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SOME NATURAL OPERATIONS BETWEEN CONNECTIONS ON FIBRED MANIFOLDS*

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Abstract. All first-order natural operators transforming 2-connections on $Y \to X$ and linear connections on X into connections on $J^1Y \to Y$ are determined. Some integrability properties of the connections are studied.
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0. INTRODUCTION

In general, the paper represents a continuation of our endeavour at a global description of the geometry of differential equations represented by connections on general fibred manifolds [8], [15]. In a strict sense, it stands for a further clarification of relations between the studied connections on various jet prolongations of the underlying fibred manifold [5], and consequently it represents generalizations of some considerations from the time-dependent mechanics [1], [2], [3], [13], [14]. While [3], [14] are concerned with connections of higher-order over one-dimensional bases, the presented results describe the situation for second-order connections over bases with an arbitrary dimension and the results can be then compared with related ones e.g. in [12] or [16]. Moreover, the adopted approach and methods are immediately applicable for natural higher-order generalizations.

In this section, we fix the notation of essential underlying structures and related notions; for detailed description of this standard material we refer e.g. to [1], [6], [7], [9], [10], [12] and particularly to our previous papers.

Thus $\pi \colon Y \to X$ is a fibred manifold with fibred coordinates (x^i, y^σ), $i = 1, \dots, n = \dim X$, $\sigma = 1, \dots, m = \dim Y - \dim X$. The first jet prolongation of π is denoted by $J^1\pi$ with the additional induced coordinates y^σ_i. Then $\pi_1 \colon J^1\pi \to X$ and $\pi_{1,0} \colon J^1\pi \to Y$ are induced projections, where the latter one is an affine bundle with the associated vector bundle $V_\pi Y \otimes \pi^*(T^*X) \to Y$. The sections of this vector bundle are π-vertical vector valued 1-forms, called *soldering forms* on π.

A *connection* on π is a section Γ of $\pi_{1,0}$. Local equations of Γ are $y^\sigma_i \circ \Gamma = \Gamma^\sigma_i(x^i, y^\lambda)$, where Γ^σ_i are the components of Γ. The horizontal form of Γ is a vector valued 1-form $h_\Gamma \colon Y \to TY \otimes \pi^*(T^*X)$. Locally, $h_\Gamma = D_{\Gamma i} \otimes dx^i$, where $D_{\Gamma i} = \partial/\partial x^i + \Gamma^\sigma_i \partial/\partial y^\sigma$ is the i-th (absolute) derivative with respect to Γ. The complementary projection to h_Γ is the vertical form $v_\Gamma = I - h_\Gamma$. The decomposition related with Γ is $TY = V_\pi Y \oplus H_\Gamma$, where the n-dimensional π–horizontal

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distribution $H_\Gamma = \mathrm{Im}\, h_\Gamma = \mathrm{span}\{D_{\Gamma i}, i = 1,\dots,n\}$. By $\Gamma\xi \in TY$ we denote the horizontal lift of a vector $\xi \in \pi^*(TX)$.

Notice that for any connections Γ, Γ_1, Γ_2 on π and a soldering form φ on π, $h_{\Gamma_1} - h_{\Gamma_2}$ is a soldering form and $h_\Gamma + \varphi$ defines again a connection on π.

We denote by $J^1\pi_1$ the second nonholonomic prolongation of π and by $\widehat{J}^2\pi \subset J^1\pi_1$ the second semiholonomic prolongation of π. If $(x^i, y^\sigma, y^\sigma_i, y^\sigma_{;i}, y^\sigma_{i;j})$ are the induced coordinates on $J^1\pi_1$, then $\widehat{J}^2\pi$ is characterized by $y^\sigma_{;i} = y^\sigma_i$. Finally, the second (holonomic) prolongation $J^2\pi$ of π has local fibred coordinates $(x^i, y^\sigma, y^\sigma_i, y^\sigma_{ij})$. Recall that $(\pi_1)_{1,0}\colon J^1\pi_1 \to J^1\pi$ or $\widehat{\pi}_{2,1}\colon \widehat{J}^2\pi \to J^1\pi$ or $\pi_{2,1}\colon J^2\pi \to J^1\pi$ are affine bundles modelled on the associated vector bundles $V_{\pi_1}J^1\pi \otimes \pi_1^*(T^*X)$ or $V_{\pi_{1,0}}J^1\pi \otimes \pi_1^*(T^*X)$ or $\pi_{1,0}^*(V_\pi Y) \otimes \pi_1^*(S^2T^*X)$ over $J^1\pi$, respectively.

A connection on π_1 is a section $\Sigma\colon J^1\pi \to J^1\pi_1$ of $(\pi_1)_{1,0}$. Due to $\widehat{J}^2\pi \subset J^1\pi_1$, a *semiholonomic connection* on π_1 is a section $\widehat{\Gamma}^{(2)}\colon J^1\pi \to \widehat{J}^2\pi$ of $\widehat{\pi}_{2,1}$ with $H_{\widehat{\Gamma}^{(2)}}$ spanned by the vector fields $\partial/\partial x^i + y^\sigma_i\,\partial/\partial y^\sigma + \widehat{\Gamma}^\sigma_{ij}\,\partial/\partial y^\sigma_j$, where $\widehat{\Gamma}^\sigma_{ij}$ need not be symmetric. Notice that evidently $H_{\widehat{\Gamma}^{(2)}}$ is a subdistribution of the canonical *Cartan distribution* $C_{\pi_{1,0}}$ on $J^1\pi$.

