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# ON SECTIONING TANGENT BUNDLES AND OTHER VECTOR BUNDLES

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## INTRODUCTION

This paper has two parts. Part One is mainly intended as a general introduction to the problem of sectioning vector bundles (in particular tangent bundles of smooth manifolds) by everywhere linearly independent sections, giving a survey of some ideas, methods and results. Part Two then records some recent progress in sectioning tangent bundles of several families of specific manifolds. We recommend reading this paper in conjunction with [72] and [42] (the latter is partially updated here; see e.g. the theorem based on the "destabilization trick" and the new $K$-theoretic computations).

## PART ONE. SPAN OF VECTOR BUNDLES AND MANIFOLDS

Consider an $n$-dimensional smooth connected manifold $M^n$. A very important object associated to $M$ is its tangent bundle $TM$. Cross-sections of this bundle are classically called *vector fields* on the manifold $M$. So a vector field on $M$ is nothing but a map (continuous, as always) $v : M \to TM$, with the property that $v(x) \in TM_x$, $TM_x$ being the tangent vector space to $M$ at $x \in M$. For instance, on any manifold $M$ we clearly have its zero-vector field sending each point $x \in M$ to the zero vector in the tangent space $TM_x$. Or, here is a less trivial example: on the odd-dimensional unit sphere $S^{2m+1} = \{(x_1, \dots, x_{2m+2}); x_1^2 + \cdots + x_{2m+2}^2 = 1\}$, one can take $v(x_1, \dots, x_{2m+2}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2m+2}, x_{2m+1})$ as a nowhere zero vector field.

Having one nowhere zero vector field on M, we can naturally ask about the existence of two everywhere linearly independent vector fields, etc. The largest number of everywhere linearly independent vector fields on a given (smooth, as always for us) manifold M is usually called the *span of the manifold $M$*; we will denote it $\operatorname{span} M$. With this notation, we just saw that span $S^{2m+1} \geq 1$, and clearly $0 \leq \operatorname{span} M \leq n$ in general.

If $\alpha$ is a finite dimensional real vector bundle, the *span of $\alpha$* (denoted $\operatorname{span}\alpha$) is defined to be its largest number of everywhere linearly independent sections. If $\alpha$ is equipped with a Euclidean metric, then $\operatorname{span}\alpha \geq k$ is the same as $\alpha = \varepsilon^k \oplus \eta$ for some vector bundle $\eta$; here and in the sequel $\varepsilon^k$ is the $k$-dimensional trivial vector bundle (we do not distinguish between a vector bundle and its isomorphism class). Of course, for a manifold $M$ we have $\operatorname{span} M = \operatorname{span} TM$.

The following problem is known as the *vector field problem.*

**PROBLEM 1.** *Find* span $M$ *for a given manifold* $M$.

It can also be stated a little differently:

**PROBLEM 1'.** *For a given* $k \geq 1$, *characterize in terms of computable invariants (characteristic classes, characteristic numbers, ...) all those manifolds* $M$, *for which* span $M \geq k$.

The span is an important geometric characteristic of manifolds. In particular, if span $M^n = n$, then of course $TM = \varepsilon^n$. A manifold such that its span is the same as its dimension admits a *global parallel motion*, and is therefore called *parallelizable*. Here is what we mean by the global parallel motion (cf. Kahn [34]): we have, in the situation under consideration, $n$ everywhere linearly independent vector fields $s_i : M \to TM$, $i = 1, \ldots, n$. Take an arbitrary point $x_0 \in M$. Clearly, the vectors $s_1(x_0), \ldots, s_n(x_0)$ form a basis of the vector space $TM_{x_0}$. Hence an arbitrary vector $v_0 \in TM_x - \{0\}$ can be written as $v_0 = \sum_{i=1}^{n} a_i s_i(x_0)$, for some $(a_1, \ldots, a_n) \in \mathbb{R}^n - \{0\}$. Then the vector field $v : M \to TM$, defined by $v(x) = \sum_{i=1}^{n} a_i s_i(x)$ is clearly nowhere zero, and the vector $v(x) \in TM_x$ can be considered, completely naturally, as a vector parallel to the vector $v_0$.

For instance, any Lie group is parallelizable: one can use (say) the right multiplication to move a basis chosen for the tangent space at the identity element to obtain a basis for the tangent space at any point.

Now let us come back to the spheres. Already in 1880's H. Poincaré, in his works devoted to curves defined by differential equations, studied singularities of vector fields (hence points of their vanishing or of linear dependence). Among other results, he found that any vector field on the 2-dimensional sphere $S^2$ has somewhere a zero-point. Then, about 1910, Brouwer [12] and Hadamard [25] independently showed that any vector field on an even-dimensional sphere is vanishing at some point. They also observed that on any odd-dimensional sphere one has a nowhere zero vector field. Hence, about 1910 it was known that span $S^{2k} = 0$, and span $S^{2k+1} \geq 1$.

Around 1920, Hurwitz [32] and Radon [55] studied systems of matrices $A_i \in O(n)$ such that $A_i A_j + A_j A_i = 0$ for all $i \neq j$, and $A_i^2 = -\mathrm{id}$ for all $i$. Of course, the last condition implies that $n$ must be even, if such a system should be non-empty. It is also clear that any such system of matrices gives the same number of linearly independent vector fields on the sphere $S^{n-1}$ as is the cardinality of the system. Indeed, one checks that the scalar product $< x, A_i(x) >= 0$ for all $i$ and all $x \in S^{n-1}$. The linear independence of $A_1(x), A_2(x)$, etc. at any $x \in S^{n-1}$ is also readily checked; indeed $< A_i(x), A_j(x) >= \delta_{ij}$.

Now write $n = (2a+1) \cdot 2^{c+4d}$, with $a, c, d \geq 0, c \leq 3$. Hurwitz and Radon found that there are $2^c + 8d - 1$, but no more, matrices with the required properties. The number $2^c + 8d$ is quite often called the *Hurwitz-Radon number*; we will use the notation $\varrho(n)$ for it. So we have at the moment: span $S^{n-1} \geq \varrho(n) - 1$.

However much time and effort was necessary until the complete answer for span $S^{n-1}$ was found.

In the meantime, around 1925 H. Hopf [31] completely solved the vector field problem for $k = 1$ (as stated in Problem 1'). His result is:

**THEOREM.** *Let $M^n$ be a closed (=compact, without boundary), smooth, connected manifold.*

*Then* span $M \geq 1$ *if and only if* $\chi(M) = 0$.

Here $\chi(M)$ is the *Euler characteristic.*

If $M$ is an open manifold, using obstruction theory, one can always see that span $M \geq 1$, and the same is true when $M$ has a boundary. Since we do not know much more about the vector field problem on open manifolds and on manifolds with boundary, in the sequel we will concentrate on smooth closed connected manifolds.

Some 10 years after the Hopf theorem, Stiefel [68] proved that any orientable 3-dimensional manifold is parallelizable (of course, no non-orientable manifold can be parallelizable).

Hence, in particular, the sphere $S^3$ is parallelizable (this is even a Lie group, so that the parallelizability is not any surprise). But for higher spheres, the parallelizability question was in general open until late fifties, when it was answered independently by Kervaire [36] and Milnor [52] using results of R. Bott on stable homotopy of the classical groups. The answer was: only $S^1$, $S^3$, and $S^7$ are parallelizable spheres. This immediately solves also the parallelizability question of real projective spaces. Indeed: a real projective space $\mathbb{R}P^{n-1}$ is obtained from $S^{n-1}$ as the quotient $S^{n-1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $S^{n-1}$ antipodally. Now, $S^1$, $S^3$, and $S^7$ can be parallelized by the Hurwitz-Radon "linear" vector fields. But those clearly induce a parallelization of $\mathbb{R}P^1 = S^1$, $\mathbb{R}P^3$ and $\mathbb{R}P^7$. On the other hand, since $S^{n-1}$ is a double covering of $\mathbb{R}P^{n-1}$, we have span $S^{n-1} \geq \operatorname{span} \mathbb{R}P^{n-1}$, and so no other projective spaces can be parallelizable.

