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CONFORMAL INVARIANT WAVE EQUATIONS FOR SPIN ZERO AND SPIN $\frac{1}{2}$ FIELDS ON DE SITTER SPACE

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ABSTRACT

Using elementary notions of the Dirac Theory for a spin $\frac{1}{2}$ particle we prove conformal invariance of a certain wave equation associated with a massless spin $\frac{1}{2}$ field on de Sitter space. With the help of an integral transform which transfers fields on de Sitter space to fields on an associated Minkowski space, we show that this equation describes a spin $\frac{1}{2}$ particle of mass $\frac{2}{R^2}$ on this Minkowski space. These results are compared with analogous results for a massless spin zero field on de Sitter space, which corresponds to a field of mass $\frac{1}{4R^2}$ on the Minkowski space.

The Conformal Invariant Wave Equations

We denote the de Sitter space by V^4 and we refer the reader to [2] for the standard definitions of V^4. We recall the embedding of V^4 into ξ^5 as the hypersurface $\xi^a\xi_a = -R^2 (\xi^a\xi_a = \xi^{0^2} - \xi^{1^2} - \xi^{2^2} - \xi^{3^2} - \xi^{4^2})$. Now, up to a set of measure zero, each element of $SO_o(2,4)$ may be expressed uniquely as the cartesian product of a point of V^4 and an element of the isotropy subgroup of the point $e_o = (0,0,0,0,1)\varepsilon V^4$ [2]. This decomposition determines an action of $SO_o(2,4)$ on V^4 as conformal transformations [2]. We denote by $\mu(g,\xi)$ the Jacobian of the transformation g at the point ξ. Let P_2^2 be the stabilizer subgroup of the point $e_o = (0,0,0,0,1)\varepsilon V^4$. Let χ_λ denote the character of P_2^2 which is defined as $\chi_\lambda(p) = |\mu(p,e_o)|^{(n-\lambda)/2} (n=4)$. We define certain multiplier representations $U^\lambda = \mathrm{Ind}_{P_2^2}^{G_2} (\chi_\lambda)$ of $G = SO_o(2,4)$ by the formula,

$$(U^\lambda(g)f)(\xi) = f(Rg^{-1}\tfrac{\xi}{R})|\mu(g,\xi)|^{(n-\lambda)/2} \tag{1}$$

where $\xi \longrightarrow Rg^{-1}\frac{\xi}{R}$ is the conformal action of $g\varepsilon SO_o(2,4)$ on $\xi\varepsilon V^4$.

Next consider the following Dirac equation for a spin $\frac{1}{2}$ field on V^4

$$\gamma^a\xi_a\psi(\xi) = \pm iR\psi(\xi);\ \{\gamma^a,\gamma^b\} = 2g^{ab}; g^{ab}=\mathrm{diag}\,(1,-1,-1,-1,-1) \tag{2}$$

Here we are using all 5 gamma matrices. ψ has just 4 components. We have 10 spin matrices,

$$S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b] \quad (3)$$

and the Γ^as ($\Gamma^a = \frac{1}{2}\gamma^a$) and S^{ab}s generate an so(2,4). The following identity may be verified [4]

$$i\Gamma^a = \frac{i}{2\mu}\xi^a - \frac{1}{2\mu}\{\xi_b, S^{ba}\} \quad (\mu = iR) \quad (4)$$

It is valid acting on any solution of the Dirac equation. Now we define representations with spin acting on solutions of (2). The representation of SU(2,2) is given by

$$(U(g)\psi)(\xi) = |\mu(g,\xi)|^{\frac{(n-\lambda)}{2} + s} D(\bar{g})\psi(Rg^{-1}\tfrac{\xi}{R}), \quad (5)$$

where $D(\bar{g})$ is the spinnor representation of g having infinitesmal generators S^{ab} and Γ^a. The represenation space consists of C^∞ functions on V^4 which satisfy in addition certain asymptotic conditions that are necessary in order to ensure the smoothness of $(U(g)\psi)(\xi)$ for all $g \in SO_0(2,4)$. The invariance of the representation space under U(g) can then be established with the help of (4). (An elementary proof that U(g) satisfies the Dirac equation is presented in the Appendix.)

The conformal invariant wave equation is

$$D^{\frac{1}{2}}\psi(\xi) = \{\frac{1}{2R^2}L^{ab}L_{ab} - \frac{1}{2R^2}\}\psi(\xi) = 0 \quad (6)$$

where $L_{ab} = M_{ab} + S_{ab}$, and $M_{ab} = i(\xi_a\partial_b - \xi_b\partial_a)$ $(\partial_a = \frac{\partial}{\partial\xi^a})$. Using eqn. (2) we show $S^{ab}M_{ab}\psi(\xi) = 0$, so that

$$D^{\frac{1}{2}}\psi(\xi) = \{\tilde{\Box} + \frac{5}{2R^2} - \frac{1}{2R^2}\}\psi(\xi) = \{\tilde{\Box} + \frac{2}{R^2}\}\psi(\xi) = 0 \quad (7)$$

Here we have used the fact that $S_{ab}S^{ab} = 5$ [3]. The last term in braces before the zero is recognized as the conformal covariant wave operator for a spin zero field of conformal weight $\frac{\lambda}{2} = 1$ on V^4. Thus $D^{\frac{1}{2}}U(g)\psi = 0$ for $\lambda = 2(1-s)$. This establishes the invariance of $D^{\frac{1}{2}}\psi = 0$ under the above representation (5) of weight $\frac{\lambda}{2} = (1-s)$.

