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ASPECTS OF AFFINE TODA FIELD THEORY

H.W.Braden, E.Corrigan, P.E.Dorey, R.Sasaki

1. Introduction

The motivation for exploring the ideas contained in this talk came principally from the recent work of Zamolodchikov [27,28] concerning the two dimensional Ising model at critical temperature perturbed by a magnetic field. At critical temperature and zero magnetic field, the Ising model is known to be associated with a conformal field theory carrying a pair of representations of the Virasoro algebra $L, \bar{L}$ each with central charge equal to one half and primary field content $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{16}, \frac{1}{16})$. The perturbed Ising model is no longer conformal but Zamolodchikov argued that it is nevertheless an integrable theory of eight spinless bosons with masses in the ratio

$$1 \quad 1.618 \quad 1.989 \quad 2.405 \quad 2.956 \quad 3.218 \quad 3.891 \quad 4.783 \tag{1.1}$$

The set of masses (1.1) were obtained by Zamolodchikov via a judicious use of information obtained by exploring the conserved quantities of the perturbed Ising model (for example, finding them to have spins $1, 7, 11, 13, 19, \ldots$, the exponents of E_8), in conjunction with the bootstrap conditions satisfied by the S-matrix elements of the conjectured theory.

Subsequently, Hollowood and Mansfield [17] pointed out that since Toda field theory is conformal the perturbation considered by Zamolodchikov might well be best regarded as a perturbation of a Toda field theory. The appropriate perturbation converts the Toda theory into an affine Toda theory which is known to be classically integrable. This work (and also that of Eguchi and Yang [13]) made it seem plausible that the theory sought by Zamolodchikov was actually affine E_8 Toda field theory. However, this connection required an imaginary value of the coupling constant. Investigations reported below concerning exact S-matrices all use a perturbative approach based on real coupling and the results differ in various ways from those thought to correspond to perturbed conformal field theory. Work by Al. Zamolodchikov [30] and recently by Klassen and Melzer [18] confirms this. Hence, a connection if it really exists is not straightforward.

A dozen years or so ago, there was a lot of interest in the development of S-matrix theory in the context of two dimensional integrable models [29] (such as sine-Gordon theory, the Gross-Neveu model, σ-models and so on). In particular, the a_n affine Toda field theory was explored by Arinshtein, Fateev and Zamolodchikov [1] and its exact S-matrix conjectured. Several sets of authors [2-4,6-8,10,21-22] have set out to explore the full set of S-matrices for all these Toda theories in order to complete the pioneering work of Arinshtein et al and to enhance the store of information concerning the two-dimensional quantum integrable theories. Certainly, there appears to be a collection of interesting observations and facts for which at present there is no ready explanation.

A further motivation is to explore the connection between conformal and perturbed conformal field theories in other contexts using similar ideas. For example, Fateev and Zamolodchikov [14] do precisely this, as do Christe and Mussardo [7] and others.[15,18,25]

The rest of this talk will be devoted to properties of the affine Toda field theory, the intention being to highlight a selection of curious properties that we feel ought to be explicable in terms of the underlying group theory but for which in most cases we have no explanation.

2. (Affine) Toda field theory

Let us begin by summarising some of the properties of classical (affine) Toda field theory.[12,20,23,26] The standard starting point is the lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi - \frac{m^2}{\beta^2}\sum_{i=1}^{r} n_i e^{\beta\alpha_i\cdot\phi} - \frac{m^2}{\beta^2} n_0 e^{\beta\alpha_0\cdot\phi} \\ &\equiv \mathcal{L}_0 + \mathcal{L}_1,\end{aligned} \tag{2.1}$$

where $\mathcal{L}_1$ denotes the final term containing α_0 and $\mathcal{L}_0$ represents the rest. The scalar field ϕ is actually a set of r real scalar fields ϕ_a $a = 1,\ldots,r$ and the vectors α_i $i = 1,\ldots,r$ are a set of simple roots for one of the simple Lie algebras of rank r. (For much of this talk we shall restrict to members of the ADE series for which all the simple roots have the same length, taken to be $\sqrt{2}$.) The extra root α_0 is the lowest root and is expressible in terms of all the other simple roots via

$$\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i,$$

where the coefficients n_i are a set of positive integers particular to the Lie algebra. The integer n_0 is always taken to be 1 and is inserted merely for uniformity of notation. The coupling constant β plays no rôle classically (it may be scaled out of the classical field equations for the stationary points of the action associated with (2.1)), but it is very important in the quantum field theory, as we shall see. The theory is formulated in Minkowski space-time. The piece $\mathcal{L}_0$ of the lagrangian is separated out because it is conformally invariant in the following sense. In terms of light-cone coordinates

$$x^{\pm} = (x \pm t) \tag{2.2}$$

the conformal transformations are given by

$$x^{\pm} \rightarrow x'^{\pm} = f^{(\pm)}(x^{\pm}) \tag{2.3}$$

and

$$\phi \rightarrow \phi'(x') = \phi(x) - \frac{\delta}{\beta} \ln(\partial_+ f^{(+)} \partial_- f^{(-)}), \tag{2.4}$$

where $\delta = \sum_{i=1}^{r} \lambda_i$ and λ_i are the fundamental weights. Then

$$\delta \cdot \alpha_i = 1 \quad i = 1, \dots, r \tag{2.5}$$

and it is easy to check that as a consequence of the transformations (2.3) and (2.4) $\mathcal{L}_0$ is scaled overall by the product $\partial_+ f^{(+)} \partial_- f^{(-)}$. The crucial feature is (2.4) and the property (2.5). It is equally straightforward to check that $\mathcal{L}_1$ is not scaled in the same way (and hence breaks the conformal symmetry), on noting that

$$-\delta \cdot \alpha_0 = \sum_{i=1}^{r} n_i = h - 1$$

where h is the Coxeter number of the algebra.

The quantisation of the conformal part of (2.1) alone leads to a quantum field theory carrying representations $L, \bar{L}$ of the Virasoro algebra for which the central charge is given by

$$c = \bar{c} = r + 48\pi |\delta|^2 \left(\frac{1}{\beta} + \frac{\beta}{4\pi} \right)^2 . \tag{2.6}$$

This is a story with a lengthy history of its own [1,16,17,19] which we cannot go into here. However, there is an interesting fact to be observed namely that c given by

(2.6) has a symmetry between 'strong' and 'weak' coupling under the interchange

$$\beta \to \frac{4\pi}{\beta}.$$

By tuning β and choosing the Lie algebra, a variety of unitary conformal theories can be found. For example, for E_8 the choice

$$\frac{\beta^2}{4\pi} = -\frac{31}{32}$$

leads to $c = \frac{1}{2}$. (The fact,

$$\delta^2 = \frac{rh(h+1)}{12}$$

is useful in this context.) Indeed, the whole unitary series of $c < 1$ representations of the Virasoro algebra can be obtained in this way.

We have already mentioned the affine theory (*i.e.* all of (2.1)) is not even classically conformal. Nevertheless, it is classically integrable and enjoys the features associated with this status. For example, there are infinitely many conserved charges Q_s in involution. These are labelled by their spin: *i.e.* transform under two-dimensional Lorentz transformations

$$x^{\pm} \to \lambda^{\pm} x^{\pm} \tag{2.7}$$

via

$$Q_s \to \lambda^{-s} Q_s. \tag{2.8}$$

The conserved quantities for the affine Toda theory based on the Lie algebra g have spins which are precisely the exponents of g modulo its Coxeter number.[12,26] Each theory also has a Lax pair with all the structure implied by that classically.