Finally, in the beginning of sixties, the vector field problem for spheres was completely solved: J. F. Adams [1], using operations in $K$-theory, showed that the Hurwitz-Radon lower bound coincides in fact with the upper bound; as a consequence, span $S^{n-1} = \varrho(n) - 1$.

Now take the Stiefel manifold $V_{n,r}$ of orthonormal $r$-frames in $\mathbb{R}^n$; of course, as a special case we have $V_{n,1} = S^{n-1}$. In 1964, W. Sutherland [70] showed that $V_{n,r}$ for $r \geq 2$ is always parallelizable. Later on others, e.g. Handel [27], Lam [45], Smith [65] (for $r \geq 3$) or Zvengrowski [77] gave other, more elementary proofs.

The spheres have many remarkable properties. One of them is that $TS^n \oplus \varepsilon^1 = \varepsilon^{n+1}$. Manifolds $M^n$ with the corresponding property $TM^n \oplus \varepsilon^1 = \varepsilon^{n+1}$ are called *stably parallelizable* (or also $\pi$-manifolds). For these, the vector field problem was solved by Kervaire, Bredon, and Kosinski. The latter two proved in 1965, in [11], the following result:

**THEOREM.** *Let $M^n$ be stably parallelizable, $n \geq 1$. Then either $M$ is parallelizable, or* span $M^n =$ span $S^n$.

*(a) If $n$ is even, $M^n$ is parallelizable* $\Longleftrightarrow \chi(M) = 0$.

*(b) If $n$ is odd and $n \notin \{1,3,7\}$, $M$ is parallelizable* $\Longleftrightarrow \widehat{\chi}_2(M) = 0$.

Here $\widehat{\chi}_2(M) = \sum_{i=0}^{[\frac{n}{2}]}(\dim H_i(M;\mathbb{Z}_2))$ (mod 2) is the so called $\mathbb{Z}_2$-*Kervaire semicharacteristic*. M. Kervaire proved (a) and "$\Leftarrow$" of (b) by different methods (using *curvatura integra*), in [35].

Now two words about the situation for those manifolds which are not necessarily stably parallelizable. If we are interested in concrete manifolds, then briefly said: apart from the trivial cases with $\chi(M) \neq 0$ (hence span $M = 0$) and apart from what we already mentioned, in the 60's and in the early 70's perhaps just results on quotient spaces of spheres are remarkable.

These quotient spaces are known as *spherical space forms*; they are smooth, closed, connected orientable manifolds of the form $S^n/G$, where $G$ is a finite subgroup of $O(n+1)$, acting freely on $S^n$. Let $G_p$ be a $p$-Sylow subgroup of $G$, where $p$ is a prime number. If $G'_p$ is another $p$-Sylow subgroup of $G$, then one can show that the quotient spaces $S^n/G_p$ and $S^n/G'_p$ are isometric. So one can introduce the *spherical p-form associated to* $S^n/G$ as $S^n/G_p$, where $G_p$ is a $p$-Sylow subgroup of $G$. It turns out that the span of an arbitrary spherical space form only depends on the kind of its associated spherical 2-forms.

Results about the span of spherical forms (in particular of lens spaces) can be found in papers by T. Yoshida ([74], [75]) and J. Becker (e.g. [9]), or also in the book by Mahammed, Piccinini, and Suter [49]. We just mention that in some cases the span of $S^n/G$ is the span of $S^n$, but in some cases it is smaller. (Clearly, always span $S^n/G \leq$ span $S^n$, because of the finitely-sheeted covering $S^n \to S^n/G$.)

It seems to us that the main research activity in late 60's and early 70's was concentrated on tryings to find theorems similar to the Hopf theorem for two, three, etc. independent vector fields. In this context, several interesting methods were developed. We will briefly comment on two of them: the so called *index method* and so called *singularity approach.*

**A) The index method.** Origins of this approach are in the Steenrod obstruction theory (see [67]). To simplify matters, let us consider an orientable closed smooth connected manifold $M^n$. In fact, choose a fixed orientation on $M$. Further, we suppose that there is a finite set $S = \{p_1, \dots, p_t\} \subset M$ and $k$ linearly independent sections of $TM|_{M\setminus S}$, say $v_1, \dots, v_k : M \setminus S \to TM|_{M\setminus S}$. We say briefly that the (ordered) $k$-tuple $(v_1, \dots, v_k)$ is a *k-field with finite singularity set* on $M$.

The Gram-Schmidt orthonormalization process is continuous, and so we can even suppose that at any $x \in M \setminus S$ the vectors $v_1(x), \dots, v_k(x)$ are mutually orthonormal.

Now take a triangulation on $M^n$ such that each (so called singular) point $p_i \in S$ lies in the interior of an $n$-simplex $\sigma_i$, and each $\sigma_i$ contains at most one point of $S$. Since each $n$-simplex $\sigma_i$ is of course contractible, the restriction $TM|_{\sigma_i}$ is trivial and we can, for each $i$, choose an orientation preserving trivialization isomorphism $TM|_{\sigma_i} \cong \sigma_i \times \mathbb{R}^n$. The boundary $\dot{\sigma}_i$ is an oriented $(n-1)$-sphere. Clearly $x \mapsto (v_1(x), \dots, v_k(x)) \in V_{n,k}$, for any $x \in S^{n-1} \cong \dot{\sigma}_i$, defines an element of the homotopy group $\pi_{n-1}(V_{n,k})$. This element is called the index of $(v_1, \dots, v_k)$ at $p_i \in S$. It is clear that the index is zero precisely when it is possible, in a small neighbourhood of $p_i$, to change (or deform) the fields $v_1, \dots, v_k$ so that the singularity disappears. Then as a global obstruction one takes $\operatorname{Index}(v_1, \ldots, v_k) := \sum_{i=1}^{t} (\text{Index at } p_i) \in \pi_{n-1}(V_{n,k})$. This is independent of the triangulation and other choices made above. But, in general, $\operatorname{Index}(v_1, \ldots, v_k)$ can be changed, if we reverse the orientation on $M$. From classical obstruction theory (see also [72]) one immediately obtains the following:

**PROPOSITION.** *Let $M^n$ be an oriented manifold with a $k$-field $(v_1, \ldots, v_k)$ with a finite singularity set. Then* $\operatorname{Index}(v_1, \ldots, v_k) = 0$ *if and only if there are $k$ everywhere linearly independent vector fields $\widetilde{v}_1, \ldots, \widetilde{v}_k : M \to TM$ so that $(\widetilde{v}_1, \ldots, \widetilde{v}_k)_{|M^{(n-2)}} = (v_1, \ldots, v_k)_{|M^{(n-2)}}$. Here $M^{(n-2)}$ is the $(n-2)$-skeleton of $M$.*

For instance, if $M$ is $(k-1)$-connected, then one can quite easily show (using obstruction theory) that there exists on $M$ a $k$-field with a finite singularity set; let us denote it $(v_1, \ldots, v_k)$, and let $\operatorname{Index}(v_1, \ldots, v_k)$ be its index. Moreover, in this situation, $\operatorname{Index}(v_1, \ldots, v_k)$ is a *primary* obstruction. That is why it is independent of the choice of $(v_1, \ldots, v_k)$. Hence in this case the index depends only on the manifold $M$, and we can write: $\operatorname{Index}(v_1, \ldots, v_k) := k\text{-INDEX}(M)$. So in this situation

$$\text{span } M \geq k \text{ if and only if } k\text{-INDEX}(M) = 0.$$

This applies, in particular, when $k = 1$ and $M$ is connected. Then it turns out that $1\text{-INDEX}(M) = \chi(M)$ (and we obtain the Hopf theorem).