The 2-connections on π (holonomic connections on π_1) are intrinsically related to the theory of second-order differential equations. Such a *2-connection* is a section $\Gamma^{(2)}\colon J^1\pi \to J^2\pi$ of $\pi_{2,1}$, locally expressed by $y^\sigma_{ij} \circ \Gamma^{(2)} = \Gamma^\sigma_{ij}$, where $\Gamma^\sigma_{ij} = \Gamma^\sigma_{ji}$ are the components of $\Gamma^{(2)}$. The horizontal form of $\Gamma^{(2)}$ is $h_{\Gamma^{(2)}}\colon J^1\pi \to TJ^1\pi \otimes \pi_1^*(T^*X)$, locally expressed by $h_{\Gamma^{(2)}} = D_{\Gamma^{(2)}i} \otimes dx^i$, where $D_{\Gamma^{(2)}i} = \partial/\partial x^i + y^\sigma_i\,\partial/\partial y^\sigma + \Gamma^\sigma_{ij}\,\partial/\partial y^\sigma_j$ is the i-th absolute derivative with respect to $\Gamma^{(2)}$. The canonical decomposition generated by $h_{\Gamma^{(2)}}$ is $TJ^1\pi = V_{\pi_1}J^1\pi \oplus H_{\Gamma^{(2)}}$, where the n-dimensional π_1–horizontal distribution $H_{\Gamma^{(2)}} = \mathrm{Im}\, h_{\Gamma^{(2)}}$ is locally generated by the vector fields $D_{\Gamma^{(2)}i}$ for $i = 1,\dots,n$.

Additionally, we consider first jet prolongations $J^1\pi_{1,0}$ or $J^1\pi_{2,1}$, i.e. the manifolds of 1-jets of local connections on π or of local 2-connections on π, respectively. The additional induced coordinates on $J^1\pi_{1,0}$ or on $J^1\pi_{2,1}$ are denoted by $z^\sigma_{ij}, z^\sigma_{i\lambda}$ or $z^\sigma_{ijk}, z^\sigma_{ij\lambda}, z^{\sigma k}_{ij\lambda}$, respectively. The vector bundle associated to $(\pi_{1,0})_{1,0}\colon J^1\pi_{1,0} \to J^1\pi$ is now evidently $V_{\pi_{1,0}}J^1\pi \otimes \pi_{1,0}^*(T^*Y) \to J^1\pi$.

Accordingly, a connection on $\pi_{1,0}$ is a section $\Xi\colon J^1\pi \to J^1\pi_{1,0}$ of $(\pi_{1,0})_{1,0}$ with the horizontal form $h_\Xi\colon J^1\pi \to TJ^1\pi \otimes \pi_{1,0}^*(T^*Y)$ locally expressed by $h_\Xi = D_{\Xi j} \otimes dx^j + D_{\Xi\lambda} \otimes dy^\lambda$, where $D_{\Xi j} = \partial/\partial x^j + \Xi^\sigma_{ij}\,\partial/\partial y^\sigma_i$, $D_{\Xi\lambda} = \partial/\partial y^\lambda + \Xi^\sigma_{i\lambda}\,\partial/\partial y^\sigma_i$ for $j = 1,\dots,n$ and $\lambda = 1,\dots,m$. The decomposition generated by h_Ξ is $TJ^1\pi = V_{\pi_{1,0}}J^1\pi \oplus H_\Xi$, where the $(n+m)$-dimensional $\pi_{1,0}$–horizontal distribution $H_\Xi = \mathrm{Im}\, h_\Xi$ is locally generated by the vector fields $D_{\Xi j}$ and $D_{\Xi\lambda}$.

1. Characteristic connections

In this section we summarize the notions and results of [5], [8] and [15] necessary for further considerations and applications.

Theorem 1 [5]. *All natural transformations of $J^1\pi_{1,0}$ into $J^1\pi_1$ over the identity* *of $J^1\pi$ form a 1-parameter family $\{f_a\}$, where*

$$
\begin{aligned}
&y^\sigma_{;i} \circ f_a = y^\sigma_i \\
&y^\sigma_{i;j} \circ f_a = z^\sigma_{ij} + z^\sigma_{i\lambda} y^\lambda_j + a(z^\sigma_{ij} - z^\sigma_{ji} + z^\sigma_{i\lambda} y^\lambda_j - z^\sigma_{j\lambda} y^\lambda_i)
\end{aligned}
\tag{1.1}
$$

for an arbitrary $a \in \mathbb{R}$.

The term $z^\sigma_{ij} + z^\sigma_{i\lambda} y^\lambda_j$ in (1.1) represents a coordinate expression of the canonical mapping $f_0 \colon J^1\pi_{1,0} \to J^1\pi_1$. This mapping has the form $j^1_y\Gamma \mapsto j^1_x(\Gamma \circ \gamma)$, where $\Gamma \colon V \subset Y \to J^1\pi$ is a local connection on π, $\Gamma(y) = j^1_x\gamma$, $\gamma \colon U \subset X \to Y$. This corresponds to the fact that $f_0(j^1_y\Gamma) = J^1(\Gamma, \mathrm{id}_X) \circ \Gamma(y)$ for any $y \in V$, where by $J^1(\Gamma, \mathrm{id}_X)$ we denote the prolongation of Γ considered as a fibred morphism over X.

The natural projections

$$s \colon \widehat{J}^2\pi \to J^2\pi \quad \text{and} \quad r \colon \widehat{J}^2\pi \to \pi^*_{1,0}(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))$$

corresponding to the canonical bundle isomorphism

$$\widehat{J}^2\pi \cong J^2\pi \times_{J^1\pi} \left[\pi^*_{1,0}(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))\right]$$

express the symmetric and antisymmetric part of every fibred coordinate $y^\sigma_{i;j}$. Consequently one can define the mappings

$$
\begin{aligned}
S &= s \circ f_0 \colon J^1\pi_{1,0} \to J^2\pi\,, \\
R &= r \circ f_0 \colon J^1\pi_{1,0} \to \pi^*_{1,0}(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))
\end{aligned}
$$

with the components

$$
\begin{aligned}
S^\sigma_{ij} &= \frac{1}{2}(z^\sigma_{ij} + z^\sigma_{ji} + z^\sigma_{i\lambda} y^\lambda_j + z^\sigma_{j\lambda} y^\lambda_i) \\
R^\sigma_{ij} &= \frac{1}{2}\left(z^\sigma_{ij} - z^\sigma_{ji} + z^\sigma_{i\lambda} y^\lambda_j - z^\sigma_{j\lambda} y^\lambda_i\right)\,,
\end{aligned}
$$

and the family of transformations (1.1) may be rewritten to

$$\{f_b\}_{b\in\mathbb{R}} \equiv \{y^\sigma_{;i} = y^\sigma_i, y^\sigma_{i;j} = S^\sigma_{ij} + b R^\sigma_{ij}\}_{b\in\mathbb{R}}\,.$$

Clearly, for each $a \in \mathbb{R}$ we get $s \circ f_a = S$ and $R_a := r \circ f_a = (1 + 2a)R$.

