ON THE GEODESIC FLOW OF A FOLIATION OF A COMPACT MANIFOLD OF NEGATIVE CONSTANT CURVATURE*

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INTRODUCTION. In [10], the dynamics of the geodesic flow (ϕ_t) of a foliation F of a Riemannian manifold M was studied. Among the others, the Lyapunov exponents of (ϕ_t) were estimated and the non-existence of totally geodesic (moreover, C^2-closed to totally geodesic) foliations of compact negatively curved Riemannian manifolds was established.

Here, we consider the flow (ϕ_t) assuming that M has negative constant curvature. We define and estimate rank of a foliation F of M and we get an estimate of the entropy of (ϕ_t). Saying that rank of F cannot be large we express the fact that F has to be rather far from being totally geodesic.

PRELIMINARIES. Let F be a C^3-foliation of an oriented C^∞-manifold M equipped with a C^3-Riemannian structure $g = \langle\cdot,\cdot\rangle$. Let $n = \dim M$ and $p = \dim F$. We assume that F is complete, i.e. that its leaves are complete with respect to the induced Riemannian structure. In this case, the geodesic flow $\phi = (\phi_t)$ of F can be considered. ϕ is the flow on SF, the unitary tangent bundle of F, defined by

$$\phi_t v = \dot{c}(t),$$

where $c : \mathbb{R} \to L$ is the geodesic on a leaf L of F satisfying $\dot{c}(0) = v$. So, ϕ coincides with the geodesic flow of L on the bundle SL for any leaf L of F.

The Levi-Civita connection on M, its curvature tensor and the sectional curvature of M are denoted here by ∇, R and K, respectively.

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The second fundamental tensor B of F takes its values in the orthogonal complement of TF, however, here it is considered as a section of the bundle $\mathrm{Hom}\,(TF \otimes TF, TM)$ which carries the connection $\tilde{\nabla}$ induced by ∇ and the orthogonal projection $TM \to TF$. We have

$$B(X,Y) = (\nabla_X Y)^{\perp}$$

and

$$(\tilde{\nabla}_Z B)(X,Y) = \nabla_Z B(X,Y) - B((\nabla_Z X)^{\top},Y) - B(X,(\nabla_Z Y)^{\top})$$

for any sections X and Y of TF and any vector field Z on M, where

$$v = v^{\top} + v^{\perp}$$

is the decomposition of a vector $v \in TM$ into the components tangent and orthogonal to F.

Let $c : \mathbb{R} \to L$ be a geodesic on a leaf L of F. Following [10], vector fields $Z = Z_{\zeta}$ along c satisfying the equation

$$(1) \qquad Z'' - 2B(Z'^{\top},\dot{c}) - (\tilde{\nabla}_Z B)(\dot{c},\dot{c}) - R(\dot{c},Z)\dot{c} = 0$$

and the initial conditions

$$(2) \qquad Z(0) = \pi_{*}\zeta \quad \text{and} \quad Z'(0) = C(\zeta),$$

where $\zeta \in TTF$, are called Jacobi fields (for F). Here, $Z' = \nabla_{\dot{c}} Z$, $\pi : TF \to M$ is the projection and $C : TTM \to TM$ is the connection map of ∇ (see [4]). Recall that Jacobi fields appear when varying a geodesic on a leaf among geodesic on (possibly different) leaves. Jacobi fields along c form a vector space (over $\mathbb{R}$) of dimension $n + p$. We denote it by J_c^F.

