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LOCALLY FINITELY GENERATED DIFFERENTIAL SPACES OF CLASS C^r

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In this paper we consider differential spaces of class C^r, which are a generalization of the concept of differential spaces introduced by Sikorski ([8], [9]). We consider differential structures of functions of class C^r with values in the field K ($K = \mathbb{R}$ or $\mathbb{C}$), where $r \in \mathbb{N} \cup \{\infty, \omega\}$, C^ω means analytical functions. In Section 2 we study some properties of differential spaces, which are locally finitely generated by a family of K-valued functions.

1. BASIC NOTIONS. Let C be a non-empty set of K-valued functions defined on a set M. Then τ_C is the weakest topology on M such that all functions of C are continuous. The family of sets $f^{-1}(Q)$, where Q is open in K, $f \in C$, is a subbasis of the topology τ_C.

For any subset A of M we denote by C_A the set of all K-valued functions f on A such that for every point $p \in A$ there exist a neighbourhood $U \in \tau_C$ of p and a function $g \in C$ such that $f|A \cap U = g|A \cap U$.

Let $C^r(K^n, K)$ be the set of all functions $\sigma : K^n \longrightarrow K$ of class C^r, where $r \in \mathbb{N} \cup \{\infty, \omega\}$, $\mathbb{N}$ is the set of natural numbers.

Denote by $sc^r C$ the set of all functions $\sigma \circ (f_1, \ldots, f_n)$, where $\sigma \in C^r(K^n, K)$, $f_1, \ldots, f_n \in C$, $n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{\infty, \omega\}$.

The set C is said to be a differential structure of class C^r on M (shortly d^r-structure) if

(a) the set C is closed with respect to localization, i.e. $C = C_M$,

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(b) the set C is closed with respect to composition with smooth functions of class C^r, i.e., $C = sc^r C$.

It is easy to verify that every d^r-structure C is a linear ring over K.

By a differential space of class C^r (shortly d^r-space), where $r \in \mathbb{N} \cup \{\infty, \omega\}$, we shall mean any pair (M,C), where M is a set and C is a d^r-structure on M. If (M,C) is a d^r-space and A is an arbitrary non-empty subset of M, then (A, C_A) is also d^r-space, which is called a d^r-subspace of (M,C).

For a set C_o of K-valued functions on M the set $C = (sc^r C_o)_M$ is the smallest d^r-structure on M including the set C_o. Then (M,C) is called the d^r-space generated by C_o. It is easy to see that $\tau_C = \tau_{C_o}$.

Let $\hat{C}_p$ be the set of germs of functions from C at p. By a vector tangent to a d^r-space (M,C) at a point p of M we shall mean any K-linear mapping $v: \hat{C}_p \longrightarrow K$ such that

$$(1.1) \quad v(\sigma \circ (\hat{f}_1, \ldots, \hat{f}_n)) = \sum_{i=1}^{n} \sigma_{|i}(\hat{f}_1(p), \ldots, \hat{f}_n(p)) \cdot v(f_i)$$

for any $\hat{f}_1, \ldots, \hat{f}_n \in \hat{C}_p$, $\sigma \in C^r(K^n, K)$.

We will denote by $T_p(M,C)$ or shortly T_pM the set of all vectors tangent to (M,C) at a point $p \in M$ and by TM the disjoint sum of all K-linear spaces T_pM, $p \in M$.

Let TC be the d^r-structure on TM generated by the set $\{f \circ \pi ;\ f \in C\} \cup \{df ;\ f \in C\}$, where $\pi : TM \longrightarrow M$ is the natural projection and $df: TM \longrightarrow K$ is the function defined by

$$(1.2) \quad (df)(v) = v(\hat{f}) \quad \text{for any } v \in TM .$$

A mapping $F: M \longrightarrow N$ of a d^r-space (M,C) into a d^r-space (N,D) is called C^r-smooth if $F^*(D) \subset C$, where $F^*(D) := \{g \circ F ;\ g \in D\}$. One can prove

<u>LEMMA 1.1</u>. Let (M,C) be a d^r-space generated by C_o, $p \in M$ an atbitrary point and $v_o: \hat{C}_{op} \longrightarrow K$ be a function satisfying the condition

$(*)$ for any $\sigma \in C^r(K^n, K)$, $\hat{f}_1, \ldots, \hat{f}_n \in \hat{C}_{op}$, $n \in \mathbb{N}$ if $\sigma \circ (\hat{f}_1, \ldots, \hat{f}_n) = 0$ then $\sum_{i=1}^{n} \sigma_{|i}(\hat{f}_1(p), \ldots, \hat{f}_n(p)) \cdot v_o(\hat{f}_i) = 0$.