As a corollary we get :

Proposition 1 [5]. *All natural transformations transforming connections on $\pi_{1,0}$ into (in fact, semiholonomic) connections on π_1 are of the form $\Xi \mapsto f_a \circ \Xi$. The only natural transformation transforming connections on $\pi_{1,0}$ into 2-connections on π is of the form $\Xi \mapsto S \circ \Xi$.*

Thus there is a unique 2-connection $\Gamma^{(2)}$ on π naturally assigned to any connection Ξ on $\pi_{1,0}$, defined by $\Gamma^{(2)} = S \circ \Xi$. For Ξ^σ_{ij}, $\Xi^\sigma_{i\lambda}$ being the components of Ξ, those of $\Gamma^{(2)}$ are in fibred coordinates expressed by

$$\Gamma^\sigma_{ij} = \frac{1}{2}\left(\Xi^\sigma_{ij} + \Xi^\sigma_{ji} + \Xi^\sigma_{i\lambda}\, y^\lambda_j + \Xi^\sigma_{j\lambda}\, y^\lambda_i\right)\,.$$

If $R \circ \Xi = 0$ then $R_a \circ \Xi = 0$ for all $a \in \mathbb{R}$. Locally it reads

$$\Xi^\sigma_{ij} - \Xi^\sigma_{ji} + \Xi^\sigma_{i\lambda}\, y^\lambda_j - \Xi^\sigma_{j\lambda}\, y^\lambda_i = 0 \,. \tag{1.2}$$

Due to the properties of the distributions $H_{\Gamma^{(2)}}$ and H_Ξ we get that if Ξ is a connection on $\pi_{1,0}$ and $\Gamma^{(2)} = S \circ \Xi$, then $H_{\Gamma^{(2)}} \subset H_\Xi$ if and only if $R \circ \Xi = 0$ and thus $H_{\Gamma^{(2)}} \subset H_\Xi$ if and only if $\Gamma^{(2)} = f_0 \circ \Xi$. Accordingly, a connection Ξ on $\pi_{1,0}$ is called *characterizable* if $R \circ \Xi = 0$ and the corresponding 2-connection $\Gamma^{(2)} = S \circ \Xi$ is called the *characteristic connection* of Ξ. Since the local conditions for Ξ to be characterizable are (1.2), the components of its characteristic connection are

$$\Gamma^\sigma_{ij} = \Xi^\sigma_{ij} + \Xi^\sigma_{i\lambda}\, y^\lambda_j \,.$$

Proposition 2 [15]. *Let Ξ be a characterizable connection on $\pi_{1,0}$ and $\Gamma^{(2)}$ its characteristic connection on π. Then $F_\Xi = 2h_\Xi - h_{\Gamma^{(2)}} - I$ is an f(3,-1) structure on $J^1\pi$ of rank $m(n+1)$.*

It can be shown that $F^2_\Xi - I = -h_{\Gamma^{(2)}}$, $F^2_\Xi + F_\Xi = 2(h_\Xi - h_{\Gamma^{(2)}})$ and $F^2_\Xi - F_\Xi = 2\, v_\Xi$. Consequently, there is a canonically determined direct sum decomposition

$$TJ^1\pi = V_{\pi_{1,0}} J^1\pi \oplus H_{\Gamma^{(2)}} \oplus H_{F_\Xi} \,, \tag{1.3}$$

where $H_{\Gamma^{(2)}} \oplus H_{F_\Xi} = H_\Xi$. The m-dimensional distribution $H_{F_\Xi} = \operatorname{Im}(h_\Xi - h_{\Gamma^{(2)}})$ is called *strong horizontal*, which means the decomposition

$$V_{\pi_1} J^1\pi = V_{\pi_{1,0}} J^1\pi \oplus H_{F_\Xi} \,.$$

A *reduced connection of type (1,0)* on π is a section $\Gamma_{(1,0)} \colon \pi^*_{1,0}(V_\pi Y) \to V_{\pi_1} J^1\pi$ linear in $\dot{y}^\sigma$, given by $\dot{y}^\sigma_i \circ \Gamma_{(1,0)} = \Gamma^\sigma_{i\lambda}(x^j, y^\sigma, y^\sigma_i)\, \dot{y}^\lambda$. In other words, $\Gamma_{(1,0)}$ represents a lift of vector fields expressed by

$$\left(j^1_x\gamma, \zeta^\sigma \frac{\partial}{\partial y^\sigma}\Big|_{\gamma(x)} \right) \stackrel{\Gamma_{(1,0)}}{\longmapsto} \zeta^\sigma \frac{\partial}{\partial y^\sigma}\Big|_{j^1_x\gamma} + \Gamma^\sigma_{i\lambda}\, \zeta^\lambda \frac{\partial}{\partial y^\sigma_i}\Big|_{j^1_x\gamma} \,,$$

and thus it generates a decomposition

$$V_{\pi_1} J^1\pi = V_{\pi_{1,0}} J^1\pi \oplus H_{\Gamma_{(1,0)}}$$

with $H_{\Gamma_{(1,0)}} = \operatorname{Im}\Gamma_{(1,0)}$ generated by the vector fields $\partial/\partial y^\lambda + \Gamma^\sigma_{i\lambda}\partial/\partial y^\sigma_i$ for $\lambda = 1, \ldots, m$.

Prop. 2 can thus be reformulated.

Proposition 3. *Any characterizable connection Ξ on $\pi_{1,0}$ splits into the direct sum of a 2-connection $\Gamma^{(2)}$ on π and a reduced connection $\Gamma_{(1,0)}$ of type $(1,0)$ on π. The decomposition is given by $H_\Xi = H_{\Gamma^{(2)}} \oplus H_{\Gamma_{(1,0)}}$, where $\Gamma^{(2)}$ is the characteristic connection of Ξ and $\Gamma_{(1,0)} = h_\Xi|_{\pi^*_{1,0}(V_\pi Y)}$.*

We refer to [15] for the importance of reduced connections (or in other words of the corresponding strong horizontal subbundles) in the theory of symmetries of the corresponding characteristic connection $\Gamma^{(2)}$.

