii). Call  $U_2'$  the component of  $U_1 \cap \Omega$  which contains  $U_2$  and let the points  $a \in U_2$  and  $b \in U_1 \setminus \Omega$  be joint by means of a polygonal line  $\Gamma$  in  $U_1$ . Take  $c \in \Gamma \cap \partial\Omega$ . The function

$$g(x) = \frac{\bar{x} - \bar{c}}{|x - c|^{m+1}}$$

is monogenic in  $\Omega$ , so, by the hypothesis made, there exists a function  $\hat{g}$ , monogenic in  $U_1$ , for which  $g = \hat{g}$  on  $U_2$  and also on  $U_2'$  by analytic continuation. Now, as  $\hat{g}$  is monogenic in  $U_1$ , we have  $\lim_{\substack{x \rightarrow c \\ x \in \Gamma}} \hat{g}(x) = \hat{g}(c)$ , while  $\lim_{\substack{x \rightarrow c \\ x \in \Gamma \cap \Omega}} |g(x)|_0 = +\infty$ , clearly a contradiction. ■

REMARK. The proof of the above theorem depends heavily upon the existence of pointwise singularities. As in the more general two Clifford-variable theory of the biregular functions (see e.g.[1]) pointwise singularities do not occur anymore, it is expected that, in analogy with complex analysis, the study of the so-called domains of biregularity will be far from trivial (see [2]).

3. Monogenic convexity

DEFINITION 3.1. Let  $\Omega$  be a domain in  $R^{m+1}$ , let  $K$  be a compact subset of  $\Omega$  and let  $F$  be a family of left-monogenic functions containing the functions  $\xi_i = x_i e_0 - x_0 e_i$ ,  $i=1, \dots, m$ . The  $F$ -convex hull of  $K$  is the set  $\hat{K}_F$  given by

$$\hat{K}_F = \{x \in \Omega : |f(x)|_0 \leq \sup_{u \in K} |f(u)|_0, \text{ for all } f \in F\} .$$

In the particular case where  $F=M(\Omega)$ , i.e. the whole family of all left monogenic functions in  $\Omega$ ,  $\hat{K}_F$  is called the monogenic hull of  $K$  and denoted by  $\hat{K}_\Omega$ .

The following properties of the  $F$ -convex hull are immediate.

PROPOSITION 3.2. Let  $\Omega, K$  and  $F$  be as in Definition 3.1. Then

- (i)  $\hat{K}_F$  is relatively closed in  $\Omega$ ;
- (ii)  $K \subset \hat{K}_F$  and  $\sup_{x \in K} |f(x)|_0 = \sup_{x \in \hat{K}_F} |f(x)|_0$  for all  $f \in F$ ;
- (iii) if  $\hat{F}_1 \subset F_2$  then  $\hat{K}_{F_2} \subset \hat{K}_{F_1}$ .

DEFINITION 3.3. A domain  $\Omega \subset R^{m+1}$  is called  $F$ -convex if for each  $K \subset \Omega$  compact, the  $F$ -convex hull  $\hat{K}_F$  is again a compact set. In the particular case where  $F=M(\Omega)$  the domain  $\Omega$  is called monogenically convex.

THEOREM 3.4. Every domain in  $R^{m+1}$  is monogenically convex.

Proof. Let  $K \subset \Omega$  be compact. In view of Proposition 3.2 it suffices to prove that  $\bar{K}_\Omega \subset \Omega$ . For  $u \in R^{m+1} \setminus \Omega$  the function

$$g(x) = \frac{\bar{x} - \bar{u}}{|x - u|^{m+1}}$$

is left-monogenic in  $\Omega$  and so

$$\sup_{x \in K} |g(x)|_0 = \sup_{x \in \hat{K}_\Omega} |g(x)|_0 \text{ or } \sup_{x \in K} \frac{1}{|x - u|^m} = \sup_{x \in \hat{K}_\Omega} \frac{1}{|x - u|^m}$$

for any  $u \in R^{m+1} \setminus \Omega$ .

$$\text{Hence } 0 < \inf_{x \in K} |x-u| = \inf_{x \in \hat{K}_\Omega} |x-u| \\ u \in R^{m+1} \setminus \Omega \quad u \in R^{m+1} \setminus \Omega$$

$$d(K, R^{m+1} \setminus \Omega) = d(\hat{K}_\Omega, R^{m+1} \setminus \Omega) > 0. \quad \blacksquare$$

#### 4. Monogenic existence domains

DEFINITION 4.1. Let  $\Omega$  be a domain in  $R^{m+1}$ .