From a straightforward perturbative point of view, given the lagrangian (2.1) we are interested in data such as the masses and couplings of various kinds. Expanding the potential part of the lagrangian up to third order, we find

$$\begin{aligned} V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i &+ \frac{m^2}{2} \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b \phi^a \phi^b \\ &+ \frac{m^2 \beta}{6} \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b \alpha_i^c \phi^a \phi^b \phi^c \\ &+ \dots \end{aligned} \tag{2.9}$$

from which we can extract a mass matrix

$$(M^2)^{ab} = m^2 \sum_0^r n_i \alpha_i^a \alpha_i^b \tag{2.10}$$

and a set of three-point couplings

$$c^{abc} = \beta m^2 \sum_0^r n_i \alpha_i^a \alpha_i^b \alpha_i^c. \tag{2.11}$$

The mass matrix has been studied before, in most cases long ago,[20] but the couplings of the mass eigenstates had not been computed until recently.[2–4,6–8] It turns out that these two pieces of data which are quite laborious to compute seem to contain the key to the quantum S-matrix.

Even at this basic level, there are some interesting facts and observations. For example, it transpires that the set of masses computed as the r eigenvalues of the mass matrix (2.10) actually constitute the Frobenius-Perron eigenvector of the Cartan matrix of the associated Lie algebra.[3,4] In other words, if we set

$$\mathbf{m} = (m_1, m_2, \ldots, m_r) \tag{2.12}$$

then

$$C\mathbf{m} = \lambda_{\min}\mathbf{m} = 4\sin^2\frac{\pi}{2h}\,\mathbf{m} \tag{2.13}$$

where C is the Cartan matrix

$$C_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} \quad i,j = 1,\ldots,r. \tag{2.14}$$

We have no explanation for this fact.

The masses are also related to the extended Cartan matrix in the following way. Let N be the diagonal matrix with entries $n_0, n_1, \ldots, n_r$ down the leading diagonal and let $\widehat{C}$ be the extended Cartan matrix

$$\widehat{C} = \alpha_i \cdot \alpha_j \quad i,j = 0,1,\ldots,r$$

(for the simply-laced cases each root has length $\sqrt{2}$), then it is not hard to verify that

$$\alpha_i \cdot (p^2 - M^2)^{-1}\alpha_j = \left((p^2 - \widehat{C}N)^{-1}\widehat{C}\right)_{ij} = \left(\widehat{C}(p^2 - N\widehat{C})^{-1}\right)_{ij} \tag{2.15}$$

demonstrating that the eigenvalues of the (mass)^2 matrix are also non-zero eigenvalues of $N\widehat{C}$. (Note, the pole at $p^2 = 0$ on the right hand side of (2.15) has a residue annihilated by $\widehat{C}$). The identity (2.15) may well be important to the development of a perturbative approach to the quantum field theory based on (2.1).

The values of the masses are listed below for convenience (for the other, non simply-laced, cases see for example ref[4]):

$$\mathbf{a_n} \text{ series} \quad m_a^2 = 4m^2 \sin^2 \frac{a\pi}{n+1} \tag{2.16}$$

$$\begin{aligned}
\mathbf{d_n} \text{ series} \quad & m_s^2 = m_{s'}^2 = 2m^2 \\
& m_a^2 = 8m^2 \sin^2 \frac{a\pi}{2(n-1)}
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\mathbf{e_6} \quad & m_l^2 \equiv m_{\bar{l}}^2 = m_1^2 = (3-\sqrt{3})m^2 \\
& m_L^2 = m_2^2 = 2(3-\sqrt{3})m^2 \\
& m_h^2 \equiv m_{\bar{h}}^2 = m_3^2 = (3+\sqrt{3})m^2 \\
& m_H^2 = m_4^2 = 2(3+\sqrt{3})m^2
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\mathbf{e_7} \quad & m_1^2 = 8m^2 \sin^2 \frac{\pi}{9} \\
& m_2^2 = 8\sqrt{3}m^2 \sin \frac{\pi}{18} \sin \frac{2\pi}{9} \\
& m_3^2 = 8m^2 \sin^2 \frac{2\pi}{9} \\
& m_4^2 = 8\sqrt{3}m^2 \sin \frac{5\pi}{18} \sin \frac{\pi}{9} \\
& m_5^2 = 8m^2 \sin^2 \frac{\pi}{3} \\
& m_6^2 = 8m^2 \sin^2 \frac{4\pi}{9} \\
& m_7^2 = 8\sqrt{3}m^2 \sin \frac{7\pi}{18} \sin \frac{4\pi}{9}
\end{aligned} \tag{2.19}$$

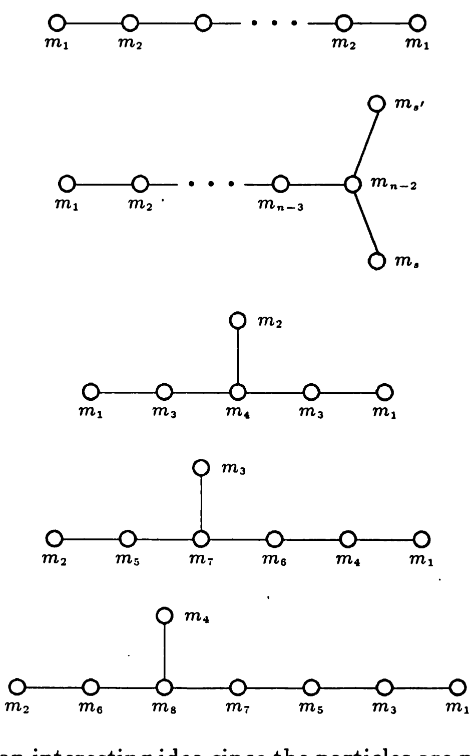
$$\begin{aligned}
\mathbf{e_8} \quad & m_1^2 = 4\sqrt{3}m^2 \sin \pi/30 \sin \pi/5 \\
& m_2^2 = 16\sqrt{3}m^2 \sin \pi/30 \sin \pi/5 \cos^2 \pi/5 \\
& m_3^2 = 16\sqrt{3}m^2 \sin \pi/30 \sin \pi/5 \cos^2 \pi/30 \\
& m_4^2 = 64\sqrt{3}m^2 \sin \pi/30 \sin \pi/5 \cos^2 \pi/5 \cos^2 7\pi/30 \\
& m_5^2 = 4\sqrt{3}m^2 \sin 11\pi/30 \sin \pi/5 \\
& m_6^2 = 4\sqrt{3}m^2 \sin 7\pi/30 \sin 2\pi/5 \\
& m_7^2 = 4\sqrt{3}m^2 \sin 13\pi/30 \sin 2\pi/5 \\
& m_8^2 = 256\sqrt{3}m^2 \sin \pi/30 \sin \pi/5 \cos^2 2\pi/15 \cos^4 \pi/5.
\end{aligned} \tag{2.20}$$

The e_8 masses are in the ratios (1.1) mentioned in the introduction. Zamolodchikov did not compute them this way, however.

The fact the classical masses can be thought of as the components of the Frobenius-Perron eigenvector of the Cartan matrix corresponding to the associated finite Lie algebra enables us to assign the particles unambiguously to spots on the appropriate Dynkin diagram.[2–4] It then becomes tempting to associate the particle with mass m_a with the corresponding fundamental representation of highest weight λ_a satisfying

$$\lambda_{\mathbf{a}} \cdot \alpha_{\mathbf{b}} = \delta_{ab} \quad b = 1, 2, \ldots, r.$$

We shall see this identification appears sensible from another point of view in a moment but first we supply a list of the diagrams for the cases whose masses appear in the above lists (2.16)–(2.20):



Actually, this is an interesting idea since the particles are not manifestly part of a multiplet; their associated representation is certainly well hidden from the point of view of the lagrangian starting point. There are mass degeneracies sometimes (in particular in the a-series where typically a particle has a conjugate partner), corresponding to the symmetries of the Dynkin diagrams. Nevertheless the particles are different, being distinguished by the conserved quantities, as we shall see, as well as by the representation to which they are tentatively associated.

Once the masses are known, the couplings between mass eigenstates can be computed in all cases. The detailed coupling tables may be found elsewhere, for example in ref[4], and will not be reproduced here. There are some universal features, however, which are worth remarking.