RESULTS. Denote by J_c^0 the subspace of J_c^F consisting of all Jacobi fields Z along a geodesic $c : \mathbb{R} \to L$ satisfying

$$(3) \qquad B(Z'^{\top},\dot{c}) = 0 \quad \text{and} \quad (\tilde{\nabla}_Z B)(\dot{c},\dot{c}) = 0$$

together with the initial conditions

$$(4) \qquad Z(0) = 0 \quad \text{and} \quad Z'(0) \perp \dot{c}(0).$$

Conditions (4) imply that $Z = Z_{\zeta}$ for some $\zeta \in TSF$ and that $\langle Z',\dot{c}\rangle \equiv 0$. Note that the scalar product $\langle Z,\dot{c}\rangle$ need not vanish identically since

$$\frac{d}{dt} \langle Z,\dot{c}\rangle = \langle Z,B(\dot{c},\dot{c})\rangle$$

in our case. This makes our situation different from that of [3], [7] and [9], for example, where the geodesic flow of a Riemannian manifold was considered.

The dimension of the space J_c^o will be called the rank of F at $v = \dot{c}(0)$ and denoted by Rank (F,v). Given a ϕ-invariant Borel measure μ on SF we define the μ-rank of F by

$$\text{Rank } (F,\mu) = \max \{m; \text{ Rank } (F,v) \geq m \text{ for } \mu\text{ - a.a. } v\}.$$

With this notation we have the following

THEOREM. Let M be a compact Riemannian manifold of constant negative curvature K. Given a complete foliation F of M we have:
(a) $\mu(\{v \in SF; \text{ Rank } (F,v) \leq \frac{1}{2}(n + p - 2)\}) = 1$ for any ϕ-invariant probability measure μ on SF.
(b) $h_\mu(\phi) \geq \sqrt{-K/2} \cdot \text{Rank } (F,\mu)$ for any ϕ-invariant smooth probability measure μ.

Here, $h_\mu(\phi)$ is the measure entropy of ϕ w.r.t. μ [5].

P r o o f. Given $v \in SF$ denote by $E^s(v)$ and $E^u(v)$ the stable and unstable space of ϕ at v, respectively. If v is a vector regular for ϕ (in the sense of the Oseledet's Multiplicative Ergodic Theorem [6], see also [5]), then $E^s(v)$ (resp., $E^u(v)$) is spanned by all vectors $\zeta \in T_vSF$ for which the Lyapunov exponent

$$\lambda(\zeta) = \lim_{t \to \pm\infty} \frac{1}{t} \log |\phi_{t*}\zeta|$$

of ϕ in the direction of ζ is negative (resp., positive).

Assume that $Z = Z_\zeta \in J_c^o$, $c : \mathbb{R} \to L$, $\dot{c}(0) = v$. Let

$$x(t) = |Z(t)|^2 \quad \text{and} \quad y(t) = |Z'(t)|^2$$

for $t \in \mathbb{R}$. From (1) and (3) we get

$$x' = 2\langle Z,Z'\rangle,$$

$$y' = 2\langle Z',Z''\rangle = 2\langle R(\dot{c},Z)\dot{c},Z'\rangle = -2K\langle Z,Z'\rangle,$$

$$x'' = 2\langle Z',Z'\rangle + 2\langle Z,Z''\rangle = 2y - 2K(|Z|^2 - \langle Z,\dot{c}\rangle^2) \geq 2y,$$

$$y'' = -2Ky - 2K\langle Z,Z''\rangle = -2Ky + 4K^2[|Z|^2 - \langle Z,\dot{c}\rangle^2] \geq -2Ky.$$

Therefore, using (4) we obtain

$$(5) \qquad y(t) \geq |Z'(0)|^2 \cos h\,(\sqrt{-2K}\,t) \geq \frac{1}{2}|Z'(0)|^2 e^{\sqrt{-2K}\,t} \qquad (t \geq 0)$$

and

$$(6) \qquad x(t) \geq \frac{1}{\sqrt{-2K}} e^{\sqrt{-2K}\,t} + at + b \qquad (t > 0)$$

for some reals a and b. This shows that the Lyapunov exponents $\lambda(\zeta)$ of the flow ϕ satisfy

$$(7) \qquad \lambda(\zeta) \geq \sqrt{-K/2}$$

for all $\zeta \in T_v SF$ such that $\zeta \neq 0$ and $Z_\zeta \in J_c^o$ with c satisfying $\dot{c}(0) = v$, $v \in SF$.