- (i)  $\Omega$  is called a weak monogenic existence domain for the monogenic function  $f$  in  $\Omega$  if for each monogenic function  $F$  in  $\Omega'$ , where  $\Omega'$  is a domain strictly containing  $\Omega$ ,  $F|_\Omega \neq f$ .
- (ii)  $\Omega$  is called a monogenic existence domain for the monogenic function  $f$  in  $\Omega$  if for each pair of domains  $U_1$  and  $U_2$  for which  $\emptyset \neq U_2 \subset \Omega \cap U_1 \subset U_1$ , and for each monogenic function  $F$  in  $U_1$ ,  $F|_{U_2} \neq f|_{U_2}$ .
- (iii)  $\Omega$  is called a (weak) monogenic existence domain if there exists a monogenic function  $f$  in  $\Omega$  for which  $\Omega$  is a (weak) monogenic domain.

REMARKS.

- (i) The first definition 4.1.(i) states that the function  $f$  cannot be extended monogenically beyond the boundary of  $\Omega$ . Definition 4.1.(ii) has a local character and implies that the result of a monogenic extension might be a multi-valued function.
- (ii) It is clear that a (weak) monogenic existence domain is also a (weak) domain of monogenicity. Moreover it is obvious that a monogenic existence domain is also a weak monogenic existence domain. Under an additional condition on  $\Omega$  both notions can be made to coincide. To be more precise :

PROPOSITION 4.2. If the domain  $\Omega$  is locally connected then for a monogenic function  $f$  in  $\Omega$  the following statements are equivalent:

- (i)  $\Omega$  is a monogenic existence domain for  $f$ ;
- (ii)  $\Omega$  is a weak monogenic existence domain for  $f$ .

As already mentioned in the introduction it is not known yet if any domain in  $R^{m+1}$  is a (weak) monogenic existence domain. Nevertheless it will be shown explicitly that a special class of monogenically convex domains are indeed monogenic existence domains.

**DEFINITION 4.3.** Call  ${}_2M(\Omega)$  the family of functions such that

- (i)  $f$  is left-monogenic in  $\Omega$ ;
- (ii)  $|f \cdot \bar{f}| = |f|^2$ , with  $|f| = 2^{-n/2} |f|_0$ ;
- (iii) all the functions  $f^{2^n}$ ,  $n \in \mathbb{N}$ , satisfy (i) and (ii).

Notice that in the complex case this family consists of all holomorphic functions in  $\Omega$ .

That  ${}_2M(\Omega)$  is not empty may be illustrated by the following examples : any monogenic function in  $\Omega$  of the form  $f(x)e_0 + g(x)e_i$  ( $0 < i < m$ ), or  $f(x)e_0 + g(x)e_i e_j$  ( $0 < i < j < m$ ),  $f$  and  $g$  being real-valued, belongs to  ${}_2M(\Omega)$ . In particular the hypercomplex variables  $\xi_i$  ( $1 \leq i < m$ ) all belong to  ${}_2M(\Omega)$ .

Now we construct in a  ${}_2M(\Omega)$ -convex domain a very peculiar monogenic function.

**THEOREM 4.4.** Let  $\Omega \neq \mathbb{R}^{m+1}$  be  ${}_2M(\Omega)$ -convex. Then there exists a monogenic function  $F$  in  $\Omega$  satisfying the following condition : for each point  $x \in \Omega$  with rational coordinates there exists a sequence

$(x^{(v)})_{v=1}^\infty$  in  $\mathring{B}(x, d(x, \partial\Omega))$  on which  $F$  is unbounded.