Firstly, it has been pointed out by a number of authors [7,3–4] that the magnitude of the coupling satisfies a universal rule (modified slightly in some of the non simply-laced cases), which may be summarised as follows. Denote the masses of the coupling particles by m_a, m_b, m_c. Particles never couple if their masses do not form a triangle of sides m_a, m_b, m_c. If the three masses do form a triangle and the coupling is non-zero then its magnitude is given by

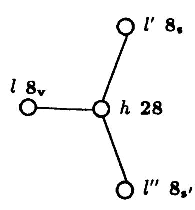
$$|c^{abc}| = \frac{4}{\sqrt{h}}\Delta^{abc} \tag{2.21}$$

where Δ^{abc} is the area of the triangle in question. Note, expression (2.21) is symmetric in the three labels. Note too, it is not sufficient for a non-zero coupling merely that the masses make a triangle. It appears the angles within the triangle must also be an integer multiple of π/h.

It is interesting to note that the assignment of representations to the particles is also reflected in the couplings though not in the most straightforward way. We note the rule (noted independently in ref[18]),

$$c^{abc} \neq 0 \quad \Rightarrow \quad (a)\otimes(b) \supset (\bar{c}) \tag{2.22}$$

for the three fundamental representations (a), (b) and (c) associated with the particles a, b and c. The implication does not go the other way except for the members of the A_n series and for D_4. For all other cases the coupling table is a subset of the Clebsch-Gordon series. The A_n cases are easily checked using the Young tableaux associated with the fundamental representations. For D_4 the diagram is



(2.23)

and the non-zero couplings are c^{hhh}, c^{llh} and $c^{ll'l''}$ in agreement with the relevant parts of the Clebsch-Gordon series:

$$\begin{aligned} \mathbf{28} \otimes \mathbf{28} &\supset \mathbf{28} \\ \mathbf{8_i} \otimes \mathbf{8_i} &\supset \mathbf{28} \\ \mathbf{8_i} \otimes \mathbf{8_j} &\supset \mathbf{8_k} \qquad \text{if } i, j, k = v, s, s' \end{aligned}$$

For D_5 it goes wrong as one may easily verify, but only for one coupling. The Clebsch-Gordon series allows

$$(2) \otimes (2) \supset (2) \tag{2.24}$$

but the coupling table does not allow the particle corresponding to this spot on the Dynkin diagram to couple to itself (despite the fact that three equal masses always form a triangle!); it appears to be important that the fusing angle defined below happens not to be an integer multiple of π/h—in this case $h = 8$ and $2\pi/3 = \frac{16}{3}(\pi/8)$. Typically, for D_n, the relevant part of the Clebsch-Gordon series is

$$(a) \otimes (b) \supset (a+b) \oplus (a+b-2) \oplus \ldots \oplus (|a-b|) \tag{2.25}$$

whereas the coupling table contains at most two possibilities for c given a and b. It is interesting, however, that the 'spinor' parts of the coupling table follow the Clebsch-Gordon series faithfully.

For the E-series the couplings are again a subset of the Clebsch-Gordon table. The relevant parts of these we have set out in tabular form together with tables giving the coupling data. The upper indices on the data in the Clebsch-Gordon part of the tables represent multiplicities; the representations are labelled in the same order as the masses (2.18), (2.19) and (2.20) which is ascending mass order left to right top to bottom in the tables. The upper indices in the coupling data tables give the phase of c^{abc} in our favoured bases for the mass eigenvectors (where there are mass degeneracies there is a choice to be made). The second row in each box of the coupling data gives the set of fusing angles—in every case a multiple of π/h for the appropriate h. (In every case in which the C-G series is truncated the fusing angle for the absent entry would fail to be an integer multiple of π/h).

For the E_6 tables the agreement is almost total. Apart from the multiplicity in the $(H) \otimes (H) \supset (H)^2$ contribution, there is only one coupling $c^{h\bar{h}L}$ which is actually zero but which would otherwise be allowed by the Clebsch-Gordon series.

We shall make a further comment on the C-G series versus fusing relations later, in the context of a perturbative approach to the quantum field theory.

3. Quantum field theory and properties of the S-matrix

The classical theory, interesting though it is, is not our main concern and we now turn to some questions within the quantum field theory associated with the affine theories.

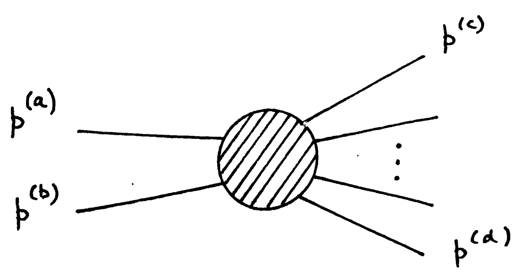
The basic entities in field theory apart from the fields themselves are the (multi) particle states which we shall take to be labelled by the momenta and species of the particles. Thus, single particle states, two-particle states and so on will be denoted by

$$\left|p^{(a)}\right\rangle; \quad \left|p^{(a)}, p^{(b)}\right\rangle, \quad \ldots \tag{3.1}$$

where the superscript denotes the particle type and p represents its momentum. The two-dimensional momentum of a particle is given conveniently in terms of its rapidity θ_a via

$$p^{(a)} = m_a(\cosh\theta_a, \sinh\theta_a). \tag{3.2}$$

We have in mind the usual idea that the particle states (3.1) make sense in a limit in which the particles are spatially well separated (such as long before or long after scattering). One of the aims of quantum field theory is to compute the probability of a particular outcome (or set of outcomes) in a scattering experiment, starting with a given initial state (usually containing two particles) which is one of (3.1) and with the possible final states being a subset of (3.1). Typically, the picture we have in mind is



(3.3)

and the task of computing the effects of the scattering process converting the initial to the final state is generally extremely difficult. However, the affine Toda field theories are very special and indeed we hope they are quantum integrable. In particular, if the conservation laws survive into the quantum theory and serve to distinguish the particles in any given theory uniquely, then the scattering picture is much simpler than (3.3) would suggest. In fact, there can be no production of extra particles at all in the final state and momenta must be preserved individually.[24] It is not even possible for the two initial particles to interchange momenta. Actually, this situation is even simpler than the sine-Gordon case (for a review see Zamolodchikov and Zamolodchikov [29]). There, the soliton and anti-soliton states are distinguished only by a conserved quantity with zero spin (the topological charge) which is not enough to forbid the exchange of momenta in soliton-anti-soliton scattering. This feature complicates the scattering theory considerably. In the very special case of affine Toda theory the only change in the two particle state as a consequence of interaction must be a phase. Thus,

$$\left|p^{(a)},\, p^{(b)}\right\rangle_{\rm out} = S_{ab}(\theta_a - \theta_b)\left|p^{(a)},\, p^{(b)}\right\rangle_{\rm in} \tag{3.4}$$

where the phases S_{ab} are functions only of the rapidity difference of the two particles (to preserve two-dimensional Lorentz invariance). There will be one such phase for each pair of particles in the theory. The totality of the the phases is referred to as the two-particle S-matrix. On general grounds, the sets of phases satisfy a number of conditions or constraints which we summarise briefly:

(a) Unitarity

The scattering matrix element $S_{ab}(\theta)$ is a phase for real θ ($\equiv \theta_a - \theta_b$)

$$S_{ab}(\theta)S^*_{ab}(\theta) = 1.$$

It is also a meromorphic function of complex θ satisfying

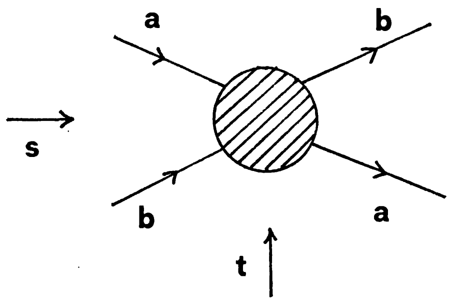
$$S_{ab}(\theta)S_{ab}(-\theta) = 1 \tag{3.5}$$

where, if θ is real,

$$S^*_{ab}(\theta) = S_{ab}(-\theta).$$

(b) Crossing

The S-matrix element $S_{ab}(\theta)$ serves also to describe the 't-channel process'