In general, if $M$ is not $(k-1)$-connected, but still has a $k$-field $(v_1, \ldots, v_k)$ with a finite singularity set, then $\operatorname{Index}(v_1, \ldots, v_k)$ can depend on $(v_1, \ldots, v_k)$. So it can then happen that the index-method does not help much in considering the question whether or not span $M \geq k$.

Nevertheless, it can also happen that this method still will be helpful. We will try to explain what we mean by this. For instance, if we have an oriented manifold $M$ and a pair $(v_1, v_2)$ of vector fields with only a finite set of singular points on $M$, then a priori we do not know if $\operatorname{Index}(v_1, v_2)$ depends or does not depend on $(v_1, v_2)$. However, E. Thomas has shown that this index depends really only on $M$. Hence also for $k = 2$, even when $M$ is not simply connected, one can speak about a $2\text{-INDEX}(M)$.

E. Thomas computed in general this $2\text{-INDEX}(M)$ for $k = 2$ (somewhat later the same was done by M. Atiyah who used certain consequences obtained from the Atiyah-Singer Index Theorem about the index of an elliptic differential operator). Now, when one takes the calculations of the $2\text{-INDEX}(M)$ together with conditions for the existence of 2-fields with finite singularity sets, then one obtains necessary and sufficient conditions for span $M \geq 2$ ($M$ closed and oriented):

| dim $M = m$ | Necessary and sufficient conditions for span $M \geq 2$ |
|---|---|
| $m \equiv 1 \pmod 4$ | $w_{m-1}(M) = 0, k(M) = 0$ |
| $m \equiv 2 \pmod 4$ | $\chi(M) = 0$ |
| $m \equiv 3 \pmod 4$ | $M$ always has span $\geq 2$ |
| $m \equiv 0 \pmod 4$, $m > 4$ | $\chi(M) = 0$, $\sigma(M) \equiv 0 \pmod 4$ |

Here $k(M) = (\sum_i \dim H_{2i}(M;\mathbb{R}))$ (mod 2) is the so called *real Kervaire semicharacteristic* (note that $k(M)-\widehat{\chi}(M) =< w_2(M) \cup w_{m-2}(M), [M] >$, by Lusztig, Milnor, and Peterson [48]), and $\sigma(M)$ is the *signature* of $M$.

The above conditions can be found for instance in the outstanding survey article by Thomas [72]. There is an index theory also for non-orientable manifolds; one can read about that in Randall [56] or also in the book by U. Koschorke [43].

**B) The singularity approach.** Let $\xi_p$ denote the Hopf line bundle over the $p$-dimensional real projective space $\mathbb{R}\mathrm{P}^p$. J. Becker [9; p. 867] and J.-P. Dax [20] have a lemma, which for the tangent bundle of a given (closed, smooth) manifold $M^n$ gives the following:

**PROPOSITION.** *Let* $1 \leq k \leq n$. *If* span $M \geq k$, *then* span $TM\widehat{\otimes}\xi_{k-1} \geq 1$. *If* $2k < n$ *and* span $TM\widehat{\otimes}\xi_{k-1} \geq 1$, *then* span $M \geq k$.

The proof of the first part is quite easy: span $M \geq k$ means we have $k$ linearly independent vector fields $v_1, \ldots, v_k : M \to TM$. Then we define $f : M \times \mathbb{R}^k \to TM$, $f(x,(a_1,\ldots,a_k)) = \sum_{i=1}^{k} a_i v_i(x)$. Clearly $f$ is a monomorphism of vector bundles. Then define $\widetilde{f} : M \times S^{k-1} \to S(TM)$, $\widetilde{f}(x,v) = \frac{f(x,v)}{\|f(x,v)\|}$ ($S(TM)$ is the sphere bundle of $TM$), and $s : M \times \mathbb{R}\mathrm{P}^{k-1} \to TM\widehat{\otimes}\xi_{k-1}$, $s(x,\{-v,v\}) := \widetilde{f}(x,v)\widehat{\otimes}v$ (this is well defined, because $\widetilde{f}(x,-v) = -\widetilde{f}(x,v)$). Then $s$ is a nowhere zero section of $TM\widehat{\otimes}\xi_{k-1}$.

To prove the second part, one first shows that there is a skew-map $m : M \times \mathbb{R}^k \to TM$ (that is: $m(x,v) \neq 0$ whenever $v \neq 0$ and $m(x,-v) = -m(x,v)$). Then, by a theorem of Haefliger and Hirsch [26], due to the hypothesis $2k < n$, we derive the existence of a monomorphism of vector bundles $\widetilde{m} : M \times \mathbb{R}^k \to TM$, which means that span $M \geq k$.

From the above proposition, if $k < \frac{n}{2}$, then span $M \geq k$ if and only if the vector bundle $TM^n\widehat{\otimes}\xi_{k-1}$ has a nowhere vanishing cross-section. We observe that sectioning vector bundles by nowhere zero sections was studied by several authors, in different theories: e.g. by J.-P. Dax [19] in the framework of a special type of bordism theory, or by M. Crabb [13], [14], [15] in the framework of cohomotopy theory and $K$-theory. All the authors derive a sort of Euler class, whose vanishing is, for $k < \frac{n}{2}$, a necessary and sufficient condition for the existence of a nowhere zero section of $TM^n\widehat{\otimes}\xi_{k-1}$, hence for span $M \geq k$. However, to compute such Euler classes is in general difficult.

An important contribution to this was provided by U. Koschorke. In the framework of a so called singularity approach, which can be used also to investigate immersions and other problems, he found an effective (at least for low values of $k$) way of attacking the vector field problem for $k < \frac{n}{2}$, hence, essentially, of computing the above mentioned Euler classes. We will briefly outline his point of view (in detail presented in [43]).

An arbitrary $k$-tuple $(s_1, \ldots, s_k)$ of vector fields on a closed connected smooth manifold $M$ defines the following vector bundle-morphism: $s : \varepsilon^k = M \times \mathbb{R}^k \to TM$,

$s(x,(a_1,\ldots,a_k)) = \sum_{i=1}^{k} a_i s_i(x)$. Then clearly the set $S = \{x \in M; \text{rank } s_x < k\}$ coincides with the set of those points, in which the fields $s_1,\ldots,s_k$ are linearly dependent; so $S$ is the *singularity set*. The set $S$ can eventually be finite. A finite set $S$ was needed for the index-method. But now one prefers to have $S$ infinite. More precisely, let us suppose that $k < \frac{n}{2}$ ($n = \dim M$). Applying a standard transversality argument, we can achieve that the rank of $s$ will be at least $k-1$ on the whole of $M$, and then the singularity set $S$ becomes a $(k-1)$-dimensional submanifold of $M$.

So then for any $x \in S$ the kernel of $s_x : \mathbb{R}^k_x \to TM_x$ is a 1-dimensional subspace in $\mathbb{R}^k$, hence an element of $\mathbb{RP}^{k-1}$. We can now define $g : S \to \mathbb{RP}^{k-1} \times M$, $g(x) = (\operatorname{Ker} s_x, x)$. In addition to this, one can find an isomorphism of vector bundles, $\bar{g}$, which describes the stable normal bundle of $S$ in $M$ by means of a certain vector bundle $g^*\Phi_M$, where $\Phi_M$ is a suitable virtual bundle over $\mathbb{RP}^{k-1} \times M$.