**Proof.** Take all points in  $\Omega$  with rational co-ordinates  $(\eta^{(v)})_{v=1}^\infty$  and arrange them as follows :

$$(w^{(v)})_{v=1}^\infty = (\eta^{(1)}, \eta^{(1)}, \eta^{(2)}, \eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta^{(1)}, \dots).$$

Put  $U_v = \mathring{B}(w^{(v)}, d(w^{(v)}, \partial\Omega))$ ; then by definition we have  $\bar{U}_1 \cap \partial\Omega \neq \emptyset$ , and so we may construct a sequence  $(w_1^{(\mu)})_{\mu=1}^\infty$  in  $U_1$  converging to a point of  $\partial\Omega$ .

Consider the compact exhaustion  $(K_\lambda)_{\lambda=1}^\infty$  of  $\Omega$ . As  $\Omega$  is  ${}_2M(\Omega)$ -convex,  $\hat{K}_{1, 2M}$  is compact; hence there exists a function  $\tilde{f}_1 \in {}_2M(\Omega)$  and a

point  $x^{(1)} = w_1^{(\mu_1)}$  such that  $x_1 \notin \hat{K}_{1, 2M}$  and  $|\tilde{f}_1(x^{(1)})|_0 > \sup_{u \in K_1} |\tilde{f}_1(u)|_0$ .

After dividing by a suitable real number we can reduce  $\tilde{f}_1$  to  $f_1 \in {}_2M(\Omega)$  for which

$$2^{-n/2} \sup_{u \in K_1} |f_1(u)|_0 < 1 < 2^{+n/2} |f_1(x^{(1)})|_0.$$

Also  $x^{(1)}$  is contained in a certain  $K_\lambda$  which is called now  $K_\lambda(x)$ .

Proceeding in the same way, we may define sequences  $(x^{(v)})_{v=1}^\infty$ ,

$(K_\lambda(x^{(v)}))_{v=1}^\infty$  and  $(f_v)_{v=1}^\infty$  such that  $x^{(v)} \in U_v \cap (K_\lambda(x^{(v+1)}) \setminus \hat{K}_\lambda(x^{(v)}), {}_2M)$

and  $2^{-n/2} \sup_{u \in K_\lambda(x^{(v)})} |f_v(u)|_0 < 1 < 2^{-n/2} |f_v(x^{(v)})|_0$ .

In view of the structure of the  ${}_2M(\Omega)$ -functions one now determines inductively a sequence of natural numbers  $(a_v)_{v=1}^\infty$  such that

$$\frac{|f_{v+1}^{a_{v+1}}(x^{(v+1)})|_0}{(v+1)^2} - \sum_{\alpha=1}^v \frac{|f_\alpha^{a_\alpha}(x^{(v+1)})|_0}{\alpha^2} > v+1.$$

The series  $\sum_{\alpha=1}^\infty \frac{f_\alpha^{a_\alpha}(x)}{\alpha^2}$  is normally convergent in  $\Omega$ , and so

represents a monogenic function in  $\Omega$ , say  $F$ . However it may be shown that  $|F(x^{(v)})|_0 \geq v - C$ ,  $C$  being a positive constant.

Now take an arbitrary point  $x$  in  $\Omega$  with rational co-ordinates; then  $x$  coincides with a certain  $\eta^{(v_0)}$  and this yields by construction a subsequence  $(w^{(v_\mu)})_{\mu=1}^\infty$  of  $(w^{(v)})_{v=1}^\infty$  such that

$w^{(v_\mu)} = \eta^{(v_0)} = x$  for all  $\mu = 1, 2, \dots$ , whence the corresponding subsequence  $(x^{(v_\mu)})_{\mu=1}^\infty$  is obviously contained in

$$U_{v_\mu} \cap \overset{\circ}{B}(w^{(v_\mu)}, d(w^{(v_\mu)}, \partial\Omega)) = \overset{\circ}{B}(x, d(x, \partial\Omega))$$

and has the property that  $|F(x^{(v_\mu)})|_0 \geq v_\mu - C$ . ■

Now we study the behaviour of  $F$  on the boundary of  $\Omega$ .