(3.6)

via analytic continuation:

$$S_{b\bar{a}}(\theta) = S_{ab}(\pi i - \theta). \tag{3.7}$$

Note, the standard Mandelstam variables s and t are given in these special cases by

$$s = \left(p^{(a)} + p^{(b)}\right)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta \tag{3.8}$$

$$t = \left(p^{(a)} - p^{(b)}\right)^2 = m_a^2 + m_b^2 - 2m_a m_b \cosh\theta \tag{3.9}$$

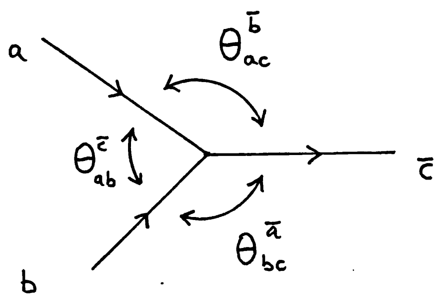
As a consequence of (a) and (b) all S-matrix elements are invariant under $\theta \to \theta + 2\pi i$ and hence may be expressed in terms of trigonometric functions.

(c) Fusing and the bootstrap

The kinematically accessible region in s (3.8), corresponds to real θ and $s \geq (m_a + m_b)^2$ (the s-channel) or to $s \leq (m_a - m_b)^2$ (the t-channel). However, the S-matrix may have bound state poles in real s at positions between the physical thresholds, corresponding in terms of θ to pure imaginary values of θ in the range $0 \leq \mathrm{Im}\theta \leq \pi$, the so-called physical strip. If two particles can bind or fuse to form a third $\bar{c}$ of mass $m_{\bar{c}}$, in the sense that their S-matrix element contains a pole, then they will do so at a particular value of rapidity $i\theta_{ab}^{\bar{c}}$ defined by

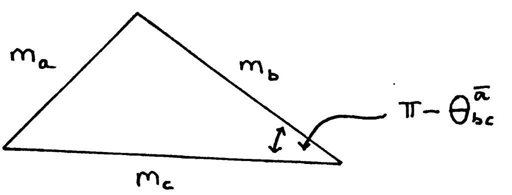
$$m_{\bar{c}}^2 = m_a^2 + m_b^2 + 2m_a m_b \cos\theta_{ab}^{\bar{c}} \tag{3.10}$$

which we refer to as the fusing angle,



(3.11)

We note that when a fusing $ab \to \bar{c}$ occurs so also can $ac \to \bar{b}$ and $bc \to \bar{a}$ at the angles represented in (3.11). Clearly, the three participating masses form a triangle



(3.12)

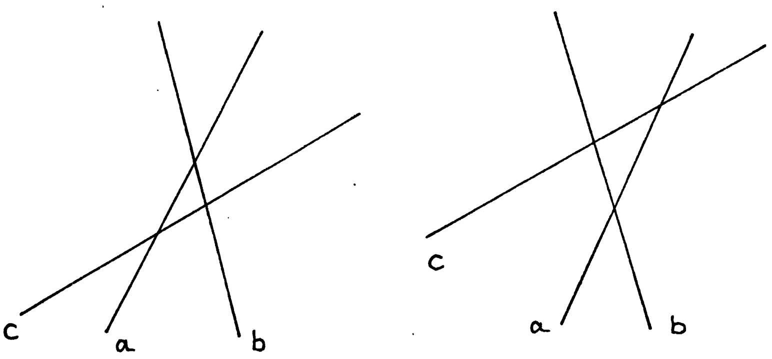
with internal angles as indicated. Equally, from (3.11) we note

$$\theta^{\bar{c}}_{ab} + \theta^{\bar{b}}_{ac} + \theta^{\bar{a}}_{bc} = 2\pi.$$

When such a fusing is possible we expect the S-matrix to have a pole. Actually, for reasons explained later, the pole may be of quite a high order. Also, for reasons to be explained briefly later, some poles (those of even order) are not indicative of bound states. Poles in θ on the physical strip correspond to s-channel or t-channel poles according to the sign of the residue.

The idea that particles can fuse below threshold to form a bound state has consequences for S-matrix elements besides the mere existence of poles in θ.

We have in mind the following picture of three particle scattering



(3.13)

in which the effect of the scattering is reasonably assumed to be the product of the three two-particle S-matrix elements either from the point of view of the left hand picture or the right hand picture (think of time flowing up the page). This assumption can be justified on the basis of the existence of an infinite nuber of conserved quantities (see for example ref [29]). That the outcome of the scattering should be the same from either point of view implies

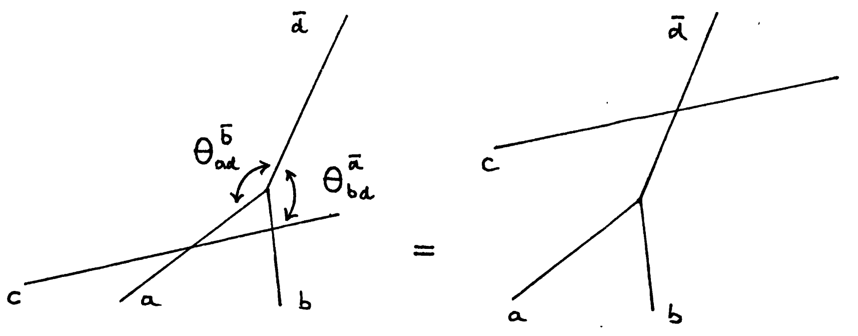
$$S_{ca}(\theta_{ca})S_{cb}(\theta_{cb})S_{ab}(\theta_{ab}) = S_{ab}(\theta_{ab})S_{cb}(\theta_{cb})S_{ca}(\theta_{ca}), \tag{3.14}$$

the Yang-Baxter equation. In the present context, since the S-matrix elements are phases for real θ, (3.14) is an identity. If some of the particles were not distinguishable (3.14) would be a non trivial set of conditions on the scattering matrix.

Now we add the following (heuristic) argument. Suppose we analytically continue in θ_{ab} to a pole position corresponding to the fusing $ab \to \bar{d}$. Then the particle c in (3.13) has the option of scattering either before, after or 'during' the fusion to the fourth particle. The last part of the statement is to be interpreted loosely; in no sense is the particle d actually produced since the kinematic region is unphysical but rather the single particle state corresponding to it domnates the S-matrix element at the unphysical value of the rapidity difference between particles a and b. This idea leads to a strong condition which appears to be borne out by calculations in particular theories. We can use it to find a precise relation between the S-matrix element $S_{c\bar{d}}$ and the S-matrix elements S_{ac} and S_{bc}. Specifically, it is

$$S_{c\bar{d}}(\theta) = S_{ac}(\theta - i\bar{\theta}^{\bar{b}}_{ad})S_{bc}(\theta + i\bar{\theta}^{\bar{a}}_{bd}) \qquad (\bar{\theta} = i\pi - \theta) \tag{3.15}$$

which is represented pictorially by



(3.16)

Eq.(3.15) defines a sort of algebra for the S-matrix elements since a similar relation must hold for every choice of three particles a, b and c and choice of fusing. The full set of this type of condition is a powerful tool for determining the S-matrix from a small set of input data. Indeed, this provided one of the main arguments used by Zamolodchikov in obtaining (1.1). One could envisage starting with (3.15) and attempting to classify all two-dimensional factorisable S-matrices. (See, for example, refs.[21–22])

Besides the conditions on the S-matrix the quantum theory ought to have a number of other features. Here we mention the conserved quantities.