The data $S$, $g$, $\bar{g}$ give a well-defined element $\omega_k := [S, g, \bar{g}]$ in the so called *normal bordism group* $\Omega_{k-1}(\mathbb{RP}^{k-1} \times M; \Phi_M)$. The element $\omega_k$ has an outstanding property: for $1 \le k < \frac{n}{2}$ we have

$$\omega_k = 0 \Longleftrightarrow \text{span } M \ge k.$$

For $k \le 4$, U. Koschorke achieved nice results in computing $\omega_k$ in terms of classical invariants such as Stiefel-Whitney classes, characteristics, etc. As an example, here is one of his theorems (see [43]), previously proved by a different method by M. Atiyah and J. Dupont [5] for orientable manifolds:

**THEOREM.** *Let $M^n$ be a manifold, $n > 6$, $n \equiv 2 \pmod 4$. Then* span $M \ge 3$ *if and only if $\chi(M) = 0$ and $w_{n-2}(M) = 0$.*

We already considered parallelizability and stable parallelizability. More generally, besides the span of a manifold one can consider its *stable span*:

$$\text{stable span}\, M := \text{span}(TM \oplus \varepsilon^r) - r = \text{span}(TM \oplus \varepsilon^1) - 1 (r \ge 1).$$

The singularity method turns out to be strong also in the study of relations between span and stable span. These two numbers can differ very much (e.g. span $S^{2k} = 0$, while stable span $S^{2k} = 2k$), but they also can coincide, as the following theorem (see [43]) indicates.

**THEOREM.** *Let $M$ be an $n$-dimensional manifold.*

(a) *If $n$ is even and $\chi(M) = 0$, then* stable span $M =$ span $M$ *(while* span $M = 0$ *if $\chi(M) \neq 0$).*
(b) *If $n \equiv 1 \pmod 4$ and $w_1(M)^2 = 0$, then* stable span $M =$ span $M$ *if $R_L(M) = 0$, and* span $M = 1$ *if $R_L(M) \neq 0$, where $R_L(M)$ is the twisted Kervaire semicharactiristic defined by Atiyah and Dupont in* [5].
(c) *If $n \equiv 3 \pmod 8$ and $w_1(M) = w_2(M) = 0$, then* stable span $M =$ span $M$ *if $\widehat{\chi}_2(M) = 0$, and* span $M = 3$ *if $\widehat{\chi}_2(M) \neq 0$.*

So another way of attacking the vector field problem is to look for the stable span and then to try to show that it is in fact the actual span. In Part Two this will be illustrated by concrete examples.

One can try to study the relation between stable span and span also using essentially a homotopy method. Take $M$ with odd dimension (if dim $M$ is even, all is clear). Then by a special case of a theorem of I. James and E. Thomas [33] there are one or two isomorphism classes of $n$-plane bundles over $M$, which are stably isomorphic to the tangent bundle $TM$. If they are two, W. Sutherland [69] defines for any $n$-plane bundle $\alpha$ stably equivalent to $TM$ a number $b_\beta(\alpha) \in \mathbb{Z}_2$, called the *Browder-Dupont invariant*; this $b_\beta$ distinguishes between those two classes of $n$-plane bundles stably isomorphic to $TM$, and $b_\beta(TM) = \widehat{\chi}_2(M)$. To compute $b_\beta(\alpha)$ is in general not easy.

Closing Part One, we would like to comment briefly on a so called *generalized vector field problem* and its generalizations. The generalized vector field problem is simply the question of what is span$(k\xi_{n-1})$ for all $n$ and $k$ (as before, $\xi_{n-1}$ is the Hopf line bundle over $\mathbb{R}P^{n-1}$). This problem was studied by several authors, e.g. Atiyah, Bott, Shapiro [4], Davis [16], Davis, Gitler, Mahowald [17], [18], Lam [44], Lam, Randall [46]. It is not yet completely solved, but it is solved for many values of $k$ and $n$, and for others there are quite good estimates (see for instance [44]).

But why is this problem called as it is called? First of all, it is a generalization of the vector field problem for projective spaces. Indeed, one has span $\mathbb{R}P^{n-1}$ = span $n\xi_{n-1} - 1$. To see that, first recall that $T\mathbb{R}P^{n-1} \oplus \varepsilon^1 = n\xi_{n-1}$. If $n$ is odd, then the Stiefel-Whitney class $w_{n-1}(\mathbb{R}P^{n-1}) = w_1^{n-1}(\xi_{n-1}) \neq 0$, which implies that span$(\mathbb{R}P^{n-1}) = 0$ and span$(n\xi_{n-1}) = 1$. If $n$ is even, then a theorem of James and Thomas [33] ensures that span$(\mathbb{R}P^{n-1})$ = stable span$(\mathbb{R}P^{n-1})$.

But in fact, the generalized vector field problem is relevant not only to the span problem on projective spaces. Indeed, whenever the tangent bundle $TM$ of a smooth manifold $M^n$ has the property that $TM \oplus \varepsilon^k = (n+k)\beta$, for a line bundle $\beta$, then there exists $t \leq n$ such that span$(TM \oplus \varepsilon^k) - k$ = stable span $M \geq$ span$(n+k)\xi_t - k$. So if stable span and span coincide, the generalized vector field problem gives an estimate for the span of $M$. Again, this will be illustrated in Part Two by a concrete example. The reason for stable span $M \geq$ span$(n+k)\xi_t - k$ is, in the above situation, that one can find $t \leq n$ and a classifying map $f : M^n \to \mathbb{R}P^t$ such that $f^*(\xi_t) = \beta$ (see Hirsch [29]), hence then $TM \oplus \varepsilon^k = (n+k)f^*(\xi_t) = f^*((n+k)\xi_t)$.

Now take the following very strong generalization:

**PROBLEM 2.** *Let $\beta$ be a vector bundle over a paracompact space and $n\beta = \overbrace{\beta \oplus \cdots \oplus \beta}^{n \text{ times}}$. Find* span $n\beta$.

This problem is closely related to the vector field problem on the class of smooth compact manifolds with a *strong almost tangent structure.* Indeed, by Goggins [24], for $r, k \geq 1$ and $n \geq (r+1)k$, a compact smooth manifold $M^n$ has a strong almost tangent structure of order $r$ and type $k$ if and only if there exist vector bundles $\beta^k$ and $\eta^{n-(r+1)k}$ over $M^n$ such that

$$TM^n = \eta^{n-(r+1)k} \oplus (r+1)\beta^k.$$

Projective Stiefel manifolds (see Part Two) are examples of such manifolds. Of course, one can consider this property also stably, and then the class of applications of solutions to the above general problem will be even bigger. For instance, projective spaces have stably strong almost tangent structure, because $\mathrm{T}\mathbb{R}\mathrm{P}^{n-1} \oplus \varepsilon^1 = n\xi_{n-1}$.

Even though the above problem might seem to be "hopeless" (mainly because of its generality), one can prove very simply the following result.

**THEOREM.** *Let $\beta$ be a vector bundle over a paracompact space $X$. Then for any $n$ we have* $\mathrm{span}(n\beta) \geq \varrho(n)$ *whenever* $\mathrm{span}(n\beta) \geq 1$.

The proof with examples of concrete applications can be found in Korbaš [40]. For a special case of Problem 2 see [39] and [41].

## PART TWO. PARALLELIZABILITY THEOREMS AND PROGRESS ON SPAN

The above mentioned theorem of Bredon and Kosinski effectively determines the span of a stably parallelizable manifold, but the question of finding the span of manifolds that are not stably parallelizable remains. This assumes added significance when one takes into account a large body of work in the period 1975–1988 which shows that parallelizability or even stable parallelizability is extremely rare among the frequently occurring families of manifolds in topology. We now recount some of this work.

The (real) *Grassmann manifold* of $k$-planes in $\mathbb{R}^n$ is written $G_{n,k}$ or $G(k,l)$, where $k + l = n$. As a homogeneous space $G_{n,k} = O(n)/(O(k) \times O(l))$, which also gives a smoothness structure to $G_{n,k}$. Its dimension is $kl$. The *Grassmann manifold $\widetilde{G}_{n,k}$ of oriented $k$-planes* in $\mathbb{R}^n$ is similarly given by $\widetilde{G}_{n,k} = SO(n)/(SO(k) \times SO(l))$, and is a double cover of $G_{n,k}$. Complex or quaternionic Grassmann manifolds are similarly defined, replacing $\mathbb{R}^n$ by $\mathbb{C}^n$ or $\mathbb{H}^n$ respectively.