PROPOSITION 4.5. With the same notations as in Theorem 4.4 we have for any  $\eta \in \partial\Omega$ ,

$$\overline{\lim}_{\substack{x \rightarrow \eta \\ x \in \Omega}} |F(x)|_0 = +\infty.$$

Proof. Take a sequence  $(u^{(\mu)})_{\mu=1}^\infty$  in  $\Omega$  converging to  $\eta$ ; then we may select a sequence  $(\eta^{(v_\mu)})_{\mu=1}^\infty$  such that  $|u^{(\mu)} - \eta^{(v_\mu)}|_0 < \frac{1}{\mu}$ .

Reviewing the construction of  $(w^{(\nu)})_{\nu=1}^{\infty}$ , we may find a subsequence  $(w^{(\nu\lambda)})_{\lambda=1}^{\infty}$  for which  $w^{(\nu\lambda)} = \eta^{(\nu\mu)}$ , and clearly

$$\lim_{\lambda \rightarrow \infty} w^{(\nu\lambda)} = \eta.$$

On the other hand, analyzing the proof of Theorem 4.4, we have that

$|x^{(\nu\lambda)} - w^{(\nu\lambda)}| < d(w^{(\nu\lambda)}, \partial\Omega)$ ,  $x^{(\nu\lambda)}$  being the point corresponding to  $w^{(\nu\lambda)}$ , and so  $\lim_{\lambda \rightarrow \infty} x^{(\nu\lambda)} = \eta$ .

As moreover  $|F(x^{(\nu\lambda)})|_0 > \nu_{\lambda} - C$ , we obtain that

$$\lim_{\lambda \rightarrow \infty} |F(x^{(\nu\lambda)})|_0 = +\infty. \blacksquare$$

In the same way as for Theorem 2.2 we may now prove

**THEOREM 4.6.** If  $\Omega$  is a  ${}_2M(\Omega)$ -convex domain in  $R^{m+1}$ , then it is a monogenic existence domain.

EXAMPLES.

(i) Take an arbitrary domain  $\omega$  in a co-ordinate plane of  $R^{m+1}$  and consider a tube domain of the form  $\Omega = \omega \times R^{m-1}$ . As each compact subset  $K$  of  $\Omega$  can be written as a subset of a certain  $K' \times R^{m-1}$ ,  $K'$  being compact in  $\omega$ , we have that  $\hat{K}'_{\Omega} \subset \hat{K}'_{\omega} \times R^{m-1}$ ,  $\hat{K}'_{\omega}$  being the holomorphically convex hull of  $K'$ . As  $\hat{K}'_{\omega}$  is compact and  $\hat{K}'_{\Omega}$  is bounded and relatively closed in  $\Omega$ , it follows that  $\hat{K}'_{\Omega}$  is compact in  $\Omega$ . So  $\Omega$  is  ${}_2M(\Omega)$ -convex and hence a monogenic existence domain.

(ii) We know from [3], Proposition 15.7.4 that the series

$$F(x) = \sum_{\alpha=0}^{\infty} p(x)^{\alpha!}, \text{ where } p(x) = \xi_1 + \dots + \xi_m, \text{ converges normally in the}$$

$$\text{tube domain } \Omega = \{x \in R^{m+1} : (x_1 + \dots + x_m)^2 + mx_0^2 < 1\}$$

Moreover it can be proved that this series becomes unbounded on  $\partial\Omega$ , whence  $\Omega$  is a monogenic existence domain for the function  $F$ .

**REMARK.** The family  ${}_2M(\Omega)$  may be replaced by any larger family of monogenic functions  $f$  which possess in  $\Omega$  an infinite number of monogenic powers  $f^{\alpha}$ ,  $\alpha \in \mathbb{I} \subset \mathbb{N}$ , satisfying the supplementary condition



$|f^\alpha| = |f|^\alpha$  for all  $\alpha \in I$ . Convexity with respect to that new family will also be sufficient for a domain to be a monogenic existence domain.

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