Classically, each of the affine Toda theories enjoys an infinite set of conserved quantities. If these survive into the quantum domain then we expect them to be a set of mutually commuting operators whose joint eigenstates are the set of particle states (3.1). Thus we expect, for example, to be able to write

$$Q_s \left|p^{(a)}\right\rangle = q^a_s\, e^{s\theta} \left|p^{(a)}\right\rangle \tag{3.17}$$

where the factor containing the rapidity is necessary to ensure the correct Lorentz transformation property (2.8). Note, the charge q^a_1 corresponding to $s = 1$ is just the mass m_a. The idea that the two particle state $\left|p^{(a)}, p^{(b)}\right\rangle$ can be dominated at some unphysical value of the rapidity by another single particle state places strong restrictions on the eigenvalues of the conserved quantities given by (3.17). Thus, referring to the diagram (3.16), we note

$$\theta_a = \theta_{\bar{d}} - i\bar{\theta}^{\bar{b}}_{ad}$$
$$\theta_b = \theta_{\bar{d}} + i\bar{\theta}^{\bar{a}}_{bd}$$

and hence, using (3.17)and

$$Q_s \left| p^{(a)}, p^{(b)} \right\rangle \approx Q_s \left| p^{(\bar{d})} \right\rangle \quad \text{when } \theta \approx i\theta^{\bar{d}}_{ab},$$

we obtain the set of relations [27–28]

$$q^a_s \, e^{-is\bar{\theta}^{\bar{b}}_{ad}} + q^b_s \, e^{is\bar{\theta}^{\bar{a}}_{bd}} = q^{\bar{d}}_s, \tag{3.18}$$

which have to be satisfied by the quantum numbers of any set of three particles related by fusing.

So far we have discussed two very different aspects of affine Toda field theory, the classical theory for which there is much detailed information including a set of masses and three (and higher) point couplings, and the quantum theory in particular the idea of the S-matrix. Classically, as we remarked, the coupling term c^{abc} in the lagrangian (2.1) is zero unless the masses of the three particles make a triangle and even then the coupling is non-zero only under certain circumstances. It is tempting to try to match the classical data with the S-matrix and see if a conjecture based on the classical information can satisfy the conditions (a), (b) and (c) above. In particlar, given the classical masses we are able to compute the fusing angles assuming the classical couplings are the relevant ones. Then it is necessary to check that the conjectured S-matrix elements satisfy the third condition (3.15). It is not difficult to carry out this procedure in all cases and we shall summarise some of the features below. What is not so easy however, is to calculate the S-matrix elements from first principles starting with the lagrangian (2.1). This step at present remains an open question. A proper understanding may come from the direction of quantum inverse scattering; a perturbative approach is hardly likely to yield more than supporting evidence for the conjectures.

One nice observation [11,18] which is worth making before we discuss the S-matrices in detail is the following generalisation of (2.13). Using the classically allowed angles derived from the non-zero three-point couplings, it is possible to verify that if for a particular theory the eigenvalues of the conserved charges for a given spin s solve the relations (3.18) and are assembled into a single vector (as we did for the masses (2.12)),

$$\mathbf{q_s} = (q_s^1, \ldots, q_s^r), \tag{3.19}$$

then the vector $\mathbf{q_s}$ is also an eigenvector of the Cartan matrix. That is,

$$C\mathbf{q_s} = \lambda_s \mathbf{q_s} \qquad \lambda_s = 4\sin^2\frac{s\pi}{2h}, \tag{3.20}$$

an elegant result. Eq(3.20) is only a statement about the ratios of the eigenvalues, the actual eigenvalues will be scaled by an (unknown) function of s and β.

4. S-matrices for the ADE series

In this section we shall give some information about the exact S-matrices constructed from the classical data via the bootstrap.[2–4,6–8,10] It happens that each of the S-matrix elements for any member of the *ade*-series of affine Toda theories is constructible from a universal building block. Although the S-matrices are merely phases for real θ, they have an intricate singularity structure for complex θ and to avoid the expressions becoming unwieldy we adopt a condensed short hand for the basic blocks, as follows. Let

$$(x) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{2h}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{2h}\right)}. \tag{4.1}$$

Then (x) is manifestly unitary, since it satisfies (3.5), but it is not crossing symmetric. The crossed partner of (x) is $-(h-x)$. Note too that the block (x) contains no dependence on the coupling constant β. It is in fact possible to satisfy the conditions (a)—(c) for any of the *ade* theories to obtain what is called the 'minimal' S-matrix, $S_{\min}$, which contains no dependence upon β at all. However, this S-matrix is not expected to follow straightforwardly from (2.1). For example, as $\beta \to 0$ we might expect the theory to be free, so that the S-matrix elements tend to unity in the limit. Arinshtein, Fateev and Zamolodchikov [0] postulated a coupling constant dependence for the theories belonging to the a-series. Their conjecture can be recast in the form of a modified building block:

$$\{x\} = \frac{(x+1)(x-1)}{(x+1-B)(x-1+B)} \tag{4.2}$$

constructed from the minimal block (4.1) and a function of β given by

$$B(\beta) = \frac{\beta^2/2\pi}{1+\beta^2/4\pi}. \tag{4.3}$$

The function $B(\beta)$ is rather special, for example we note

$$B\left(\frac{4\pi}{\beta}\right) = 2 - B(\beta)$$

and hence

$$\{x\}_B = \{x\}_{2-B}. \tag{4.4}$$

The symmetry between the weak and strong coupling demonstrated by (4.4) is striking and reminiscent of the symmetry of the central charge exhibited earlier in (2.6). The function $B(\beta)$ has been chosen also so that no extra poles in θ are introduced into the physical strip. It is obvious from the definition (4.2) that $\{x\} \to 1$ as $\beta \to 0$.

In terms of the modified block (4.2) the conjectured S-matrices satisfying all the conditions (a)—(c) are:

a_n

$$S_{ab}(\theta) = \{|a-b|+1\}\{|a-b|+3\}\ldots\{a+b-3\}\{a+b-1\} \tag{4.5}$$

d_n

$$\begin{aligned} S_{ab}(\theta) &= \prod_{\substack{|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{p\}\{2(n-1)-p\} \qquad a,b = 1,\ldots,n-2 \\ S_{ss}(\theta) = S_{s's'}(\theta) &= \{1\}\{5\}\{9\}\ldots\{2n-3\} \\ S_{ss'}(\theta) &= \{3\}\{7\}\{11\}\ldots\{2n-5\} \\ S_{s'a}(\theta) = S_{sa}(\theta) &= \prod_{\substack{0 \\ \text{step } 2}}^{2a-2} \{n-a+p\} \qquad a = 1,2,3\ldots n-2. \end{aligned} \tag{4.6}$$

In the d-series, notice that, for n even, all the S-matrices are crossing symmetric whereas, for n odd, the elements S_{ss} and $S_{ss'}$ exchange under crossing. This emphasises that for odd n the particle s' should be regarded as the anti-particle of s. This also fits naturally with the representation assignment to s and s' mentioned earlier.

In either case, the minimal S-matrices are obtained by deleting those factors containing the dependence upon the coupling constant β.

e_n

Consult the tables at the end of the article

For the members of the e-series we have prepared tables. In each case there is one containing the minimal S-matrix (in conjunction with the relevant subset of the Clebsch-Gordon series) and one containing the complete S-matrix (in conjunction with the fusing tables). For the latter, we have employed a further notational device: a single integer represents the crossing symmetric combination

$$x \equiv \{x\}\{h-x\}$$

in the full S-matrix tables, whereas the notation $[x]$ represents the crossing symmetric combination

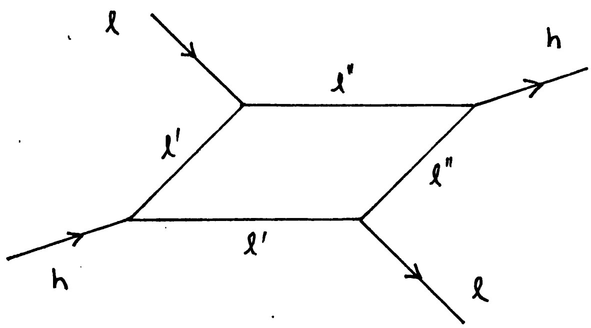
$$(x)(h-x)$$

in the minimal tables.