Then there exists a unique vector v tangent to (M,C) at p such that $v | C_o = v_o$.

<u>Proof</u>. Let $v: \hat{C}_p \longrightarrow K$ be the mapping defined by $v(\hat{f}) =$

$= \sum_{i=1}^{n} \sigma_{|i}(\hat{f}_1(p),\dots,\hat{f}_n(p)) \cdot v_o(\hat{f}_i)$, for $\hat{f} \in \hat{C}_p$, where $\hat{f}_1,\dots,\hat{f}_n \in \hat{C}_{op}$ and $\sigma \in C^r(K^n,K)$ are such germs that there is an open neighbourhood $U \in \mathcal{T}_C$ of p and $f|U = \sigma \circ (f_1,\dots,f_n)|U$. From (*) is follows the correctness of the definition of the vector v .□

Now, let (M,C) be a d^r-space, $r \in \mathbb{N} \cup \{\infty,\omega\}$, generated by a set C_o . A vector field tangent to (M,C) is a mapping $X: M \longrightarrow TM$ such that $\pi \circ X = id_M$. Let us put $\forall_{n \in \mathbb{N}}\ n \leqslant \infty \leqslant \omega$. For any vector field X tangent to (M,C) and $f \in C$ let $Xf: M \longrightarrow K$ be the function given by $(Xf)(p) := X(p)(\hat{f})$ for $p \in M$. A vector field X tangent to (M,C) is called C^t-smooth, $(t \leqslant r)$, if $\forall_{f \in C} Xf \in H_{t-1}$, where $H_i := (sc^i\ C_o)_M$ for $i \in \mathbb{N} \cup \{\infty,\omega\}$ and H_o is the set of all K-valued continous functions on the topological space $(M,\mathcal{T}_C)$. It is easy to verify that $X: M \longrightarrow TM$ is a C^t-smooth vector field tangent to (M,C) if and only if $X^*(TC) \subset H_{t-1}$.

Denote by $\mathcal{X}^t(M)$ the set of all C^t-smooth vector fields tangent to (M,C). It is clear that $\mathcal{X}^t(M)$ is a H_{t-1}-module.

A d^r-space (M,C) has a constant differential dimension n if for any $p \in M$ there exist a neighbourhood $U \in \mathcal{T}_C$ of p and C^r-smooth vector fields $X_1,\dots,X_n \in \mathcal{X}^r(U)$ such that for any $q \in U$ the sequence $X_1(q),\dots,X_n(q)$ is a vector basis of $T_q(M,C)$ and $X_1,\dots,X_n$ is a basis of $(H_{r-1})_U$ -module $\mathcal{X}^r(U)$.

If M is a C^r-manifold, $C^r(M)$ the set of all C^r-functions on M, then $(M,C^r(M))$ is a d^r-space of constant differential dimension.

<u>2.MAIN RESULTS</u>. Let (M,C) be a d^r-space, $r \in \mathbb{N} \cup \{\infty,\omega\}$. (M,C) is said to be finitely generated by a set $C_o = \{f_1,\dots,f_n\}$ if $C = (scC_o)_M$ ([4]) .

Let N be a non-empty subset of K^n, $n \in \mathbb{N}$ and $D := C^r(K^n,K)_N$. It is easy to observe that (N,D) is a finitely generated d^r-space by the set $\{\pi_1|N,\dots,\pi_n|N\}$, where $\pi_i: K^n \longrightarrow K$ is the projection onto the i-th coordinate for $i = 1,\dots,n$. The natural imbedding $\iota_N: N \longrightarrow K^n$ is a smooth mapping of (N,D) into $(K^n,C^r(K^n,K))$.

Let $p \in N$ be an arbitrary point and $I_p: T_p(K^n,C^r(K^n,K)) \longrightarrow K^n$

be the natural isomorphism given by

(2.1) $I_p(v) = (v(\hat{\pi}_1), \dots, v(\hat{\pi}_n))$ for $v \in T_p(K^n)$.

It is evident that the composition $L_p = I_p \circ (\iota_N)_{*p} : T_p(N,D) \longrightarrow K^n$ is injective.