The Grassmann manifolds are just special cases of *flag manifolds*

$$G(n_1, n_2, \ldots, n_r) = O(n)/(O(n_1) \times \cdots \times O(n_r))$$

(namely $r = 2$). *Oriented flag manifolds* $\widetilde{G}(n_1, n_2, \ldots, n_r) = SO(n)/(SO(n_1) \times \cdots \times SO(n_r))$ may also be defined. Here $n = n_1 + \cdots + n_r$ throughout.

**THEOREM.** *$G_{n,k}$ is parallelizable only for $G_{2,1} = \mathbb{R}P^1$, $G_{4,1} = G_{4,3} = \mathbb{R}P^3$, and $G_{8,1} = G_{8,7} = \mathbb{R}P^7$. None of the others is even stably parallelizable.*

Proofs were given by Yoshida [76], Hiller, Stong [28], Bartík, Korbaš [6], and Trew, Zvengrowski [73], the latter also covering the complex and quaternionic cases.

**THEOREM.** *Among $\widetilde{G}_{n,k}$ only $\widetilde{G}_{2,1} = S^1$, $\widetilde{G}_{4,1} = \widetilde{G}_{4,3} = S^3$, $\widetilde{G}_{8,1} = \widetilde{G}_{8,7} = S^7$ and $\widetilde{G}_{6,3}$ are parallelizable. Of the remaining ones only $\widetilde{G}_{4,2}$ is stably parallelizable.*

This theorem was first stated and partially proved by I. Miatello and R. Miatello [51], and completely proved in the dissertation of P. Sankaran (cf. [57]). Recently $K^*(\widetilde{G}_{n,k})$ has been computed by Sankaran and Zvengrowski, leading to a very short proof of this result which we will outline below (cf. the following section on Grassmann manifolds).

**THEOREM.** *For $r > 2$ only the flag manifold $G(1, 1, \dots, 1)$ is parallelizable. The others are not stably parallelizable.*

The proof is due to Korbaš [37], and Sankaran, Zvengrowski [59], the latter also covering the complex and quaternionic cases.

**THEOREM.** *For $r > 2$ the oriented flag manifold $\widetilde{G}(n_1, \dots, n_r)$, where without loss of generality we assume $n_1 \geq n_2 \geq \dots$, is parallelizable if and only if $n_2 = \dots = 1$, or $n_1 = n_2 = \dots = 3$, or $\{n_1, n_2, \dots\} = \{3, 1\}$, or $\{n_1, n_2, \dots\} = \{2, 1\}$ with at least two $n_i = 1$ in the latter case. Of the rest only $\widetilde{G}(2, \dots, 2)$ and $\widetilde{G}(2, \dots, 2, 1)$ are stably parallelizable.*

Here the proof is due to I. Miatello, R. Miatello [51], and Sankaran, Zvengrowski [60].

The *projective Stiefel manifolds* were introduced by Baum, Browder [8], Gitler, and Handel [21], [22], [23]. They are obtained from the ordinary Stiefel manifold $V_{n,k}$ ($k < n$) by identifying any $k$-frame $(v_1, \dots, v_k)$ of vectors in $\mathbb{R}^n$ with its antipodal $k$-frame $(-v_1, \dots, -v_k)$, giving $X_{n,k}$ double covered by $V_{n,k}$. Note $X_{n,1} = \mathbb{R}\mathrm{P}^{n-1}$.

**THEOREM.** *The projective Stiefel manifold $X_{n,k}$ is parallelizable if $(n, k)$ equals $(n, n-1)$, $(2m, 2m-2)$, $(16, 8)$, and if $n = 2$, 4, or 8. None of the remaining ones are stably parallelizable, with the possible exception of the undecided case $X_{12,8}$.*

This was proved by computing $K^*(X_{4m,k})$ in [3].

Similar results hold for the *partially oriented flag* manifolds [60] and for homogeneous spaces obtained as quotients of other Lie groups such as $U(n)$, $Sp(n)$, and the exceptional Lie groups (Singhof [62], Singhof, Wemmer [63], [64]); we simply refer the reader to those papers for the details. Having established that in an intuitive sense the vast majority of the manifolds that are frequently used in geometry are not stably parallelizable, we now turn our attention to more specific manifolds, in particular the Grassmann and projective Stiefel manifolds, and examine some of the methods that have led to successful calculations of the span of many of these (not stably parallelizable) manifolds, e.g. results such as span $G_{6,3} = 7$ and span $X_{16,5} = 58$. In most cases these results were not known or known only in much weaker form prior to 1990, e.g. in the above two cases only $3 \leq$ span $G_{6,3} \leq 7$ and $10 \leq$ span $X_{16,5} \leq 61$ would have been known.

The main tools in deducing these sharper results are:

(a) $K$-theory,
(b) use of Koschorke's work to deduce (if possible) span $M$ = stable span $M$ for a manifold $M$, then determining stable span $M$,
(c) techniques specific to a given manifold or family of manifolds,
(d) other techniques, some already discussed in Part One above.

We elaborate briefly on (a)-(d). It is to be expected that $K$-theory, which essentially classifies vector bundles over a given space up to stable equivalence, will be useful in the situations we are studying. We will usually refer to complex $K$-theory $K^*(X)$ rather than the real $K$-theory $KO^*(X)$ due to the difficulties of computing $KO^*(X)$, although the real $K$-theory may be more informative. An effective tool for finding $K^*(X)$ has proved to be the Hodgkin spectral sequence [30]. In 1993 it was used to compute $K^*(X_{n,r})$ for all $n$, $r$ by N. Barufatti and D. Hacon [7], as well as to compute $K^*(\widetilde{G}_{n,k})$ for all $n$, $k$ by Sankaran and Zvengrowski [61]. We will return to these results shortly.

Regarding (b), this idea was already discussed in Part One. It may be of interest, however, to point out that there was a conjecture (due to V. Eagle) that naturally generalizes the Bredon-Kosinski theorem. For any smooth $M^n$, this conjecture stated that span $M$ = span $S^n$ or span $M$ = stable span $M$. Counterexamples were provided by Koschorke. A fairly simple one is $M^{13} = \mathbb{RP}^2 \times S^{11}$. As an exercise the reader may enjoy proving stable span $M = 11$ (Stiefel-Whitney classes will help) and span $M \geq 3$. To see that span $M = 3$ one uses results in the last chapter of [43]. Notice span $S^{13} = 1 \neq$ span $M \neq$ stable span $M$.

We will deal with (c) individually for the families of manifolds to be studied. As far as (d), in addition to methods described in Part One such as characteristic classes, James-Thomas numbers, etc., we mention the following source of bounds on the stable span (see Korbaš, Zvengrowski [42]).

**THEOREM.** *Let $M^n$ be a closed, smooth, connected manifold.*

(a) stable span $M > 0$ *if and only if* $\chi(M)$ *is even.*

(b) *If $M$ is oriented and $n \equiv 0 \pmod 4$, then* stable span $M \geq r$ *implies that the signature $\sigma(M)$ is divisible by $b_r \in \mathbb{N}$, where $b_1 = 2$, $b_2 = 4$, $b_3 = 8$, $b_4 = b_5 = b_6 = b_7 = 16$, $b_8 = 32$, and $b_{r+8} = 16 b_r$ defines $b_n$ for $n \geq 1$.*

**Grassmann Manifolds.**

For $G_{n,k}$ there is an obvious $k$-plane bundle $\gamma$. Its orthogonal complement gives the $l$-plane bundle $\gamma^\perp$. It is known (cf. Milnor, Stasheff [53]) that $\tau = \gamma \otimes \gamma^\perp = \mathrm{Hom}(\gamma, \gamma^\perp)$ gives the tangent bundle. There is also a canonical line bundle $\xi_{n,k}$ over $G_{n,k}$, arising from the double cover $\widetilde{G}_{n,k} \to G_{n,k}$. For $\widetilde{G}_{n,k}$ one again has $\tau = \gamma \otimes \gamma^\perp$, where $\gamma$ and $\gamma^\perp$ are now oriented bundles. In all cases $\gamma \oplus \gamma^\perp = \varepsilon^n$ is clear.