For a detailed example, it is quite instructive to inspect one of the cases more closely, say d_4. There, we recall the diagram (2.23) illustrating the three equal mass particles (mass $\sqrt{2}m$) and a heavier particle (mass $\sqrt{6}m$). The minimal S-matrix is

$$\begin{aligned}
S_{ll'}(\theta) &= \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{3}\right)\sinh\left(\frac{\theta}{2}+\frac{i\pi}{6}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{3}\right)\sinh\left(\frac{\theta}{2}-\frac{i\pi}{6}\right)} \\
S_{ll}(\theta) &= -S_{ll'}(\theta) \\
S_{lh}(\theta) &= \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{12}\right)\sinh\left(\frac{\theta}{2}+\frac{5i\pi}{12}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{12}\right)\sinh\left(\frac{\theta}{2}-\frac{5i\pi}{12}\right)}\left(\frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)}\right)^2 \qquad (4.7) \\
S_{hh}(\theta) &= \left(S_{ll}(\theta)\right)^3, \qquad (4.8)
\end{aligned}$$

the full S-matrix being given by (4.6) for $n=4$. It is not difficult to verify that poles appear in expected places, corresponding to the classically allowed couplings. However, there is one extra even order pole (in (4.7)) and a cubic pole (in (4.8)). From the point of view of the full S-matrix, the existence of these higher order poles is actually predicted [9,2-4,7] on the basis of perturbation theory. For example, the Feynman diagram



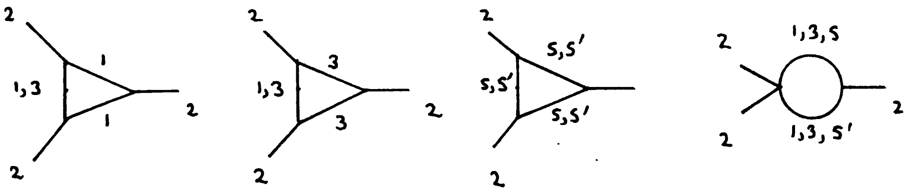
(4.9)

contributes a double pole at $\theta = i\pi/2$. Note, there is no reason to expect this pole to correspond to a bound state. The cubic pole comes from a pair of triangle vertex corrections. Indeed it is likely that all higher order poles in any S-matrix element (and the order can get quite high in the e_8 table), can be explained on the basis of perturbation theory, although the checking of pole residues has not been carried out in complete detail in most cases.

5. Further comments

Having conjectured the S-matrices for the affine Toda field theories at least for the *ade*-series one has to find a proof. In the absence of a proper proof—and we do not expect to find one using perturbation theory—we can at least seek to demonstrate the plausibility of the conjecture by verifying certain features to low order in perturative field theory. We have mentioned the pole singularities already, but it is also necessary to check the absence of production (*i.e.* true compatibility with the conservation laws), the coupling constant dependence (this was originally conjectured by Arinshtein, Fateev and Zamolodchikov on the basis of the observation that the a_1 theory is also the sinh-Gordon theory whose S-matrix might be expected to be the same as the already known S-matrix for the lightest breather in the sine-Gordon theory; nevertheless, it is necessary to verify that the function $B(\beta)$ of eq(4.3) really is universal. This has been checked by two of us [5] to one loop order. It is fascinating to note that the classical mass ratios appear to be relevant in the quantum theory, giving the positions of the poles in the conjectured S-matrices. Once again it is necessary to check that finite renormalisation effects do not spoil this feature. This has also been done to one loop and found to be correct.[3–4,8]

Within the perturbation theory there are further interesting observations. For example, we mentioned earlier that the classical couplings are provided by a subset of the Clebsch-Gordon series. However, it is perfectly possible to find one loop or higher diagrams within the perturbation theory which appear to permit couplings disallowed at tree level. For example, in the case of d_5 there was one coupling allowed by the Clebsch-Gordon series but actually absent, the $22 \to 2$ coupling or fusing. There are several diagrams at order β^3 which would permit this:



However, they cancel precisely when all three momenta are on mass-shell ($p^2 = m_2^2$)! In other words, the striking feature concerning the tree-level couplings is protected even at higher order. Presumably there is a deeper significance to the connection with the Clebsch-Gordon series than we have uncovered so far. This feature also persists in other cases where we have checked.

Klassen and Melzer[18] have pointed out another interesting fact about the minimal S-matrices and their Lie algebra connection. Suppose we define a matrix N_{ab} by setting

$$N_{ab} = -\frac{1}{2\pi i}\left[\ln S_{ab}(\theta)\right]_{\theta=-\infty}^{\theta=\infty}$$

then it transpires that for each of the theories we have discussed

$$N = 2C^{-1} - I \tag{5.1}$$

where C is the Cartan matrix. For each of the theories in which all the particles are self-conjugate, N_{ab} simply counts the $[x]$ blocks contributing to S_{ab}. The intricate pole structure implied by the bootstrap (3.15) is clearly crucial to this result. They needed to make use of this fact in their computation of finite scaling effects.

Finally, we remark briefly that the non simply-laced cases (*i.e.* those connected with the $bcfg$-series and those connected with the twisted affine diagrams [20,23]) certainly do not fall into the same consistent pattern as the ade-series [2–4,6–8,21]. The S-matrices that have been constructed on the basis of the classical data cannot follow from the lagrangian field theory. The situation in these cases must be more subtle and needs further clarification.

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TABLE: E_8 C-G Series and Minimal S-matrices