Let $\Gamma^{(2)}$ be an integrable 2-connection on π. A (generally local) connection Ξ on $\pi_{1,0}$ is called an *integral* of $\Gamma^{(2)}$ if Ξ is integrable and $\Gamma^{(2)}$ is its characteristic connection.

We have shown in [8] (in terms of the so-called *fields of paths*) that the meaning of searching for integrals of a given $\Gamma^{(2)}$ rests upon the possibility of transferring the problem of solving second-order equations related to $\Gamma^{(2)}$ to that of solving a family of first-order equations for integral sections of Ξ. In the same paper, the possibility for a local integral of $\Gamma^{(2)}$ to be constructed by means of a set of independent first integrals of $H_{\Gamma^{(2)}}$, was presented.

The following questions naturally appear in terms of the above considerations. First, whether there exist transformations 'converse' in some sense to those of Theorem 1; in particular: is there a possibility to assign a (global) characterizable connection Ξ on $\pi_{1,0}$ to a given 2-connection $\Gamma^{(2)}$ on π ? And secondly: what conditions (if any) must be satisfied for Ξ to be a (global) integral of $\Gamma^{(2)}$?

It is worth mentioning here the existing results closely related to these questions. For $\dim X = 1$, the fibred coordinates on $J^2\pi$ or $J^1\pi_{1,0}$ are denoted by $(t, q^\sigma, q^\sigma_{(1)}, q^\sigma_{(2)})$ or $(t, q^\sigma, q^\sigma_{(1)}, z^\sigma, z^\sigma_\lambda)$, respectively. Accordingly, the components of $\Gamma^{(2)}$ or Ξ are $\Gamma^\sigma_{(2)}$ or Ξ^σ, Ξ^σ_λ, respectively.

In [14], the following assertion was proved (for the sake of brevity, we present the 'first-order case' only).

Proposition 4. *Let $\pi\colon Y \to X$ be an arbitrary fibred manifold with $\dim X = 1$. Let $\Gamma^{(2)}\colon J^1\pi \to J^2\pi$ be a 2-connection on π, let $\Omega = \omega\, dt$ be a volume form on X. Then there is a connection $\Xi\colon J^1\pi \to J^1\pi_{1,0}$ on $\pi_{1,0}$ whose characterizable connection is $\Gamma^{(2)}$, called a natural dynamical connection of type Ω on $J^1\pi$. The components of Ξ are*

$$\Xi^\sigma_\lambda = \frac{1}{2}\left(\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} - \frac{d\omega}{dt}\frac{1}{\omega}\,\delta^\sigma_\lambda\right)$$

and

$$\Xi^\sigma = \Gamma^\sigma_{(2)} + \frac{1}{2}\left(\frac{d\omega}{dt}\frac{1}{\omega}\,q^\sigma_{(1)} - \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}\,q^\lambda_{(1)}\right) = \Gamma^\sigma_{(2)} - \frac{1}{2}\,\Xi^\sigma_\lambda q^\lambda_{(1)}.$$

The proposition was proved locally by means of constructing the corresponding f(3,-1) structure rather then Ξ. In the proof, natural affinors (vector-valued one forms) were used (for the case of $\mathbb{R} \times T^1M$ see [4] and for the general situation see [12]). What is interesting in this respect is the fact that these affinors are just the 'differences' of the connections on $\pi_{1,0}$, i.e. the sections of the corresponding associated vector bundle (soldering forms).

Supposing $X = \mathbb{R}$ and (t) to be a global canonical coordinate on $\mathbb{R}$ and using a canonical volume form $\Omega = dt$, one obtains the results for $J^1\pi = \mathbb{R} \times TM$ ([1], [2], [3] etc.):

$$\Xi^\sigma_\lambda = \frac{1}{2}\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} \quad , \quad \Xi^\sigma = \Gamma^\sigma_{(2)} - \frac{1}{2}\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}\,q^\lambda_{(1)} \,.$$

The motivation for the results we will present is the fact that the functions

$$\Lambda(t) = -\frac{d\omega}{dt}\frac{1}{\omega}$$

are transformed in the same way as the components of a linear connection Λ on $\tau_X\colon TX \to X$, or equivalently $\Lambda^*(t) = -\Lambda(t)$ like the components of the dual connection Λ^* on $\tau_X^*\colon T^*X \to X$. In keeping with previous ideas and formalism, a *linear connection* on $\tau_X\colon TX \to X$ (or briefly on X) is a section $\Lambda\colon TX \to J^1\tau_X$, locally given by

$$(x^i, \dot{x}^i, \dot{x}^i_j) \circ \Lambda = (x^i, \dot{x}^i, \Lambda^i_{jk}(x^\ell)\dot{x}^k)$$

and the *dual connection* Λ^* of Λ is again a linear connection, now on the dual bundle $\tau_X^*\colon T^*X \to X$, the components of which are $\Lambda^{*i}_{jk} = -\Lambda^i_{jk}$ (see [10]).

2. New results

According to the general theory of natural operations in differential geometry [6], natural operators generalize the concept of a geometrical construction. In this context we can pose the question of determining all natural operators (i.e. all possible geometric constructions of a prescribed type). In particular, we look for all geometrical constructions of a connection on $\pi_{1,0}\colon J^1\pi \to Y$ by means of a 2–connection on $\pi\colon Y \to X$ and a linear connection on X. We will use the concept of a natural operator from [6].

The following assertion represents the main result of this paper.

Theorem 2. *All natural operators transforming a 2-connection $\Gamma^{(2)}$ on π and a linear connection Λ on X into the connection Ξ on $\pi_{1,0}$ being of the first order in $\Gamma^{(2)}$ and of the zero order in Λ are of the form*

$$\Xi^\Lambda_a = g^\Lambda_a \circ j^1\Gamma^{(2)} \tag{2.1}$$

where $g^\Lambda_a\colon J^1\pi_{2,1} \to J^1\pi_{1,0}$ is a fibred morphism over $J^1\pi$ locally expressed by

$$\begin{aligned} z^\sigma_{i\lambda} &= \frac{1}{2}(z^{\sigma k}_{ik\lambda} + \delta^\sigma_\lambda \Lambda^k_{ki}) + a\delta^\sigma_\lambda(\Lambda^k_{ik} - \Lambda^k_{ki}) \\ z^\sigma_{ij} &= y^\sigma_{ij} - z^\sigma_{i\lambda} y^\lambda_j \end{aligned} \tag{2.2}$$

for any $a \in \mathbb{R}$.