Let us put $O^r(N) := \{f \in C^r(K^n,K);\ f|N = 0\}$. Consider a K-linear subspace $N_p = \{h \in K^n ;\ f_{|h}(p) = 0$ for any $f \in O^r(N)\}$,where $f_{|h}(p)$ is the directional derivative of f in the direction of h .

PROPOSITION 2.1. $N_p = L_p(T_p(N,D))$.

Proof. We first show that $L_p(T_p(N,D)) \subset N_p$. Let $h \in L_p(T_p(N,D))$. There is a vector $v \in T_p(N,D)$ such that $L_p(v) = h$. For any $f \in O^r(N)$ we have $f \circ \iota_N = 0$. Thus $v(\widehat{f \circ \iota_N}) = 0$. It is easy to see that

$$v(\widehat{f \circ \iota_N}) = v(f \circ (\widehat{\pi_1|N}, \dots, \widehat{\pi_n|N})) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot v(\widehat{\pi_i|N}) =$$

$$= (\text{grad } f)(p) \cdot h = f_{|h}(p) .$$

Thus $f_{|h}(p) = 0$ for any $f \in O^r(N)$ or equivalently $h \in N_p$.

Let now $h \in N_p$. It means that $f_{|h}(p) = 0$ for any $f \in O^r(N)$. It is evident that the following condition is satisfied:

(*) for any $\sigma \in C^r(K^n,K)$, $n \in \mathbb{N}$ if $\sigma \circ (\pi_1|N, \dots, \pi_n|N) = 0$

then $\sum_{i=1}^{n} \sigma_{|i}(p) \cdot h_i = 0$.

In fact, since $\sigma \in O^r(N)$, $\sigma_{|h}(p) = 0$ or equivalently $\sum_{i=1}^{n} \sigma_{|i}(p) \cdot h_i = 0$. From Lemma 1.1 it follows that there exists a unique vector $v_h \in T_p(N)$ such that $v_h(\widehat{\pi_i|N}) = h_i$ for $i = 1, \dots, n$. Of course $L_p(v_h) = h$. This proves the inclusion $N_p \subset L_p(T_p(N,D))$. □

Now, let us put $G_p = \{(\text{grad } f)(p) ;\ f \in O^r(N)\}$. Of course G_p is a K-linear subspace of K^n. One can prove

PROPOSITION 2.2. $G_p \oplus N_p = K^n$ and G_p is g-orthogonal to N_p with respect to the metric g defined by

(2.2) $g(x,x') = \sum_{i=1}^{n} x_i \cdot x'_i$ for $x,x' \in K^n$.

Proof. The proof is almost trivial. It is easy to see that $N_p = \{h \in K^n;\ (\text{grad } f)(p) \cdot h = 0$ for any $f \in O^r(N)\} = G_p^{\perp}$. Since g is non-degenerate, $G_p \oplus N_p = K^n$. □

COROLLARY 2.1. The following conditions are equivalent:

(i) $\dim T_pN = n$,

(ii) $f_{|h}\, p = 0$ for any $f \in O^r(N)$,

(iii) $\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0$ for any $f \in O^r(N)$.

Proof. From Proposition 2.2 it follows that $\dim N_p = n$ iff $\dim G_p = 0$. It is clear that $\dim G_p = 0$ iff $(\mathrm{grad}\, f)(p) = 0$ for any $f \in O^r(N)$. This is equivalent to (ii) and (iii) . □

PROPOSITION 2.3. If $\dim T_pN = k \geq 1$, then there exists an open neighbourhood $U \in \mathcal{T}_D$ of the point p and a k-dimensional C^r-surface $S \subset K^n$ including U and $D_U = C^r(S)_U$, where $C^r(S) = C^r(K^n, K)_S$. Moreover, the integer $k = \dim T_pN$ is the smallest dimension of such a C^r-surface S.

Proof. L_p is an isomorphism of T_pN onto N_p. Thus $\dim T_pN = \dim N_p = k$. From Proposition 2.2 it follows that $\dim G_p = n-k$. Let $u_1, \ldots, u_{n-k} \in K^n$ be a vector basis of G_p. There exist functions $f_1, \ldots, f_{n-k} \in O^r(N)$ such that $v_i = (\mathrm{grad}\, f_i)(p)$ for $i=1, \ldots, n-k$. Since rank $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq n-k \\ 1 \leq j \leq n}} = n-k$, the mapping $(f_1, \ldots, f_{n-k}) : K^n \longrightarrow K^{n-k}$ is regular at p. There is a neighbourhood V open is topK^n of p such that rank $\left(\frac{\partial f_i}{\partial x_j}(q)\right)_{\substack{1 \leq i \leq n-k \\ 1 \leq j \leq n}}$ $= n-k$ for $q \in V$.