1 2 3	1 2 3 4	1 2 3 4 5	2 3 4 5 6	3 4 5 6 7	4 5 6 7	5 6 7 8	7 8
	$1\ 2\ 3\ 4$ $5\ 6$	$1\ 2\ 3^2\ 4$ $5\ 6$	$1\ 2\ 3\ 4$ $5\ 6\ 7$	$2\ 3\ 4\ 5^2$ $6\ 7$	$2\ 3\ 4\ 5$ $6^2\ 7\ 8$	$4\ 5\ 6\ 7^2$ 8	$6\ 7\ 8^3$
		$1\ 2^2\ 3^2\ 4^2$ $5^2\ 6\ 7$	$1\ 2\ 3^2\ 4^2$ $5^2\ 6^2\ 7$	$1\ 2\ 3^2\ 4^2$ $5^3\ 6^3\ 7^2\ 8$	$2\ 3\ 4^2\ 5^3$ $6^3\ 7^3\ 8$	$3\ 4\ 5^2$ $6^3\ 7^4\ 8^3$	$5\ 6\ 7^3\ 8^5$
$-[20]\ [12]\ [2]$			$1\ 2\ 3^2\ 4$ $5^2\ 6^2\ 7\ 8$	$1\ 2\ 3^2\ 4^2$ $5^3\ 6^3\ 7^3\ 8$	$1\ 2\ 3^2\ 4^2$ $5^3\ 6^3\ 7^3\ 8^2$	$2\ 3\ 4\ 5^3$ $6^3\ 7^4\ 8^4$	$4\ 5\ 6^2r\ 7^4$ 8^6
$[24]\ [18]$ $[14]\ [8]$	$-[24]\ [20]\ [14]$ $[8]\ [2]\ [12]^2$			$1\ 2^2\ 3^3\ 4^3$ $5^6\ 6^5\ 7^6\ 8^3$	$1\ 2\ 3^3\ 4^3$ $5^5\ 6^6\ 7^6\ 8^5$	$1\ 2\ 3^2\ 4^3$ $5^6\ 6^6\ 7^{10}\ 8^9$	$3\ 4\ 5^3\ 6^5$ $7^9\ 8^{17}$
$[29]\ [21]\ [13]$ $[3]\ [11]^2$	$[25]\ [19]\ [9]$ $[15]\ [7]^2\ [13]^2$	$-[22]\ [20]^3\ [14]$ $[12]^3\ [4]\ [28]^2$			$1\ 2^2\ 3^3\ 4^3$ $5^6\ 6^6\ 7^7\ 8^6$	$1\ 2\ 3^3\ 4^3$ $5^6\ 6^7\ 7^{10}\ 8^{11}$	$2\ 3\ 4^2\ 5^5$ $6^6\ 7^{11}\ 8^{17}$
$[25]\ [21]\ [17]$ $[11]\ [7]\ [15]$	$[27]\ [23]\ [5]$ $[15]\ [19]^2\ [9]^2$ $[13]^2$	$[26]\ [16]^3\ [20]^2$ $[12]^2\ [6]^2\ [8]^2$	$-[26]\ [20]^3\ [16]^3$ $[12]^3\ [2]\ [6]^2$ $[8]^2$			$1\ 2^2\ 3^4\ 4^4$ $5^{10}\ 6^{10}\ 7^{17}\ 8^{22}$	$1\ 2\ 3^3\ 4^4$ $5^9\ 6^{11}\ 7^{22}\ 8^{38}$
$[28]\ [22]\ [14]$ $[4]\ [10]^2\ [12]^2$	$[26]\ [16]^3\ [12]^2$ $[10]^2\ [6]^2\ [8]^2$	$[29]\ [23]\ [21]^3$ $[13]^3\ [5]\ [15]$ $[3]^2\ 11]^4$	$[27]\ [23]^3\ [19]^3$ $[9]^3\ [13]^4\ [15]^2$ $[5]^2$	$-[22]^3\ [20]^5\ [12]^5$ $[16]^4\ [6]^2\ [4]^2$ $[2]^2$			$1\ 2^3\ 3^5\ 4^6$ $5^{17}\ 6^{17}\ 7^{38}\ 8^{70}$
$[25]\ [19]\ [9]$ $[15]\ [7]^2\ [13]^2$	$[29]\ [25]\ [19]^3$ $[13]^3\ [3]\ [9]^2$ $[15]\ [7]^2$	$[26]\ [24]^3\ [18]^3$ $[8]^3\ [14]^4\ [10]^2$	$[28]\ [22]^3\ [20]^4$ $[18]^4\ [16]^4\ [6]^2$ $[4]^2$	$[27]\ [25]^3\ [17]^5$ $[15]^3\ [11]^4\ [9]^4$ $[7]^4$	$-[24]^3\ [20]^5\ [14]^5$ $[28]^2\ [26]^2\ [12]^6$ $[8]^4$		
$[27]\ [23]\ [5]$ $[15]\ [9]^2\ [11]^2$ $[13]^2$	$[27]\ [21]^3\ [17]^3$ $[11]^3\ [7]^2\ [15]^2$ $[5]^2$	$[28]\ [22]^3\ [24]^2$ $[26]^2\ [12]^4\ [10]^4$ $[14]^4$	$[28]\ [24]^3\ [18]^5$ $[14]^5\ [22]^4\ [10]^4$ $[4]^2$	$[29]\ [25]^3\ [21]^5$ $[27]^2\ [23]^4\ [19]^6$ $[15]^3\ [13]^6$	$[28]\ [26]^3\ [22]^5$ $[16]^7\ [24]^4\ [20]^6$ $[12]^6$	$-[26]^3\ [24]^5\ [20]^7$ $[28]^2\ [22]^6\ [16]^8$ $[12]^8$	
$[26]\ [16]^3\ [10]^2$ $[8]^2\ [6]^2\ [12]^2$	$[28]\ [22]^3\ [14]^4$ $[12]^4\ [10]^4\ [6]^2$ $[4]^2$	$[27]\ [25]^3\ [17]^5$ $[15]^3\ [11]^4\ [9]^4$ $[7]^4$	$[29]\ [25]^3\ [21]^5$ $[27]^2\ [15]^3\ [13]^6$ $[11]^6\ [7]^4$	$[28]\ [26]^3\ [24]^5$ $[18]^7\ [20]^6\ [22]^6$ $[16]^8$	$[29]\ [27]^3\ [23]^5$ $[21]^7\ [25]^4\ [19]^8$ $[17]^8\ [15]^4$	$[29]\ [27]^3\ [25]^5$ $[23]^7\ [19]^9\ [21]^8$ $[17]^{10}\ [15]^5$	$-[28]^3\ [26]^5\ [24]^7$ $[22]^9\ [20]^{11}\ [18]^{12}$ $[16]^{12}$

TABLE: E_7 **C-G Series and Minimal S-matrices**

2 4	1 3	2 4 5	1 3 6	3 6	4 5 7	6
	2 4 5	1 3 6	2 4 5	2 4 5 7	3 6	5 7
		2 4 5 7	1 3 6	1 3 6	2 4 5 7	3 6
[10] [2]			$2\ 4^2\ 5\ 7$	$2\ 4\ 5^2\ 7$	$1\ 3\ 6^2$	$4\ 5\ 7^3$
[5] [7]	−[12] [10] [2]			$2\ 4^2\ 5^2$ 7^2	$1\ 3\ 6^2$	$2\ 4\ 5^2$ 7^3
[14] [10] [6]	[15] [13] [11] [9]	$[10]^2\ [6]^2$ [4] [2]			$2\ 4^2\ 5^2$ 7^3	$1\ 3\ 6^3$
[9] [11] [1] [3]	$[8]\ [6]^2\ [4]$	[15] [9] $[7]^2\ [5]^2$	$-[16]^2\ [12]$ $[8]^3\ [4]$			$2\ 4^3\ 5^3$ 7^7
$[2]\ [4]\ [10]^2$	$[13]\ [11]^2\ [9]$ [3] [1]	$[16]\ [14]^2$ $[12]^2\ [10]^3$	$[13]^3\ [9]$ $[7]^3\ [3]$	$-[14]^2\ [12]^3$ $[10]^4\ [2]^2$		
[16] [14] $[8]^2\ [6]$	$[15]\ [13]^2$ $[11]^2\ [9]$	$[8]^3\ [6]^3$ $[4]^2\ [2]$	$[17]\ [15]^2\ [9]^2$ $[7]^3\ [5]^2$	$[16]\ [14]^3$ $[12]^4\ [10]^4$	$[8]^5\ [6]^4$ $[4]^3\ [2]^2$	
$[15]\ [13]^2$ $[11]^2\ [9]$	$[14]^2\ [10]^3$ $[6]^2\ [2]$	$[11]^4\ [8]^2\ [5]^3$ $[3]^2\ [1]$	$[14]^3\ [10]^4$ $[6]^4\ [2]$	$[15]^3\ [11]^5\ [9]^3$ $[5]^4\ [1]$	$[9]^3\ [7]^6\ [5]^5$ $[3]^3\ [1]$	$-[16]^3\ [12]^7$ $[8]^8\ [4]^5$

TABLE: E_6 **C-G Series and Minimal S-matrices**

$\bar{l}\ \bar{h}$	L	$l\ h$	$\bar{l}\ \bar{h}$	$L\ H$	h
	$l\ h$	$\bar{l}\ \bar{h}$	$L\ H$	$l\ h$	$\bar{h}$
		$L\ H$	$l\ h$	$\bar{l}\ \bar{h}$	$L\ H$
(8) (6) (2)			$\bar{l}\ \bar{h}$	$L\ H$	$l\ h$
−(10) (6) (4)	(8) (6) (2)			$l\ h$	$\bar{l}\ \bar{h}$
[9] [7]	[9] [7]	−[10] [8] [6]			$L\ H^2$
(11) (5) [9] [7]	(7) (1) [9] [7]	[10] $[8]^2$ [6]	(8) (6) (2) [10] $[8]^2$ [6]		
(7) (1) [9] [7]	(11) (5) [9] [7]	[10] $[8]^2$ [6]	−(10) (6) (4) [10] $[8]^2$ [6]	(8) (6) (2) [10] $[8]^2$ [6]	
[10] $[8]^2$ [6]	[10] $[8]^2$ [6]	[11] $[9]^2$ $[7]^3$	[11] $[9]^3$ $[7]^4$	[11] $[9]^3$ $[7]^4$	−$[10]^3$ $[8]^5$ $[6]^3$