We now outline the results for $K^*(\widetilde{G}_{n,k})$. First define the graded algebra $H_{s,t}$ by $H_{s,t} = \mathbb{Z}[p_1, \ldots, p_s, q_1, \ldots, q_t]/\sim$, where $n = k + l$, $k = 2s$ or $2s+1$, $l = 2t$ or $2t+1$, deg $p_i = i =$ deg $q_i$, and the relations are given by the single graded relation $(1 + p_1 + \cdots + p_s)(1 + q_1 + \cdots + q_t) = 1$. Also recall $K^*$ is graded by $\mathbb{Z}_2$ with $K(X) = K^0(X)$. Using the Hodgkin spectral sequence [30] as well as results of Pittie [54], Sankaran and Zvengrowski [61] show:

(a) for $k, l$ both odd
$$K^*(G_{n,k}) \approx H_{s,t} \otimes E_{\mathbb{Z}}[X],$$
where $H_{s,t} = K^0$, $X \in K^1$, and $E_{\mathbb{Z}}$ denotes the exterior algebra over $\mathbb{Z}$,

(b) for $k, l$ both even $K^1 = 0$, $K^0$ is torsion free given by
$$K^0 \approx H_{s,t}[(\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}]/\sim,$$

(c) for mixed parities, say $k$ even and $l$ odd, again $K^1 = 0$, $K^0$ is torsion free, and $K^0 \approx H_{s,t}[(\Delta_s^+)^2, \Delta_{s,t}]/\sim$,

(d) in all cases $H_{s,t}$ is a subalgebra of $K(\widetilde{G}_{n,k}) = K^0(\widetilde{G}_{n,k})$.

*Remark*: We have not specified the generators and relations in (b) or (c) to avoid going into too many details.

To utilize these results the so called $\alpha$-construction is helpful. It identifies the (complexifications of) $\gamma$, $\gamma^\perp$, and hence $\tau$, in $K(\widetilde{G}_{n,k})$, as given in the next lemma.

**LEMMA.** $[c\tau_{n,k}] = p_1q_1 + lp_1 + kq_1 + kl \in K(\widetilde{G}_{n,k})$.

Since $p_1 + q_1 = 0$, we obtain $[c\tau_{n,k}] = p_1q_1 - (k-l)p_1 + kl$. Now suppose $\widetilde{G}_{n,k}$ to be stably parallelizable, then $[c\tau_{n,k}] = kl$, or $0 = p_1q_1 + (l-k)p_1$. Next suppose $k, l \geq 2$ to eliminate the well known case of the spheres. Then $k = l$ (in grading 1) and $p_1q_1 = 0$ (grading 2). Since $p_2 + p_1q_1 + q_2 = 0$ is the only relation in $H_{s,t}$ in grading 2, $p_1q_1 = 0$ can only hold if $s = t = 1$ since $p_2, q_2$ are then absent. This leaves $k = l = 2$ (i.e. $\widetilde{G}_{4,2}$) and $k = l = 3$ (i.e. $\widetilde{G}_{6,3}$) as the only possibilities.

Since $\widetilde{G}_{4,2} = S^2 \times S^2$ is clearly stably parallelizable and not parallelizable, only the case $\widetilde{G}_{6,3}$ remains. Here is a short proof that it is stably parallelizable; the Bredon-Kosinski theorem and the mod 2 Kervaire semi-characteristic could then be used to show parallelizability. Note $\gamma, \gamma^\perp$ are oriented of dimension 3. We have

$$15\varepsilon = \lambda^2(6\varepsilon) = \lambda^2(\gamma \oplus \gamma^\perp) = \lambda^2\gamma \oplus \lambda^1\gamma \otimes \lambda^1\gamma^\perp \oplus \lambda^2\gamma^\perp,$$

upon taking the second exterior power $\lambda^2$. Recall $\lambda^1 = id$, and by Hodge duality $\lambda^2\gamma = \lambda^1\gamma = \gamma$, similarly for $\gamma^\perp$. We find

$$15\varepsilon = \gamma \oplus \gamma \otimes \gamma^\perp \oplus \gamma^\perp = 6\varepsilon \oplus \tau.$$

It is often interesting to ask when a real vector bundle $\xi$ has an underlying complex structure, i.e. $\xi$ is a $2n$-dimensional real vector bundle obtained from an $n$-dimensional complex vector bundle $\eta$ by simply forgetting the complex action. If this holds for the tangent bundle $\tau_M$ of a manifold $M$, we call $M$ almost complex. Clearly $M$ must then be even dimensional. If it holds stably for $\tau_M$, i.e. $\tau_M \oplus s\varepsilon$ comes from a complex vector bundle for some $s$, we call M weakly almost complex.

The question of which $\widetilde{G}_{n,k}$ are almost complex goes back to Steenrod [67], where it is shown for $k = 1$ only $S^2$ and $S^6$ are almost complex. For any $n$, $\widetilde{G}_{n,2}$ is almost complex [10]. Culminating recent progress by Sankaran [58] and Tang Zi-Zhou [71], the $K^*(\widetilde{G}_{n,k})$ calculation gives the following complete solution to this question.

**THEOREM.** *For $k, l \geq 3$, no $\widetilde{G}_{n,k}$ is almost complex and only $\widetilde{G}_{6,3}$ is weakly almost complex.*

As a corollary, it is not hard to obtain the following theorem for the oriented flag manifolds.

**THEOREM.** *Let $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$, $s \geq 3$, $n = \sum n_i$, and $M = \widetilde{G}(n_1, \ldots, n_s)$. There is a weak almost complex structure on $M$ if and only if $n_1 \leq 3$ or $n_2 \leq 2$.*

Let us now turn to the span of the Grassmann manifolds $G_{n,k} = G(k,l)$. From the Euler characteristic and Hopf criterion one sees that span $G(k,l) > 0$ if and only if $k, l$

are both odd. Nevertheless it will be useful to consider $G(k,l)$ for all parities of $k,l$, since we wish to use stable span. In fact the obvious inclusions $G(k-1,l) \subset G(k,l)$ and $G(k-1,l-1) \subset G(k,l)$ lead to the following easy but useful result (see Korbaš, Zvengrowski [42]).

**PROPOSITION.** Stable span $G(k,l) \leq n-1+$ stable span $G(k-1,l)$, *and*

$$\text{stable span } G(k,l) \leq n-1+\text{stable span } G(k-1,l-1).$$

*Remark*: All the above comments and the proposition hold for $\widetilde{G}_{n,k}$ as well as $G_{n,k}$, and the same is true for the next proposition, due to Leite and Miatello [47] (a short proof can be found in [42; p. 14]), which is useful in getting a lower bound for span $G_{n,k}$.

**PROPOSITION.** *For $k,l$ both odd,* span $G_{n,k} \geq \operatorname{span} S^{n-1} = \varrho(n)-1$.

Before stating the next theorem, which partially makes use of the above ideas, we note that it is explained in more detail in Korbaš, Zvengrowski [42], and we define $\alpha(s)$, for $s \in \mathbb{N}$, as the number of 1's in the dyadic expansion of $s$, and introduce $a_i$ recursively by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 7$, $a_{i+4} = 8 + a_i$ and $c_i$ by $c_0 = 1, c_1 = 2, c_2 = 3, c_3 = 7, c_{i+4} = c_i + 8$.