Consider the set $S = \{q \in V \;;\; f_1(q) = f_2(q) = \ldots = f_{n-k}(q) = 0\}$. From the implicit theorem ([1], [7], [10]) it follows that S is a k-dimensional C^r-surface in K^n. Of course, the set $U = M \cap V$ is open in $\mathcal{T}_D$ and $U \subset S$. Clearly $D_U = C^r(S)_U$. □

PROPOSITION 2.4. If $\dim T_pN = 0$ then the point p is isolated in N.

Proof. Suppose that p is not isolated in N. Then there exists a sequence (p_i) of points of N different from p and convergent to p . Consider the sequence $h_n := \frac{p_n - p}{|p_n - p|}$, $n \in \mathbb{N}$, of points such that $h_n = |1|$ for any $n \in \mathbb{N}$. There exists a subsequence (h_{n_i}) convergent to a point $h \in K^n$ and $|h| = 1$. One can easy see that for any $f \in C^r(K^n, K)$

$$\lim_{i \to \infty} \frac{f(p_{n_i}) - f(p)}{|p_{n_i} - p|} = f_{|h}(p) .$$

Thus for any $f \in O^r(N)$, since $f|N = 0$, we have

$$f_{|h}(p) = \lim_{i\to\infty} \frac{f(p_{n_i}) - f(p)}{|p_{n_i} - p|} = 0 .$$

Hence $h \in N_p$ and $h \neq 0$. Thus $\dim N_p \geqslant 1$, which contradics $\dim T_pN = \dim N_p = 0$. $\square$

Now let $\mathcal{D}_1^r$ ([12]) denote the class of all d^r-spaces (M,C) which fulfills the condition:

(* *) for any $p \in M$ there exist a set $U \ni p$ open in $\mathcal{T}_C$ and a C^r-manifold $\tilde{M}$ such that U is contained in the set of points of $\tilde{M}$, $\dim \tilde{M} = \dim T_p(M,C)$ and $C_U = C^r(\tilde{M})_U$.

From Proposition 2.3 and 2.4 it follows that $(N,D) \in \mathcal{D}_1^r$.

Now consider a d^r-space (M,C) finitely generated by a set $C_o = \{f_1,\dots,f_n\}$. Let $\Phi : M \longrightarrow K^n$ be the smooth mapping defined by

(2.3) $\Phi(p) = (f_1(p),\dots,f_n(p))$ for $p \in M$.

Let $\tilde{\Phi} : (M,C) \longrightarrow (\Phi(M), C^r(K^n,K)_{\Phi(M)})$ be the mapping Φ onto the image $\Phi(M)$. Similarly to Lemma 2.1 in ([4]) one can prove

LEMMA 2.1. Let (M,C) be a d^r-space finitely generated by the set $C_o = \{f_1,\dots,f_n\}$. Then:

(i) the empty set and the sets of the form $\Phi^{-1}(A)$ make a base of the topology $\mathcal{T}_C$, where A is an arbitrary set from the base of the Tikhonov topology of K^n,

(ii) the mapping $\tilde{\Phi} : (M,C) \longrightarrow (\Phi(M), C^r(K^n,K)_{\Phi(M)})$ is open ,

(iii) $\mathcal{T}_C$ is the Hausdorff topology iff $\tilde{\Phi} : M \longrightarrow \Phi(M)$ is a homeomorphism.

PROPOSITION 2.5. If (M,C) is a finitely generated d^r-space by the set $C_o = \{f_1,\dots,f_n\}$, then the mapping $\tilde{\Phi}^* : C^r(K^n,K)_{\Phi(M)} \longrightarrow C$ is an isomorphism between linear rings. If $\mathcal{T}_C$ is a Hausdorff topology, then the mapping

$$\tilde{\Phi} : (M,C) \longrightarrow (\Phi(M), C^r(K^n,K)_{\Phi(M)})$$

is a diffeomorphism.