The Generalized Fourier Transform

Now, on the space of solutions of (6), there is defined a linear representation of $SO_0(1,4)$, which is obtained from the above representation of the conformal group by restriction to the subgroup. We may also construct a multiplier representation of $SO_0(1,4)$ for spin ½ fields on T^3, the unit mass hyperboloid[3]. The equivalence of this representation of $SO_0(1,4)$ on T^3 and the one on the space of solutions of (6) is established with the help of the following integral transform: [3]

$$\psi(\xi) = (\pi\psi)(\xi) = c^\lambda(\varkappa)\int_{T^3} dT^3\, \phi(\tfrac{p}{m})|\mu(g(\xi)\zeta(p))|^{-\nu-\frac{3}{2}-s} \quad (s=\tfrac{1}{2})$$
$$(\varkappa = (\nu, s),\ \lambda = \tfrac{1}{R},\ \nu = \tfrac{m}{R}) \quad (8)$$

Here $\phi: T^3 \longrightarrow C^4$, $\frac{p^\mu}{m} \in T^3$, $\zeta(p) = (\frac{p^\mu}{m}, 1)$. $c^\lambda(\varkappa)$ is a constant which is related to the Plancherel measure on V^4 in the s = o case. (For s = o this integral transform intertwines the representation of $SO_o(1,4)$ on solutions of the zero mass wave equation on V^4 with a certain multiplier representation on T^3.) We have established the following key result [3]:

$$(\tilde{\Box}_s \Pi^\nu \phi)\ (\xi) = [\Pi^\nu\{B_\mu B^\mu - \frac{\lambda^2}{2} L_{\mu\nu} L^{\mu\nu}\}\phi](\xi) = [\Pi^\nu\{P^2 + \frac{9}{4}\lambda^2 - \lambda^2 s(s+1)\}\ \phi\ ]\ (\xi) \tag{9}$$

where

$$B_\mu = P_\mu + \frac{\lambda}{2m}\{P^\rho, L_{\rho\mu}\}\ ,\ L_{\mu\nu} = M_{\mu\nu} + S_{\mu\nu}\ ,\ P^2 = P_\mu P^\mu .$$

$\tilde{\Box}_o$ = Laplace-Beltrami operator on V^4. $\tilde{\Box}_{1/2} = \frac{1}{2R^2} L_{ab} L^{ab}$. Using eqn. (9) it follows that a massless spin zero field on V^4 corresponds, under the equivalence, to a spin zero field of mass $-\frac{1}{4R^2}$, and a massless spin ½ field on V^4 corresponds to a spin ½ field of mass $-\frac{2}{R^2}$ in the associated momentum space. Furthermore we may transfer these fields on the momentum space to fields on Minkowski space with the help of the usual Fourier transform for spin zero and spin ½ fields.

The conformal mass is specified by the requirement of invariance of the particular wave equation under $SO_o(2,4)$. We have also introduced another mass, which is defined by the following equations:

$$\left.\begin{aligned} M^2 &= P_\mu P^\mu \\ P^\mu &= f^\mu(\frac{1}{\Lambda_o} B^\rho,\ L^{\rho\nu}) \end{aligned}\right\} \tag{10}$$

where B^ρ and $L^{\rho\nu}$ are the generators of the $SO_o(1,4)$ subgroup. The f^μs are inverse relations to

$$\frac{1}{\Lambda_o} B_\mu = \frac{1}{2\Lambda_o}\{\ P^\nu\ ,\ \ L_{\nu\mu}\}\ - \frac{\rho}{\Lambda_o}\ P_\mu \tag{11}$$

They should be specifically obtainable as a special case of a more general relation, which expresses the generators of translations of motion groups in terms of generators of the associated semisimple groups [1].

APPENDIX:

With the help of (4) we can rewrite (5) as [5]:

$$(U(g)\psi)\ (\xi) = e^{-i(\frac{\omega^{ab}}{2} L_{ab} - \omega^{6a}\frac{1}{\mu} B_a)}\psi\ (\xi) \tag{12}$$

where

$$B_a = B_a^o - \frac{1}{2\mu}\{\xi^b\ ,\ S_{ba}\}$$

and

$$B^o_a = \frac{1}{2\mu}\{\xi^b, M_{ba}\} + \frac{i}{\mu}\ \xi_a$$

The invariance of the representation space is now a direct consequence of (12) since both L_{ab} and B_a commute with the Dirac equation (2).

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