Proof. Denote by $G^3_{n,m}$ the group of all 3–jets at the origin of the diffeomorphisms $\overline{x}^i = \overline{x}^i(x)$, $\overline{y}^\sigma = \overline{y}^\sigma(x,y)$ of $\mathbb{R}^{n+m}$ preserving the origin and the canonical fibration $\mathbb{R}^{n+m} \to \mathbb{R}^n$. Then the local coordinates on $G^3_{n,m}$ which correspond to the partial derivatives of $\overline{x}^i$ and $\overline{y}^\sigma$ at the origin are

$$(a^i_j, a^i_{jk}, a^i_{jk\ell}, a^\sigma_i, a^\sigma_{ij}, a^\sigma_{ijk}, a^\sigma_\lambda, a^\sigma_{\lambda i}, a^\sigma_{\lambda ij}, a^\sigma_{\lambda\rho}, a^\sigma_{\lambda\rho i}, a^\sigma_{\lambda\rho\tau}). \tag{2.3}$$

We shall denote by tilde the coordinates of the element inverse to (2.3) in $G^3_{n,m}$. By [6] and [7] there is a canonical bijection between natural operators and the equivariant maps of the corresponding standard fibres. Hence we have to determine all $G^3_{n,m}$– equivariant maps

$$\begin{aligned} z^\sigma_{i\lambda} &= f^\sigma_{i\lambda}(y^\sigma_i, y^\sigma_{ij}, z^\sigma_{ijk}, z^\sigma_{ij\lambda}, z^{\sigma k}_{ij\lambda}, \Lambda^i_{jk}) \\ z^\sigma_{ij} &= f^\sigma_{ij}(y^\sigma_i, y^\sigma_{ij}, z^\sigma_{ijk}, z^\sigma_{ij\lambda}, z^{\sigma k}_{ij\lambda}, \Lambda^i_{jk}) \end{aligned} \tag{2.4}$$

which express the coordinate form of the natural operators in question. Using standard evaluations we compute the following transformation laws

$$\begin{aligned}
\overline{\Lambda}^i_{jk} &= a^i_\ell \Lambda^\ell_{mn} \tilde{a}^m_j \tilde{a}^n_k + a^i_{\ell m} \tilde{a}^\ell_j \tilde{a}^m_k \\
\overline{y}^\sigma_i &= a^\sigma_\lambda y^\lambda_j \tilde{a}^j_i + a^\sigma_j \tilde{a}^j_i \\
\overline{y}^\sigma_{ij} &= a^\sigma_\lambda y^\lambda_{k\ell} \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_{\lambda\rho} y^\lambda_k y^\rho_\ell \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_{\lambda k} y^\lambda_\ell \tilde{a}^k_i \tilde{a}^\ell_j + \\
&\quad + a^\sigma_{\lambda\ell} y^\lambda_k \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_\lambda y^\lambda_k \tilde{a}^k_{ij} + a^\sigma_{k\ell} \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_k \tilde{a}^k_{ij} \\
\overline{z}^\sigma_{i\lambda} &= a^\sigma_\rho z^\rho_{j\tau} \tilde{a}^\tau_\lambda \tilde{a}^j_i + a^\sigma_{\rho\tau} y^\rho_j \tilde{a}^\tau_\lambda \tilde{a}^j_i + a^\sigma_{\rho j} \tilde{a}^\rho_\lambda \tilde{a}^j_i \\
\overline{z}^\sigma_{ij} &= a^\sigma_\lambda z^\lambda_{k\ell} \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_\lambda z^\lambda_{k\rho} \tilde{a}^\rho_j \tilde{a}^k_i + a^\sigma_{\lambda\ell} y^\lambda_k \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_{\lambda\rho} y^\lambda_k \tilde{a}^\rho_j \tilde{a}^k_i + \\
&\quad + a^\sigma_\lambda y^\lambda_k \tilde{a}^k_{ij} + a^\sigma_{k\ell} \tilde{a}^k_i \tilde{a}^\ell_j + a^\sigma_{k\lambda} \tilde{a}^\lambda_j \tilde{a}^k_i + a^\sigma_k \tilde{a}^k_{ij} \\
\overline{z}^{\sigma k}_{ij\lambda} &= \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_\tau \tilde{a}^f_i \tilde{a}^h_j z^{\tau\ell}_{fh\rho} + \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_{\rho\tau} y^\tau_h \tilde{a}^\ell_i \tilde{a}^h_j + \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_{\tau\rho} y^\tau_h \tilde{a}^h_i \tilde{a}^\ell_j + \\
&\quad + \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_{\rho h} \tilde{a}^h_i \tilde{a}^\ell_j + \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_{\rho h} \tilde{a}^\ell_i \tilde{a}^h_j + \tilde{a}^\rho_\lambda a^k_\ell a^\sigma_\rho \tilde{a}^\ell_{ij} \\
\overline{z}^\sigma_{ijk} &= a^\sigma_\lambda \tilde{a}^f_i \tilde{a}^\ell_j \tilde{a}^h_k z^\lambda_{f\ell h} + \dots \\
\overline{z}^\sigma_{ij\lambda} &= \tilde{a}^\tau_\lambda a^\sigma_\rho \tilde{a}^k_i \tilde{a}^\ell_j z^\rho_{k\ell\tau} + \dots ,
\end{aligned}$$

where for $\overline{z}^\sigma_{ijk}$ and $\overline{z}^\sigma_{ij\lambda}$ we shall need only the first terms. The homotheties $\tilde{a}^i_j = k\delta^i_j$, $a^\sigma_\lambda = \delta^\sigma_\lambda$ with other $a's$ vanishing will be called the base homotheties. Quite analogously, the fibre homotheties will be characterized by $\tilde{a}^i_j = \delta^i_j$, $a^\sigma_\lambda = k\delta^\sigma_\lambda$. Consider first the map $f^\sigma_{i\lambda}$ from (2.4). Using the equivariance with respect to the base homotheties we obtain a homogeneity condition

$$k f^\sigma_{i\lambda} = f^\sigma_{i\lambda}(k y^\sigma_i, k^2 y^\sigma_{ij}, k^3 z^\sigma_{ijk}, k^2 z^\sigma_{ij\lambda}, k z^{\sigma k}_{ij\lambda}, k\Lambda^i_{jk}) .$$