Let Λ be the set of all points of SF regular with respect to ϕ. Then $\mu(\Lambda) = 1$ for any ϕ-invariant probability measure μ on SF ([6], see also [5]).

Let $v \in \Lambda$. From (7) it follows that

$$\dim E^u(v) \geq \operatorname{Rank}(F,v).$$

Also, if $\sigma : SF \to SF$ is given by $\sigma(v) = -v$, then

$$\phi_{-t} \circ \sigma = \sigma \circ \phi_t \qquad (t \in \mathbb{R}).$$

Therefore,

$$\sigma_* E^s(v) = E^u(-v)$$

and

$$\dim E^s(v) \geq \operatorname{Rank}(F,-v) = \operatorname{Rank}(F,v).$$

Consequently,

$$2\operatorname{Rank}(F,v) \leq \dim E^s(v) + \dim E^u(v) \leq \dim SF - 1 = n + p - 2$$

when $v \in \Lambda$. This proves (a).

To prove (b) recall the Pesin's inequality ([8], see also [5])

$$(8) \qquad h_\mu(\psi) \geq \int_X \chi(\psi,x)\,d\mu(x)$$

which holds for any C^2-flow ψ on a compact manifold X and for any smooth (i.e. absolutely continuous w.r.t. the Lebesgue measure) ψ-invariant measure μ. Here, $\chi(\psi,x)$ is the sum of all positive Lyapunov exponents of ψ at x counted together with their multiciplities.

In our case, inequality (7) shows that

$$\chi(\phi,v) \geq \sqrt{-K/2} \cdot \text{Rank}\,(F,v) \geq \sqrt{-K/2} \cdot \text{Rank}\,(F,\mu) \tag{9}$$

μ-a.e. if μ is a ϕ-invariant measure on SF.

Comparing (8) and (9) ends the proof.

FINAL REMARKS. A. We expect that the statement (a) of our Theorem could be proved under less restrictive assumptions on M, for example when M is locally symmetric and negatively curved.

B. In [2], the rank of a compact Riemannian manifold of non-positive curvature M is defined as the minimal dimension of the space of all parallel Jacobi fields along a given geodesic. Ballmann [1] proved that if M is irreducible and of rank at least 2, then M is locally symmetric. Following this idea one could search for the minimal number m such that if

$$\text{Rank}\,(F) \;=\; \min\{\text{Rank}\,(F,v),\ \ v \in SF\}$$

exceed m, then - under some assumptions on M - F has to be totally geodesic ($B \equiv 0$).

C. If the set of all smooth ϕ-invariant probability measures on SF is non-empty, then Theorem (b) implies that

$$h_{top}(\phi) \geq \sqrt{-K/2} \cdot \text{Rank}\,(F) \tag{10}$$

where $h_{top}(\phi)$ denotes the topological entropy of ϕ. In [10], we showed that non-trivial smooth ϕ-invariant measures exist when F is transversely minimal, i.e. when trace of the second fundamental tensor of the orthogonal complement of F vanishes. So, inequality (10) holds for transversely minimal foliations of compact manifolds of constant curvature $K < 0$. However, the existence of such foliations seems to be an open problem.

D. If $p = n$ (codim $F = 0$), then $B \equiv 0$, Rank $F = n - 1$ and inequality (10) takes the form

$$h_{top}(\phi) \geq \sqrt{-K/2} \cdot (n - 1).$$

However, it is not too hard to show that (see, for example, [9]) that in this case

$$h_{top}(\phi) \geq \sqrt{-K} \cdot (n - 1).$$

The reason for our estimate is weaker than the last one is that mentioned in Introduction: We could not use the fact that $Z \perp \dot{c}$ all the time if $Z(0) \perp \dot{c}(0)$ and $Z'(0) \perp \dot{c}(0)$. So, we were able to show only that $x'' \geq 2y$, not that $x'' \geq 2y - 2Kx$.

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