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ON PECULIARITY OF THE P=2 CASE AMONG THE P-STATE ISING-LIKE MODELS*

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Abstract

New indications of the p=2 case peculiarity among p-state lattice models are found while studing a relation between Rademacher and hyperbolic functions.

I. Introduction.

Consider the well known formula (Vietá)

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \qquad 0 \neq x \in \mathbb{R} \ . \tag{V}$$

In [1] Mark Kac has shown how a specifically wise derivation of (V) reveals the stochastic properties of numbers via their dyadic representation.

At the same time - such representations are 1:1 related to properties of Walsh functions which form a base of a specific example of maximally graded algebra [2] where

$$\Gamma \equiv Z_2 \oplus \ldots \oplus Z_2 \oplus \ldots \equiv Z(2^{\infty})$$

Algebras of the kind and generalized Walsh functions are already known to be of big importance for the theory of lattice spin systems with the grading group being in general that of $Z(p^{\infty})$;

$$Z(p^{\infty}) = Z_p \oplus \ldots \oplus Z_p \oplus \ldots$$

as described in [2,3].

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The case p=2 corresponds in the context to Ising model which is a very peculiar one (simply - it is solved!) among all Potts models with $Z(p^{\infty})$ grading group (p ≡ prime number; once for all, in this note).

Question

"why this is so ?"

we shall try to trace further reasons for that apart from the ones discussed already in [2,3,4]

Procedure

For that to do, we shall try to generalize some of the very first observations of M. Kac in [1] and see why this or that is not possible.

Remark

We shall work rather with hypergeometric functions (generalized cosh functions [2,4]) than with Mikusiński's generalization of sin and cos trygonometry from Z_2 to Z_p case, because partition functions of Potts models are polynomials in generalized cosh functions.

Result

The first results, however partial, serve as additional (to those found in [2,4]) indications "why this is so".

II. Rademacher and hyperbolic functions.

After recalling some prerequisities concerning real Rademacher step functions - a relation between them and hyperbolic function f_o is found.

Def.

$$r_o\ ,\qquad r_o(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ -1 & \frac{1}{2} < x < 1 \end{cases} \qquad ;\ r_o(x+1) = r_o(x);$$

$$r_n;\qquad r_n(x) = r_o(2^n x)$$

Name

$\{r_n\}_o^{\infty}$ - the set of Rademacher functions.

Observation 1.

$\{r_n\}_o^{\infty}$ - the set of orthonormal functions.

Now let x_i: $[0,1] \to Z_2$;

$$x \to x_i(x) \; ; \qquad x = \sum_{i=1}^{\infty} \frac{x_i(x)}{2^i}$$

where, in the case of two dyadic representation of x, allways the finite one is chosen.
Then one has an

Observation 2.

$$[0,1] \ni x \; ; \qquad r_n(x) \equiv 1 - 2\,x_n(x) \equiv (-1)^{x_n} \; .$$

We now extend the definition of Rademacher functions from the case p=2 to arbitrary prime p.

Def.

Def.

$$r_k(x) = \omega^{x_k} \qquad ; \qquad \text{where}$$

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p^k} \qquad ; \qquad x_k \in Z_p \quad \text{and} \quad \omega = \exp\{\frac{i2\pi}{p}\} \quad .$$

∎

Name

$\{r_k\}_1^{\infty}$ - p-Rademacher functions.

Observation

These functions form a set of orthonormal function.

Observation 3.

$$[0,1]\ni x \quad r_n(x) = \sum_{i\in Z_p} \omega^i (p-1)(x_n-1)\cdot \ldots \cdot (x_n-i+1)\cdot$$
$$\cdot (x_n-i-1)\cdot \ldots \cdot (x_n-p+1),$$

where Z_p is considered now to be the field.

Proof

It is enough to take into account:

a) Wilson's theorem $\Leftrightarrow$ $(p-1)! = p-1 \pmod{p}$ iff p-prime.

b) $(-1)\cdot(-2)\cdot\ldots\cdot(-p+1) = (p-1)! \pmod{p}$.

∎

Observation 4.

r_k is a linear function of x_k iff p=2 (see Observation 2)

Consequence 1.

For p=2 one has

$$\lim_{n\to\infty} \int_0^1 \exp\left(ix \sum_{k=1}^{n} \frac{r_k(t)}{2^k}\right) = \frac{\sin x}{x}$$

what leads to (V), as at the same time

$$\lim_{n\to\infty} \int_0^1 \exp\left(ix \sum_{k=1}^{n} \frac{r_k(t)}{2^k} \right) = \prod_{k=1}^{\infty} \cos\frac{x}{2^k}$$

■

Proof

(see [1]).