**THEOREM.** *(1) If $\alpha(s)+\alpha(t)-\alpha(s+t)=0$, then both the span of $\tilde{G}(4s+1,4t+1)$ and the span of $G(4s+1,4t+1)$ are 1. For the stable span we then have*

$$\begin{aligned} \text{stable span } \tilde{G}(4s+1,4t+1) &\leqq 4(s+t)+2, \text{ and} \\ \text{stable span } G(4s+1,4t+1) &\leqq 4(s+t)+1. \end{aligned}$$

*(2) If $\alpha(s)+\alpha(t)-\alpha(s+t)=i$ for some $i \geqq 1$, then the stable span and the span of $\tilde{G}(4s+1,4t+1)$ coincide. Moreover, we have then*

$$\begin{aligned} 2 \leqq \text{span } \tilde{G}(4s+1,4t+1) &\leqq 4(s+t)+1+\text{stable span } \tilde{G}(4s,4t) \text{ and} \\ &\text{stable span } \tilde{G}(4s,4t) \leqq c_i, \quad \text{together with} \\ 1 \leqq \text{(stable) span } G(4s+1,4t+1) &\leqq 4(s+t)+1+\text{stable span } G(4s,4t) \text{ and} \\ &\text{stable span } G(4s,4t) \leqq a_i. \end{aligned}$$

*(3) The stable span and the span of $\tilde{G}(4s+3,4t+3)$ always coincide. In addition to this,*

$$\begin{aligned} 2 &\leqq \text{span } \tilde{G}(4s+3,4t+3) \leqq 8(s+t+1)+\text{stable span } \tilde{G}(4s+1,4t+1), \\ 1 &\leqq \text{(stable) span } G(4s+3,4t+3) \leqq 8(s+t+1)+\text{stable span } G(4s+1,4t+1). \end{aligned}$$

*(4) Each of the manifolds $\tilde{G}(4s+1,4t+3)$, $\tilde{G}(4s+3,4t+1)$, $G(4s+1,4t+3)$ and $G(4s+3,4t+1)$ has span at least $\rho(4(s+t+1))-1(\geq 3)$.*

*If* $\alpha(s)+\alpha(t)-\alpha(s+t)=i$ *for some* $i \geqq 0$, *then we have*

*(stable) span* $\tilde{G}(4s+1,4t+3) \leqq 4(s+t)+3+$ *stable span* $\tilde{G}(4s,4t+2)$ *and*

*stable span* $\tilde{G}(4s,4t+2) \leqq c_i$, *together with*

*(stable) span* $\tilde{G}(4s+3,4t+1) \leqq 4(s+t)+3+$ *stable span* $\tilde{G}(4s+2,4t)$ *and*

*stable span* $\tilde{G}(4s+2,4t) \leqq c_i$.

*If* $\alpha(s)+\alpha(t)-\alpha(s+t)=0$, *then one has*

$$\text{(stable) span } G(4s+1,4t+3) \leqq 4(s+t)+3 \text{ and}$$
$$\text{(stable) span } G(4s+3,4t+1) \leqq 4(s+t)+3.$$

*If* $\alpha(s)+\alpha(t)-\alpha(s+t)=i$ *for some* $i \geqq 1$, *then one has*

$$\text{(stable) span } G(4s+1,4t+3) \leqq 4(s+t)+3+a_i \text{ and}$$
$$\text{(stable) span } G(4s+3,4t+1) \leqq 4(s+t)+3+a_i.$$

*(5) If* $u \geqq 1$, *then stable span* $G(3,4\cdot 2^u+1)=$ *span* $G(3,4\cdot 2^u+1)=3$. *If* $m$ *is* 0 *or* $2^u$ *for* $u \geqq 1$, *then both the stable span and the span of* $G(3,8m+5)$ *are* 7.
*(6) Span* $G(3,3)=7$, stable span $G(2,2)=2$.
*(7) If* $\binom{n}{r}$ *is odd, then stable span* $G(r,n-r)=0$.

*Remark*: While this theorem provides much information about span $G_{n,k}$ and span $\widetilde{G}_{n,k}$, and even gives the exact span for several infinite families, it is nevertheless fairly weak in many cases. For example it gives span $G(3,5)=7$, but only $1 \leq$ span $G(3,7) \leq 17$. One can in fact show span $G(3,7) \leq 7$ by calculating $w_{14} \neq 0$ (using formulas of Korbaš [38]).

**Projective Stiefel Manifolds.**

To date the most accurate results on span have been obtained for the projective Stiefel manifolds although a reasonable conjecture for span $X_{n,r}$ has yet to be found. The results are of importance for line bundles over any $CW$-complex because of a certain universal property of $X_{n,r}$ (given as the next proposition). The relevant vector bundles are $\tau(X_{n,r})=\tau_{n,r}$ (of dimension = dim $X_{n,r}=-\binom{r+1}{2}+nr$), the obvious orthogonal complement bundle $\beta_{n,r}$ of dimension $n-r$, the canonical line bundle $\xi_{n,r}$ associated to the double cover $V_{n,r}\to X_{n,r}$ and $\beta'_{n,r}=\beta_{n,r}\otimes\xi_{n,r}$. If no confusion is possible we omit $n,r$, and also write $\xi_{n,1}=\xi_{n-1}$ for the special case of $X_{n,1}=\mathbb{R}\mathrm{P}^{n-1}$.

The basic relations among these bundles are as follows:

(i) $\beta_{n,r}\oplus r\xi=n\varepsilon^1$, $\beta'_{n,r}\oplus r\varepsilon^1=n\xi_{n,r}$,
(ii) $p^*\beta'_{n,r}=\beta'_{n,r+1}\oplus\varepsilon^1$, where $p: X_{n,r+1}\to X_{n,r}$ is the standard fibration,
(iii) $\tau_{n,r+1}=p^*\tau_{n,r}\oplus\beta'_{n,r+1}$,
(iv) $\tau_{n,r}=r\beta'_{n,r}\oplus\binom{r}{2}\varepsilon^1$,
(v) $\tau_{n,r}\oplus\binom{r+1}{2}\varepsilon^1=nr\xi_{n,r}$,
(vi) $\tau_{n,r}\oplus\lambda^2\beta_{n,r}=\binom{n}{2}\varepsilon^1$.

These relations were first derived in Lam [45] and Zvengrowski [77]. The second relation in (i) shows that $n\xi$ always admits $r$ sections over $X_{n,r}$. Indeed this is universal for this property, as the following result ([21], [66]) shows.

**PROPOSITION.** *For any finite CW-complex $Y$ and line bundle $\eta$ over $Y$, $n\eta$ admits $r$ sections if and only if there is a map $f: Y \to X_{n,r}$ such that $f^*(\xi_{n,r}) = \eta$.*

Relations (ii) and (iii) describe the pull backs under $p$; (iv) shows that $X_{n,r}$ has a strong almost tangent structure of order $r-1$ and type $n-r$, as defined in Part One above. It also shows span $X_{n,r} \geq \binom{r}{2}$, a lower bound which is weak unless $r$ is nearly equal to $n$. Relation (v) is extremely useful since it gives a simple form for the stable tangent bundle, while (vi) gives a normal bundle.

Combining (v) with the previous proposition gives the following:

**PROPOSITION.** *If stable span $X_{n,r} \geq t$, then there is a map $f: X_{n,r} \to X_{nr,\binom{r+1}{2}+t}$ such that $f^*(\xi_{nr,\binom{r+1}{2}+t}) = \xi_{n,r}$ (and conversely).*

This furnishes a useful source of upper bounds for the stable span, and hence also for span $X_{n,r}$. One simply shows that for $t$ too large such a map is impossible, using any convenient cohomology theory. The $\mathbb{Z}_2$-cohomology of $X_{n,r}$ together with Steenrod operations is known through the work of Baum, Browder [8], Gitler, Handel [21], and Antoniano [2], while the $K$-theory is known through Antoniano, Gitler, Ucci, and Zvengrowski [3], and Barufatti, Hacon [7].

On the other hand (v) can also be used to furnish a lower bound for the stable span of $X_{n,r}$, as follows.

**DEFINITION.** $k_{n,r} = -\binom{r+1}{2} + \mathrm{span}(nr\xi_{n-1})$.

We recall (Part One) that the span of multiples of the line bundle $\xi_{n-1}$ over $\mathbb{R}P^{n-1}$ is a well studied problem, under the name of the generalized vector field problem, and $k_{n,r}$ is thus computable, e.g. from Lam's tables [44].