By the homogeneous function theorem [6] $f^\sigma_{i\lambda}$ is independent of y^σ_{ij}, z^σ_{ijk}, $z^\sigma_{ij\lambda}$ and linear in y^σ_i, $z^{\sigma k}_{ij\lambda}$, Λ^i_{jk}. Next, the fibre homotheties yield

$$f^\sigma_{i\lambda} = f^\sigma_{i\lambda}(k y^\sigma_i, z^{\sigma k}_{ij\lambda}, \Lambda^i_{jk})$$

so that $f^\sigma_{i\lambda}$ is linear in $z^{\sigma k}_{ij\lambda}$, Λ^i_{jk}. Denote further by $G \subset G^3_{n,m}$ the subgroup with arbitrary a^i_j, a^σ_λ and with other $a's$ vanishing. Then the equivariance with respect to G implies that $f^\sigma_{i\lambda}$ correspond to the $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$–invariant tensors. Taking into account the symmetry of $z^{\sigma k}_{ij\lambda}$ in i, j and applying the invariant tensor theorem [6] we find that

$$f^\sigma_{i\lambda} = a z^{\sigma k}_{ik\lambda} + b\delta^\sigma_\lambda \Lambda^k_{ik} + c\delta^\sigma_\lambda \Lambda^k_{ki} + d\delta^\sigma_\lambda z^{\rho k}_{ik\rho}$$

with real parameters a, b, c, d. Finally, the full equivariance with respect to the subgroup $L \subset G^3_{n,m}$ characterized by $a^i_j = \delta^i_j$, $a^\sigma_\lambda = \delta^\sigma_\lambda$ reads

$$\begin{aligned}
a z^{\sigma k}_{ik\lambda} &+ b\delta^\sigma_\lambda \Lambda^k_{ik} + c\delta^\sigma_\lambda \Lambda^k_{ki} + d\delta^\sigma_\lambda z^{\rho k}_{ik\rho} + a^\sigma_{\lambda\rho} y^\rho_i + a^\sigma_{\lambda i} \\
&= a(z^{\sigma k}_{ik\lambda} + a^\sigma_{\lambda\rho} y^\rho_i + a^\sigma_{\rho\lambda} y^\rho_i + a^\sigma_{\lambda i} + a^\sigma_{\lambda i} + \delta^\sigma_\lambda \tilde{a}^k_{ik}) \\
&+ d\delta^\sigma_\lambda (z^{\rho k}_{ik\rho} + a^\rho_{\rho\tau} y^\tau_i + a^\rho_{\rho\tau} y^\tau_i + a^\rho_{\rho i} + a^\rho_{\rho i} + \tilde{a}^k_{ik}) \\
&+ b\delta^\sigma_\lambda \Lambda^k_{ik} + b\delta^\sigma_\lambda a^k_{ik} + c\delta^\sigma_\lambda \Lambda^k_{ki} + c\delta^\sigma_\lambda a^k_{ki}
\end{aligned}$$

This implies $a = \frac{1}{2}$, $d = 0$, $b + c = a$ which corresponds to the first equation of (2.2). Applying the same procedure to f^σ_{ij} we obtain

$$f^\sigma_{ij} = a_1 y^\sigma_{ij} + a_2 y^\sigma_i z^{\rho k}_{jk\rho} + a_3 y^\rho_i z^{\sigma k}_{jk\rho} + a_4 y^\sigma_j z^{\rho k}_{ik\rho} + a_5 y^\rho_j z^{\sigma k}_{ik\rho} + a_6 y^\sigma_k z^{\rho k}_{ij\rho} + a_7 y^\rho_k z^{\sigma k}_{ij\rho}$$
$$+ a_8 y^\sigma_i \Lambda^k_{jk} + a_9 y^\sigma_i \Lambda^k_{kj} + a_{10} y^\sigma_j \Lambda^k_{ik} + a_{11} y^\sigma_j \Lambda^k_{ki} + a_{12} y^\sigma_k \Lambda^k_{ij} + a_{13} y^\sigma_k \Lambda^k_{ji} \, .$$

The equivariance with respect to L then leads to such relations among $a_1, \dots, a_{13}$, which correspond to the second equation of (2.2). □

The verification of the following assertion is an easy replica of the proof of Theorem 2.

Proposition 5. *There is no first order natural operator transforming 2-connections on π into connections on $\pi_{1,0}$.*

It should be mentioned that the presence of a linear connection Λ on X is not surprising since linear connections on the base manifold play an important role in many other geometrical constructions on jet spaces, see [6].

Remark 1. In Theorem 2 we have discussed natural operators of the first order in $\Gamma^{(2)}$ and of the zero order in Λ. Using homotheties one can easily prove that zero is the maximal finite order in Λ. In other words, the connection Ξ on $\pi_{1,0}$ cannot depend on the higher order derivatives $D^\alpha \Lambda^k_{ij}$, where the multiindex α satisfies $|\alpha| \geqslant 1$.

Let Λ be a linear connection on X with the torsion T. Contracting T one obtains a 1-form $\widehat{T} = T_i\, dx^i$ with $T_i = T^k_{ik} = \Lambda^k_{ik} - \Lambda^k_{ki}$. Moreover, the following assertion appears.

Proposition 6. *All natural operators transforming linear connections on X into 1-forms on X are of the form*

$$\Lambda \mapsto k\widehat{T} \, , \ k \in \mathbb{R} \, .$$

Proof. Denote by $Q = \Lambda^1 \mathbb{R}^{m*}$ the standard fibre of $\Lambda^1 T^*$ and by $F_0 = (QP^1\mathbb{R}^m)_0$ the standard fibre of the bundle QP^1 of linear connections.