TABLE: E_6 **COUPLINGS AND S-MATRICES**

$\bar{l}^{-i}\ \ h^{-i}$ 8 2	L 6	$l\ \ h$ 9 5	$\bar{l}^{-i}\ \ \bar{h}^{-i}$ 11 7	$L\ \ H$ 9 3	h 10
	$l^{i}\ \ h^{i}$ 8 2	$\bar{l}\ \ \bar{h}$ 9 5	$L\ \ H$ 9 3	$l^{i}\ \ h^{i}$ 11 7	$\bar{h}$ 10
		$L\ \ H$ 8 2	l 10	$\bar{l}$ 10	$L\ \ H^{-}$ 11 7
{1} {7}			$\bar{l}^{-i}\ \ \bar{h}^{i}$ 10 8	H^{-} 6	$l\ \ h^{-}$ 11 9
{5} {11}	{1} {7}			$l^{-i}\ \ h^{i}$ 10 8	$\bar{l}\ \ \bar{h}^{-}$ 11 9
4	4	1 5			$L^{-}\ \ H$ 10 8
{4} {6} {10}	{2} {6} {8}	3 5	{1} {3} {5} $\{7\}^2$ {9}		
{2} {6} {8}	{4} {6} {10}	3 5	{3} $\{5\}^2$ {7} {9} {11}	{1} {3} {5} $\{7\}^2$ {9}	
3 5	3 5	2 4 6	2 4^2 6	2 4^2 6	1 3^2 5^3

TABLE: E_7 COUPLINGS AND S-MATRICES

$2^- \; 4^-$ $10 \; 2$	$1^- \; 3$ $13 \; 7$	$2 \; 4 \; 5$ $14 \; 10 \; 6$	$1^- \; 3 \; 6$ $17 \; 11 \; 3$	$3 \; 6$ $14 \; 8$	$4 \; 5 \; 7$ $16 \; 12 \; 4$	6 15
	$2^- \; 4^- \; 5^-$ $12 \; 8 \; 2$	$1 \; 3^- \; 6^-$ $15 \; 11 \; 5$	$2^- \; 5^-$ $14 \; 8$	$2^- \; 4^- \; 7^-$ $17 \; 13 \; 3$	3^- 15	$5^- \; 7$ $16 \; 10$
		$2^- \; 7^-$ $14 \; 2$	1 15	$1 \; 6$ $16 \; 10$	$2^- \; 5 \; 7$ $16 \; 12 \; 8$	$3^- \; 6$ $17 \; 13$
$1 \; \{9\}$			$4^- \; 5 \; 7^-$ $12 \; 10 \; 4$	$2^- \; 4 \; 7$ $15 \; 13 \; 7$	$1 \; 6$ $17 \; 11$	$4^- \; 5$ $16 \; 14$
6	$1 \; 7$			5 12	$1 \; 3$ $16 \; 14$	$2^- \; 4 \; 7^-$ $17 \; 15 \; 11$
$5 \; \{9\}$	$4 \; 8$	$1 \; 5 \; 7 \; \{9\}$			$4 \; 7^-$ $14 \; 10$	$1 \; 3 \; 6^-$ $17 \; 15 \; 13$
$2 \; 8$	$5 \; 7$	$4 \; 6 \; 8$	$1 \; 3 \; 7 \; 9$			$2 \; 5^- \; 7$ $16 \; 14 \; 12$
$5 \; 7$	$2 \; 6 \; 8$	$3 \; 5 \; 7 \; 9$	$4 \; 6^2 \; 8$	$1 \; 3 \; 5 \; 7^2$ 9		
$3 \; 7 \; \{9\}$	$4 \; 6 \; 8$	$3 \; 5 \; 7^2 \; \{9\}$	$2 \; 4 \; 6 \; 8^2$	$3 \; 5^2 \; 7^2 \; 9$	$1 \; 3 \; 5^2 \; 7^2$ $\{9\}^3$	
$4 \; 6 \; 8$	$3 \; 5 \; 7 \; 9$	$2 \; 4 \; 6^2 \; 8^2$	$3 \; 5^2 \; 7^2 \; 9$	$2 \; 4^2 \; 6^2 \; 8^3$	$2 \; 4^2 \; 6^3 \; 8^3$	$1 \; 3^2 \; 5^3 \; 7^4$ 9^2

TABLE: E_8 COUPLINGS AND S-MATRICES

1^- 2 3^- 20 12 2	1 2^- 3 4^- 24 18 14 8	1^- 2 4 5 29 21 13 3	2^- 3 4^- 5^- 6 25 21 17 11 7	3 4^- 6 7^- 28 22 14 4	4 5 7 25 19 9	5^- 6 8 27 23 5	7 8 26 16
	1^- 2 4 5 6^- 24 20 14 8 2	1 3 6 25 19 9	1^- 2 7 27 23 5	2 6 26 16	2^- 3 5 7 8 29 25 19 13 3	4 6 7^- 8 27 21 17 11	6 7 28 22
		2 3 5 6 7 22 20 14 12 4	1 5 26 16	1 3 4 7 8 29 23 21 13 5	2 3 6 8 26 24 18 8	3 5 28 22	5 6 8^- 27 25 17
1 11			1^- 4^- 6 7^- 8^- 26 20 16 12 2	1^- 3 5^- 8 27 23 19 9	1 4 28 22	2 4^- 7 8^- 28 24 18 14	4^- 5 7^- 29 25 21
7 13	1 7 11 13			4^- 5 8^- 22 20 12	1 2 7^- 27 25 17	1^- 3 6^- 29 25 21	3 4 5^- 8^- 28 26 24 18
2 10 12	6 8 12 14	1 3 9 11^2 13			3 6^- 8^- 24 20 14	1 2 5^- 8^- 28 26 22 16	2 3 6^- 7^- 29 27 23 21
6 10 14	4 8 10 12 14	5 7 9 11 13 15	1 5 7 9 11^2 13 15			2^- 4 7^- 26 24 20	1 2 4^- 6^- 8 29 27 25 23 19
3 9 11 13	5 7 9 11 13 15	2 4 8 10^2 12^2 14	4 6 8^2 10 12^2 14^2	1 3 5 7 9^2 11^3 13^2 15			1 3^- 5^- 7 8 28 26 24 22 20
6 8 12 14	2 6 8 10 12^2 14	5 7^2 9 11 13^2 15	3 5 7 9^2 11^2 13^2 15	4 6^2 8^2 10^2 12^2 14^3	1 3 5 7^2 9^2 11^3 13^3 15		
4 8 10 12 14	4 6 8 10^2 12 14^2	3 5 7 9^2 11^2 13^2 15	3 5 7^2 9^2 11^2 13^3 15	2 4 6^2 8^2 10^3 12^3 14^3	3 5^2 7^2 9^3 11^3 13^3 15^2	1 3 5^2 7^3 9^3 11^4 13^4 15^2	
5 7 9 11 13 15	3 7 5 9^2 11^2 13^2 15	4 6^2 8^2 10^2 12^2 14^3	2 4 6^2 8^2 10^3 12^3 14^3	3 5^2 7^3 9^3 11^3 13^4 15^2	2 4^2 6^2 8^3 10^4 12^4 14^4	2 4^2 6^3 8^4 10^4 12^5 14^5	1 3^2 5^3 7^4 9^5 11^6 13^6 15^3