Comment:

The second formula of Consequence 1. exhibits the relation between 2-Rademacher functions and cos function.
This relation can be generalized to arbitrary p-prime.

Def.

$$f_i(x) = \frac{1}{p} \sum_{k=0}^{p-1} \omega^{-ki} \exp\{\omega^k x\} \quad ; \qquad x \in \mathbb{R}, \quad i \in Z_p$$

■

Name

$\{f_i\}_{i\in Z_p}$ ⇔ the set of hyperbolic functions (generalized cosh functions)

Generalization 1. (first step)

$$\int_0^1 \exp\left\{ \sum_{k=1}^{n} c_k r_k(t) \right\} dt = \prod_{k=1}^{n} f_o(c_k) \quad ; \qquad c_k \in \mathbb{R}$$

■

Proof

$$\int_0^1 \exp\left\{ \sum_{k=1}^{n} c_k r_k(t) \right\} dt = \frac{1}{p^n} \sum_{\{l(k)\}} \exp\left\{ \omega^{l(k)} c_k \right\} =$$

$$= \prod_{k=1}^{n} \frac{1}{p} \sum_{l=1}^{p-1} \exp\left\{ \omega^l c_k \right\} \quad ;$$

where

$$\{l(k)\}_1^n \in Z_p \oplus \ldots \oplus Z_p \quad \text{(n-summands)}$$

Generalization 2. (second step)

Let us first define:

Def

$$w[\text{" "}] \; : [0,1] \to [0,1] \quad ;$$

$$[0,1]\ni x \to w["x"] \equiv \sum_{n=1}^{\infty} \frac{w[x_n]}{p^n} \qquad \text{where}$$

$$x = \sum_{n=1}^{\infty} \frac{x_n}{p^n} \quad \text{and} \quad w : Z_p \to Z_p .$$

■

Name

$w["\ \ "]$ = p-adic function on [0,1].

Def

$$w_k : Z_p \to Z_p \ ;$$

$$Z_p \ni \alpha \to W_k(\alpha) \equiv (p-1)\cdot\alpha\cdot(\alpha-1)\cdot\ldots\cdot\overset{\vee}{(\alpha-k)}\cdot\ldots\cdot(\alpha-p+1) \ ;$$

where "v" sign over factor $(\alpha-k)$ means: "the factor $(\alpha-k)$ is absent".

Observation 5.

$$\sum_{n=1}^{\infty} \frac{r_n(x)}{p^n} \equiv \sum_{k=0}^{p-1} \omega^k \, W_k["x"]$$

■

Observation 6.

$$\forall \ k \in Z_p \qquad W_k["x"] \equiv W_k[x] \qquad \underline{\text{iff}} \ \ p=2.$$

Comment:

Observation 6 ≡ Observation 4.

Generalization

$$\int_0^1 \exp\left\{x \sum_{k=0}^{p-1} w_k["t"]\right\} dt = \prod_{s=1}^{\infty} f_o\left(\frac{x}{p^s}\right)$$

III. Towards generalization of Vietá's formula.

In II we have produced, in a sense a "half of such a formula" i.e. we have generalized only the second formula from Consequence 1. As linearity of r_n in x_n argument is crucial to have the left-hand side of (V) we feel hopeless in the case of p>2.

However, in this respect we may proceed in another way; trying to generalize the formula (1.1) in [1] i.e.

$$\sin x = 2^n \sin \frac{x}{2^n} \prod_{k=1}^{n} \cos\frac{x}{2^k} \quad . \tag{S}$$

We do not feel it to be an easy thing for us, hence we take now p=3.
For that case one has:

<u>Lemma</u>

$$3f_o(x)f_1(x)f_2(x) = \frac{1}{3}\left[f_o(3x) - 1\right]$$

with $x \in A$ where A is any associative algebra with unity; $\dim A < \infty$.

The proof consists of just straightforward - though slightly tricky - calculation.

In order to follow the scheme leading to (S) in ordinary case; one should now know how to express also $f_1(3x)$ and $f_2(3x)$ in terms of $f_o(x)$, $f_1(x)$ and $f_2(x)$.

Once this is known the (V) formula for the case of p=3 is at hand.

Up to now however this is not known to us.

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