Step 1. By [6] the zero order operators $QP^1 \rightsquigarrow \Lambda^1 T^*$ correspond to the G^2_m-equivariant maps $F_0 \to Q$ of the form $\omega_i = \omega_i(\Lambda^i_{jk})$, where ω_i are the induced coordinates on Q. Using the equivariance with respect to the homotheties we get $k\omega_i = \omega_i(k\Lambda^i_{jk})$, which implies that ω_i are linear in Λ^i_{jk}, i.e. $\omega_i = k_1 \Lambda^\ell_{\ell i} + k_2 \Lambda^\ell_{i\ell}$, $k_i \in \mathbb{R}$. The full equivariance then leads to the relation $k_1 = -k_2$. Hence $\omega_i = k(\Lambda^\ell_{\ell i} - \Lambda^\ell_{i\ell})$, which is the coordinate form of our assertion.

Step 2. Using homotheties one easily evaluates that the r-th order natural operators are reduced to the case 1 for any $r > 0$.

Step 3. By [6] every natural operator $QP^1 \rightsquigarrow \Lambda^1 T^*$ has finite order. □

According to [12], any linear connection Λ on X thus generates canonically a vector-valued 1-form

$$S_\Lambda = T_i \frac{\partial}{\partial y^\sigma_i} \otimes (dy^\sigma - y^\sigma_j\, dx^j) \, . \tag{2.5}$$

Clearly, S_Λ is a soldering form on $\pi_{1,0}$ or in other words a *deformation* of the connections on $\pi_{1,0}$, trivial if and only if Λ is torsion free. The above connection Ξ_a^Λ from (2.2) can be then written as

$$\Xi_a^\Lambda = \Xi_0^\Lambda + a\, S_\Lambda\ , \tag{2.6}$$

where the components of Ξ_0^Λ are by (2.2)

$$\begin{aligned}\Xi_{i\lambda}^\sigma &= \frac{1}{2}\left(\frac{\partial \Gamma_{ik}^\sigma}{\partial y_k^\lambda} + \delta_\lambda^\sigma \Lambda_{ki}^k\right)\\ \Xi_{ij}^\sigma &= \Gamma_{ij}^\sigma - \Xi_{i\lambda}^\sigma y_j^\lambda\end{aligned} \tag{2.7}$$

for Γ_{ij}^σ being the components of $\Gamma^{(2)}$. Recall in this context the result of [6]: all natural operators transforming linear connections on X into themselves form a 3–parameter family

$$\tilde{\Lambda} = \Lambda + k_1\, T + k_2\, I \otimes \widehat{T} + k_3\, \widehat{T} \otimes I \tag{2.8}$$

where $k_1, k_2, k_3 \in \mathbb{R}$, $\widehat{T}$ is a contracted torsion of Λ and I is the identity tensor of $TX \otimes T^*X$. Then the term $a\, S_\Lambda$ in (2.6) expresses the difference between Ξ_0^Λ and $\Xi_0^{\tilde{\Lambda}}$ corresponding to (2.8).

Remark 2. It is easy to see a geometrical interpretation of Ξ_0^Λ in the case of one-dimensional base X. In this situation, for any volume form $\Omega = \omega dt$ on X which is an integral section of the dual connection Λ^* (i.e. $\Lambda^* \circ \Omega = j^1\Omega$), the natural dynamical connection of type Ω (see Prop. 4) is just Ξ_0^Λ.

Proposition 7. *A connection Ξ_a^Λ from (2.1) is characterizable with the characteristic connection $\Gamma^{(2)}$ for any Λ and a.*

Proof. Immediately from the second part of (2.2) we obtain (1.2). □

The whole situation can be described by the following commutative diagram:

$$\begin{array}{ccccc} J^1\pi_{1,0} & = & J^1\pi_{1,0} & = & J^1\pi_{1,0} \\ {\scriptstyle f_b}\big\downarrow & & {\scriptstyle \Xi_a^\Lambda}\big\uparrow & & {\scriptstyle g_a^\Lambda}\big\uparrow \\ J^2\pi & \xleftarrow{\Gamma^{(2)}} & J^1\pi & \xrightarrow{j^1\Gamma^{(2)}} & J^1\pi_{2,1}\ . \end{array}$$

Notice that the class of characterizable connections Ξ on $\pi_{1,0}$ with the same characteristic $\Gamma^{(2)}$ is wide (any such Ξ will be called *associated* to $\Gamma^{(2)}$). In fact, it is easy to see that there is a family of natural linear morphisms of $V_{\pi_{1,0}} J^1\pi \otimes \pi_{1,0}^*(T^*Y)$ into $\pi_{1,0}^*(V_\pi Y) \otimes \pi_1^*(T^*X \otimes T^*X)$ over the identity of $J^1\pi$, induced by $\{f_a\}$ from Theorem 1. Using the alternative description by S and R, the corresponding associated transformations read

$$\overline{f}_b = \overline{S}_{ij}^\sigma + b\overline{R}_{ij}^\sigma\ ,$$

where

$$\overline{S}^{\sigma}_{ij} = \frac{1}{2}(\varphi^{\sigma}_{ij} + \varphi^{\sigma}_{ji} + \varphi^{\sigma}_{i\lambda} y^{\lambda}_{j} + \varphi^{\sigma}_{j\lambda} y^{\lambda}_{i})$$
$$\overline{R}^{\sigma}_{ij} = \frac{1}{2}\left(\varphi^{\sigma}_{ij} - \varphi^{\sigma}_{ji} + \varphi^{\sigma}_{i\lambda} y^{\lambda}_{j} - \varphi^{\sigma}_{j\lambda} y^{\lambda}_{i}\right) \tag{2.9}$$

with

$$\overline{S}^{\sigma}_{ij} \circ \varphi \in \pi^*_{1,0}(V_\pi Y) \otimes \pi^*_1(S^2 T^* X)$$
$$\overline{R}^{\sigma}_{ij} \circ \varphi \in \pi^*_{1,0}(V_\pi Y) \otimes \pi^*_1(\Lambda^2 T^* X)$$

for

$$\varphi = \frac{\partial}{\partial y^{\sigma}_{i}} \otimes (\varphi^{\sigma}_{ij}\, dx^j + \varphi^{\sigma}_{i\lambda}\, dy^{\lambda}) \colon J^1\pi \to V_{\pi_{1,0}} J^1\pi \otimes \pi^*_{1,0}(T^*Y)\ .$$

Then φ can be called *admissible deformation* on $\pi_{1,0}$ if and only if $\varphi \in \ker \overline{f}_b$, since just these deformations do not change the characteristic connection when added to the given associated Ξ; the local conditions for φ to be admissible are

$$\varphi^{\sigma}_{ij} + \varphi^{\sigma}_{i\lambda} y^{\lambda}_{j} = 0 \tag{2.10}$$

for any σ, i, j. Remark that for any 1-form $\lambda = \lambda_i\, dx^i$ on X, $S_\lambda = \lambda_i \partial/\partial y^{\sigma}_{i} \otimes (dy^{\sigma} - y^{\sigma}_{j}\, dx^j)$ is admissible.