Proof. Since $\tilde{\Phi}$ is a surjection, $\tilde{\Phi}^*$ is a monomorphism. Now we will prove that $\tilde{\Phi}^*$ is "onto". For any $f \in C$, let $\sigma_f : \Phi(M) \longrightarrow K$ be defined by

(2.4) $\quad \sigma_f(q) = f(p) \quad$ for $q \in \Phi(M)$,

where $p \in M$ is such that $q = \Phi(p)$. Cleary,

(2.5) $\quad \sigma_f \circ \tilde{\Phi} = f$.

It remains to show that $\sigma_f \in C^r(K^n,K)_{\Phi(M)}$. Fix $q \in \Phi(M)$ and choose $p \in M$ such that $\Phi(p) = q$. There exist an open neighbourhood $V \in \tau_C$ of p and a function $\sigma \in C^r(K^n,K)$ such that

(2.6) $\quad f|V = \sigma \circ \tilde{\Phi}|V$.

From (2.5) and (2.6) we have

$$\sigma \circ \tilde{\Phi}|V = \sigma_f \circ \tilde{\Phi}|V .$$

Hence $\sigma_f|\tilde{\Phi}(V) = \sigma|\tilde{\Phi}(V)$. Evidently from Lemma 2.1 it follows that $\tilde{\Phi}(V)$ is an open set containing q . Thus $\sigma_f \in C^r(K^n,K)_{\Phi(M)}$.

If τ_C is a Hausdorff topology, then by Lemma 2.1 $\tilde{\Phi}$ is a homeomorphism. It remains to show $\tilde{\Phi}^{-1}$ is smooth. In fact, it results from the following equalities:

(2.7) $\quad f_i \circ \tilde{\Phi}^{-1} = \pi_i|\Phi(M) \quad$ for $i = 1,\ldots,n$.

This finishes the proof. □

A d^r-space (M,C) is said to be locally finitely generated if for every $p \in M$ there exists an open neighbourhood $V \ni p$ such that the d^r-subspace (V,C_V) is finitely generated.

Let $\mathcal{L}^r$ denote the class of all locally finitely generated Hausdorff d^r-spaces.

PROPOSITION 2.6. $\mathcal{L}^r = \mathcal{D}_1^r$.

Proof. If (M,C) is of class $\mathcal{D}_1^r$, then for any $p \in M$ there exist a set $U \ni p$ open in τ_C and C^r-manifold $\tilde{M}$ such that $U \subset \tilde{M}$, $\dim \tilde{M} = \dim T_p(M,C)$ and $C_U = C^r(\tilde{M})_U$. Since $\tilde{M}$ is locally finitely generated, (U,C_U) is also locally finitely generated as a d^r-subspace of $\tilde{M}$. Thus (M,C) belongs to $\mathcal{L}^r$. We have proved the inclusuon $\mathcal{D}_1^r \subset \mathcal{L}^r$.

Now let (M,C) a locally finitely generated d^r-space. For any $p \in M$ there exist an open neighbourhood V of p and functions $g_i\colon V \longrightarrow K$, $i = 1,\ldots,n$ such that $C_V = (sc^r\{g_1,\ldots,g_n\})_V$. From Proposition 2.5 it follows that $\Psi = (g_1,\ldots,g_n)$ is a diffeomorphism of (V,C_V) onto $(\Psi(V), C^r(K^n,K)_{\Psi(V)})$. Let $\dim T_p(M,C) = k$. Then $\dim T_{\Psi(p)}\Psi(V) = k$. From Proposition 2.3 it follows that there exist an open neighbourhood $W \in$ top $\Psi(V)$ of $\Psi(p)$ and a k-dimensional C^r-surface $S \subset K^n$ such that $C^r(K^n,K)_W = C^r(S)_W$.