**PROPOSITION.** *Stable span $X_{n,r} \geq k_{n,r}$.*

The proof is immediate from isomorphism (v) and

$$\mathrm{span}(nr\xi_{n,r}) = \mathrm{span}(nrp^*(\xi_{n-1})) \geq \mathrm{span}(nr\xi_{n-1}).$$

Unfortunately a lower bound for stable span does not in general imply anything for the span, but using theorems from [43; §20] it is not hard to prove the following:

**LEMMA.** *One has* span $X_{n,r} =$ stable span $X_{n,r}$ *for:*

(a) *$n$ even, $r \equiv 0, 2, 3, 4, 7 \pmod 8$,*
(b) *$n$ odd, $r \equiv 0, 1, 4, 5, 6 \pmod 8$,*
(c) $(n, r) = (4m, 8q+5)$, $(4m, 16p+6)$, $(8m, 16p+9)$, $(4m+2, 8q+1)$, $(8n-1, 16p+7)$.

*Remark*: The lemma is also trivially true whenever $X_{n,r}$ is parallelizable, and for $r = 1$ by [33].

It is now plausible to make two conjectures.
(A) span $X_{n,r}$ = stable span $X_{n,r}$ for all $(n,r)$,
(B) span $X_{n,r} \geq k_{n,r}$.
The above lemma shows (A) to be true at least 70% of the time, in an obvious asymptotic sense, and the proposition before that shows $(A)$ implies $(B)$. However we now give an elementary proof of big part of $(B)$ using a destabilization trick.

**THEOREM.** *Except possibly for* $r = 2$ *and* $n$ *odd,* span $X_{n,r} \geq k_{n,r}$.

*Proof.* The case $r = 1$ and the case $r = 2$ with $n$ even are already proved in the above lemma and the remark following; let us consider $r \geq 3$. Then we have $X_{n,r} \overset{p}{\twoheadrightarrow} X_{n,r-1} \overset{q}{\twoheadrightarrow} X_{n,1} = \mathbb{R}\mathrm{P}^{n-1}$, and write $\pi = q \circ p$, also $r - 3 \geq 0$. From (ii)

$$p^*(r\beta'_{n,r-1}) = rp^*(\beta'_{n,r-1}) = r\beta'_{n,r} \oplus r\varepsilon^1,$$

hence (using (iv))

$$\tau_{n,r} = p^*(r\beta'_{n,r-1}) \oplus \frac{r(r-3)}{2}\varepsilon^1 \quad (\text{note } r + \frac{r(r-3)}{2} = \binom{r}{2}). \tag{1}$$

By definition, for some bundle $\alpha$ over $\mathbb{R}\mathrm{P}^{n-1}$,

$$nr\xi_{n-1} = (\binom{r+1}{2} + k_{n,r})\varepsilon^1 \oplus \alpha. \tag{2}$$

Now, multiplying (i) by $r$,

$$r\beta'_{n,r-1} \oplus r(r-1)\varepsilon^1 = nr\xi_{n,r-1} = q^*(nr\xi_{n-1}),$$

$$r\beta'_{n,r-1} \oplus r(r-1)\varepsilon^1 = (\binom{r+1}{2} + k_{n,r})\varepsilon^1 \oplus q^*\alpha. \tag{3}$$

Formula (3) is in dim $nr$ and can be destabilized to dim $X_{n,r-1}+1$. Now $nr - \binom{r+1}{2} =$ dim $X_{n,r} \geq$ dim $X_{n,r-1} + 1$, hence we can cancel $\binom{r+1}{2}\varepsilon^1$ and obtain

$$r\beta'_{n,r-1} \oplus \frac{r(r-3)}{2}\varepsilon^1 = k_{n,r}\varepsilon^1 \oplus q^*(\alpha). \tag{4}$$

Taking $p^*$ of (4) and using (1), $\tau_{n,r} = k_{n,r}\varepsilon^1 \oplus \pi^*(\alpha)$. □

As a corollary we now have the following information on lower bounds (adding the obvious Hurwitz-Radon lower bound to the previous ones).

**COROLLARY.** *(a)* Span $X_{n,2} \geq 1$. *(b) Except possibly for* $r = 2$ *and* $n$ *odd* span $X_{n,r} \geq \max\{\binom{r}{2}, \varrho(n) - 1, k_{n,r}\}$.

Using this source of lower bounds, and the previously mentioned source of upper bounds with $H^*(\ ;\mathbb{Z}_2)$ (and primary and secondary cohomology operations), or with

$K(\ )$ (also using the ring structure here), very accurate results for span $X_{n,r}$ are possible. We remark that Corollary (a) can be improved to span $X_{n,2} \geq 4$, for $n \geq 4$.

Because of the fundamental nature of the line bundle $\xi_{n,r}$, its $K$-theoretic order is important to know. From the recent work of Barufatti, Hacon [7], the "*complex*" order is now known to be $2^{a(n,r)}$ (meaning $2^{a(n,r)} \cdot (c\xi_{n,r})$ is stably trivial), where $a(n,r)$ is given as follows. Let $k = [\frac{n-r}{2}]$, $n = 2m$ or $n = 2m+1$. For $n$ even or $r$ even, $a(n,r) = \min\{[\frac{n-1}{2}], 2j-1+\nu_2\binom{m}{j} : j \geq k+1\}$. For $n, r$ both odd, $a(n,r) = \min\{[\frac{n-1}{2}], 2k+\nu_2\binom{m}{k}, 2j-1+\nu_2\binom{m}{j} : j \geq k+1\}$. Here, for any integer $t \neq 0$, $\nu_2(t)$ is the largest exponent such that $t$ is divisible by $2^{\nu_2(t)}$.

Now let $2^{b(n,r)}$ denote the "*real*" order of $\xi_{n,r}$ (meaning $2^{b(n,r)} \cdot \xi_{n,r}$ is stably trivial). From the well known fact that complexification followed by realification is just multiplication by 2, it follows that $b(n,r) = a(n,r) + \varepsilon(n,r)$, where $\varepsilon = 0$ or $\varepsilon = 1$. It would be of great interest to determine all $\varepsilon(n,r)$, and the following recent result of Sankaran and Zvengrowski will do this about 70% of the time.

**DEFINITION.** *If* $\min\{2j-1+\nu_2\binom{m}{j} : j \geq k+1\} = a(n,r)$ *when $n$ or $r$ is even, or* $\min\{2k+\nu_2\binom{m}{k}, 2j-1+\nu_2\binom{m}{j} : j \geq k+1\} = a(n,r)$, *when $n, r$ both are odd, we will say $(n,r)$ is in the upper range (or $r$ is in the upper range for $n$).*

**THEOREM.** *For $(n,r)$ in the upper range, or for $n \equiv 0, \pm 1 \pmod 8$, $a(n,r) = b(n,r)$ (i.e. $\varepsilon(n,r) = 0$).*

*Remark*: $\varepsilon(n,r)$ can certainly equal 1, e.g. when $r = 1$ and $n \not\equiv 0, \pm 1 \pmod 8$.

Using this theorem has led to the complete determination of span $X_{n,n-s}$, for $s = 1, 2, 3$. We will just quote the results, writing $d = \dim X_{n,r}$.

**THEOREM.** *(a) The manifolds $X_{n,n-1}$ and $X_{2m,2m-2}$ are parallelizable.*

*(b)* st. span $X_{n,n-2} = d - 2^a$ *when* $n = 2^a \times (\textit{odd}) + 1, a \geq 1$.

*(c)* st. span $X_{n,n-3} = \begin{cases} d-2, & n \equiv 0, 3 \pmod 4 \\ d+2-2^a, & n = 2^a \times (\textit{odd}) + b, b = 1 \text{ or } b = 2, a \geq 2. \end{cases}$

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