Following the ideas mentioned in the previous section, we finally discuss the integrability of connections (and thus of corresponding equations) under consideration. Integrability conditions for a connection Γ on π, meaning equivalently the involutiveness of the corresponding horizontal distribution H_Γ, can be expressed among others by the vanishing of the Frölicher-Nijenhuis bracket $[h_\Gamma, h_\Gamma]$ or equivalently of the Lie bracket $[D_{\Gamma i}, D_{\Gamma j}]$ for $i, j = 1, \dots, n$. Consequently, the analogous integrability conditions for other connections in question will be applied. In particular, a connection Ξ on $\pi_{1,0}$ is integrable if and only if

$$[D_{\Xi i}, D_{\Xi j}] = 0 \tag{2.11}$$
$$[D_{\Xi \lambda}, D_{\Xi \sigma}] = 0 \tag{2.12}$$
$$[D_{\Xi i}, D_{\Xi \lambda}] = 0 \tag{2.13}$$

for any i, j, σ, λ.

Let Ξ be characterizable and $\Gamma^{(2)}$ its characteristic connection. Then for any $i = 1, \dots, n$

$$D_{\Gamma^{(2)} i} = D_{\Xi i} + y^{\lambda}_{i} D_{\Xi \lambda} \tag{2.14}$$

and as a consequence

$$\begin{aligned}[D_{\Gamma^{(2)} i}, D_{\Gamma^{(2)} j}] &= \\ &= y^{\lambda}_{i} y^{\sigma}_{j} [D_{\Xi\lambda}, D_{\Xi\sigma}] + y^{\lambda}_{j} [D_{\Xi i}, D_{\Xi\lambda}] - y^{\lambda}_{i} [D_{\Xi j}, D_{\Xi\lambda}] + [D_{\Xi i}, D_{\Xi j}]\end{aligned} \tag{2.15}$$

for any $i, j = 1, \dots, n$, which means that if Ξ is integrable so is $\Gamma^{(2)}$ (we refer to [8] for an alternative procedure).

Conversely, if Ξ is associated with an integrable $\Gamma^{(2)}$, then it is not possible to say much about the integrability of Ξ in general. Suppose then additionally the involutivity of the strong horizontal distribution H_{F_Ξ}, which means just (2.12) for all σ, λ; recall that the involutivity both of $H_{\Gamma^{(2)}}$ and H_{F_Ξ} is generated e.g. by the vanishing of $N_{F_\Xi} = [F_\Xi, F_\Xi]$ (called the *integrability* of the f(3,-1)-structure F_Ξ). Due to $H_\Xi = H_{\Gamma^{(2)}} \oplus H_{F_\Xi}$, Ξ is then integrable if and only if

$$[D_{\Gamma^{(2)}i}, D_{\Xi\lambda}] \in H_\Xi$$

for any i and λ. Under the above assumptions and by means of (2.14) we get

$$[D_{\Gamma^{(2)}i}, D_{\Xi\lambda}] = [D_{\Xi i}, D_{\Xi\lambda}] - \Xi^\sigma_{i\lambda} D_{\Xi\sigma}$$

and thus the following assertion can be presented.

Proposition 8. *Let $\Gamma^{(2)}$ be an integrable 2-connection on π, let Ξ be a connection on $\pi_{1,0}$ associated with $\Gamma^{(2)}$. If the strong horizontal distribution H_{F_Ξ} of Ξ is involutive then Ξ is integrable if and only if $[D_{\Xi i}, D_{\Xi\lambda}] = 0$ for any i, λ.*

In such situation, (2.13) locally reads

$$D_{\Gamma^{(2)}i}(\Xi^\sigma_{k\lambda}) - \frac{\partial \Gamma^\sigma_{ki}}{\partial y^\lambda} + \left(\Xi^\sigma_{k\alpha}\delta^\ell_i - \frac{\partial \Gamma^\sigma_{ki}}{\partial y^\alpha_\ell} \right) \Xi^\alpha_{\ell\lambda} = 0 \tag{2.16}$$

for any $i, k = 1, \dots, n$ and $\sigma, \lambda = 1, \dots, m$.

Let $\dim X = 1$. Then any 2-connection $\Gamma^{(2)}$ on π is integrable (as a system of ordinary differential equations), (2.11) holds trivially and consequently (2.15) means that (2.12) holds for any connection Ξ on $\pi_{1,0}$. Accordingly, Ξ is integrable if and only if (2.13) holds. Then analogously to (2.16) we obtain local conditions

$$D_{\Gamma^{(2)}}(\Xi^\sigma_\lambda) - \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda} + \left(\Xi^\sigma_\alpha - \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\alpha_{(1)}} \right) \Xi^\alpha_\lambda = 0 \tag{2.17}$$

for any $\sigma, \lambda = 1, \dots, m$. As a consequence we get :

Corollary 1. *Let $\dim X = 1$, let $\Gamma^{(2)}$ be a 2-connection on π and Λ a linear connection on X. Then the connection Ξ^Λ_0 is a (global) integral of $\Gamma^{(2)}$ if and only if*

$$D_{\Gamma^{(2)}} \left(\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} \right) - \frac{1}{2} \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\alpha_{(1)}} \frac{\partial \Gamma^\alpha_{(2)}}{\partial q^\lambda_{(1)}} - 2 \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda} + \left(\frac{d\Lambda}{dt} + \frac{1}{2}\Lambda^2 \right) \delta^\sigma_\lambda = 0$$

for any $\sigma, \lambda = 1, \dots, m$.

Briefly, the searching for global integrals of $\Gamma^{(2)}$ can be parameterized by linear connections on the base X.

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