Let $\tilde{S} = \psi^{-1}(W) \cup (S \setminus W) \times \{W\}$ and let $F: \tilde{S} \longrightarrow S$ be the mapping defined by

$$(2.8) \qquad F(q) = \begin{cases} \psi(q) & \text{when } q \in \psi^{-1}(W) \\ q' & \text{when } q = (q', W) \text{ and } q' \in S \setminus W . \end{cases}$$

Cleary, F is a bijection. It is easy to see that $C^r(\tilde{S}) := F^*(C^r(S))$ ia a d^r-structure on $\tilde{S}$ such that F is a diffeomorphism of $\tilde{S}$ onto S. Obviously, $\dim \tilde{S} = \dim S = k$ and $\psi^{-1}(W) \subset \tilde{S}$. Moreover, $C_{\psi^{-1}(W)} = C^r(\tilde{S})_{\psi^{-1}(W)}$, because $F|\psi^{-1}(W) = \psi|\psi^{-1}(W)$. Therefore $(M,C) \in \mathcal{L}^r$ and $\mathcal{L}^r \subset \mathcal{D}^r_1$. □

PROPOSITION 2.7. Let $N \subset K^n$ be a subset such that $\dim T_p(N,D) = n$ for every $p \in N$, where $D := C^r(K^n,K)_N$. Then (N,D) has a differential dimension n.

Proof. Let us put $X_i = \frac{\partial}{\partial x_i}$ for $i = 1,\dots,n$. Of course, $X_1,\dots,X_n$ is a global basis of $C^r(K^n,K)$-module $\mathcal{X}^r(K^n)$. It is evident that $(\iota_N)_{*p} : T_p(N,D) \longrightarrow T_p(K^n)$ is an isomorphism for every $p \in N$. Let us put

$$(2.9) \qquad Y_i(x) = (\iota_N)^{-1}_{*\,x}(X_i(x)) \quad \text{for } x \in N,\ i = 1,\dots,n .$$

It remains to prove that $Y_1,\dots,Y_n$ is a basis of H_{r-1}-module $\mathcal{X}^r(N)$, where $H_{r-1} = (sc^{r-1}\{\pi_1|N,\dots,\pi_n|N\})_N$. It is easy to see that $Y_i(\pi_j|N) = \delta_{ij}$ for $i,j = 1,\dots,n$. Evidently every d^r-smooth vector field $Z \in \mathcal{X}^r(N)$ may be presented in the form $Z = \sum_{i=1}^{n} \varphi^i Y_i$, where $\varphi^i = Z(\pi_i|N) \in H_{r-1}$, $i = 1,\dots,n$. Of course, $Y_1(x),\dots,Y_n(x)$ is a basis of $T_x(N,D)$ for every $x \in N$. □

COROLLARY 2.2. The sequence $Y_1,\dots,Y_n$ defined by (2.9) is a basis of H_{t-1}-module $\mathcal{X}^t(N)$ for any $t \le r$.

Proof. Let $W \in \mathcal{X}^t(N)$. Since $Y_1(x),\dots,Y_n(x)$ is a vector basis of $T_x(N,D)$, $W(x)$ for any $x \in N$ may be uniquely presented in the form $W(x) = \sum_{i=1}^{n} \psi^i(x) Y_i(x)$, where ψ^i is a K-valued function defined on N, $i = 1,\dots,n$. Hence and from (2.9) we have

$$W(x)(\widehat{\pi_i|N}) = \psi^i(x) \quad \text{for } x \in N,\ i = 1,\dots,n .$$

Thus $\psi^i = W(\pi_i|N) \in H_{t-1}$ for $i = 1,\dots,n$. This finishes the proof. □

COROLLARY 2.3. If (M,C) is a space of class $\mathcal{D}^r_1$ such that $\dim T_p(M,C) = n$ for any $p \in M$, then (M,C) has a differential dimension n.

Proof. This is a consequence of Proposition 2.6 and 2.7.□

EXAMPLE 1. Let $N \subset \mathbb{R}^2$ be the graph of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is of class C^2 but not C^3. The d^r-space (N,D) with $D = C^r(\mathbb{R}^2,\mathbb{R})_N$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, has a differential dimension 2 for $r \geq 3$ and has a differential dimension 1 for $1 \leq r < 2$. It results easily from Proposition 2.3 and Proposition 2.7.

EXAMPLE 2. Let $N \subset K^n$ be a dense subset, $D = C^r(K^n,K)_N$. Then (N,D) has a differential dimension n for $r \in \mathbb{N} \cup \{\infty, \omega\}$.

EXAMPLE 3. Let $N \subset \mathbb{R}^2$ be the graph of the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 0 , \\ x^2 & \text{for } x < 0 . \end{cases}$$

The d^r-space (N,D), where $D = C^r(\mathbb{R}^2,\mathbb{R})_N$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, is a 1-dimensional C^r-manifold for $1 \leq r \leq 2$, but $\dim T_{(0,0)}(N,D) = 2$ for $r \geq 3$.

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