Christian Gross

Cohomology and connection on $S1$-bundles

In: Jan Slovák (ed.): Proceedings of the 15th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1996. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 43. pp. [123]--132.

Persistent URL: http://dml.cz/dmlcz/701581

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 123-132

COHOMOLOGY AND CONNECTION ON $\mathbb{S}^1$-BUNDLES

Christian Gross

1. INTRODUCTION

Given any fiber bundle $B(M, F, G)$, the projection $\pi: B \to M$ induces an homomorphism of the DE-RHAM cohomology groups $\pi^*: H^*(M) \to H^*(B)$, since the exterior derivative d of differential forms commutes with pullbacks. Nevertheless this homomorphism neither needs to be injective nor surjective, as the example of the HOPF fibration $\pi: \mathbb{S}^3 \to \mathbb{S}^2$ shows. In general, spectral sequences are needed to construct the cohomology of the bundle from those of the base and the fiber, and to answer the question whether some closed $\omega \in \mathcal{A}(F)$ lives to the bundle and one thus finds $[\tilde{\omega}] \in H^*(B)$, such that ω is the restriction of $\tilde{\omega}$ to the fibers.

Now let any connection Γ on the associated principal bundle $P(M, G)$ be given. Γ defines horizontal and vertical projections of differential forms on any associated bundle $B(M, F, G)$. It is only natural to ask if — given any $\omega \in \mathcal{A}(F)$ that lives to the bundle — such $\tilde{\omega}$ can be found, which is anyhow adapted to Γ. In view of this question we will prove an answer for $G \cong \mathbb{S}^1$: given any left LIE group action $L: \mathbb{S}^1 \times F \to F$, such an adapted $\tilde{\omega}$ exists for any $\mathbb{S}^1$-invariant $\omega \in \mathcal{A}(F)$ only if ω lives to any $\mathbb{S}^1$-bundle, that comes along with L, independently of the base manifold and the transition functions. On the other hand, $\tilde{\omega}$ exists, if the zig-zag produced by ω in the spectral sequence, is of a certain form.

Finally, we apply our results to the skyrmion bundle in theoretical nuclear physics, which generalizes the ungauged SKYRME model[8] in order ot treat interactions not only between baryons and mesons but also with electromagnetic fields, that are described by a MAXWELL connection on the associated principal bundle.

2. CONNECTIONS ON PRINCIPAL $\mathbb{S}^1$-BUNDLES

For any fiber bundle $B(M, F, G)$ with bundle manifold B, base manifold M, fiber F and LIE group G, let $\pi: B \to M$ denote the projection onto the base and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ a bundle atlas, where $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of M and $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F: b \mapsto (\pi(b), \pi_\alpha(b))$ define local projections $\pi_\alpha: \pi^{-1}(U_\alpha) \to F$ onto the fiber. For the left effective LIE group action $L: G \times F \to F$, we write $L_g(f) = \tau_f(g)$, where $L_g: F \to F$ and $\tau_f: G \to F$ are differentiable for all $g \in G$ and $f \in F$. For all $\alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, $g_{\alpha\beta}: U_{\alpha\beta} \to G$ means the C^∞-map defined by $g_{\alpha\beta}(x) := \pi_\alpha|_{\pi^{-1}(\{x\})} \circ (\pi_\beta|_{\pi^{-1}(\{x\})})^{-1}: F \to F$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Recall that any connection Γ on a principal bundle is uniquely defined by a connection 1-form $\omega^\Gamma \in \mathcal{A}_1(P, L(G))$. For $G \cong \mathbb{S}^1$ we can identify $L(G)$ and $\mathbb{R}$. Then the curvature 2-form $\Omega^\Gamma \in \mathcal{A}_2(P)$ simply reads $\Omega^\Gamma = d\omega^\Gamma$. If $\sigma_{\alpha,f}: U_\alpha \to \pi^{-1}(U_\alpha): x \mapsto \psi_\alpha^{-1}(x, f)$ denote local sections for all $\alpha \in A$ and $f \in F$, then ω^Γ and Ω^Γ define local 1-forms $A^\alpha \in \mathcal{A}_1(U_\alpha)$ and 2-forms $F^\alpha \in \mathcal{A}_2(U_\alpha)$ by

$$A^\alpha = \sigma^\star_{\alpha,0}(\omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_1(U_\alpha), \qquad F^\alpha = \sigma^\star_{\alpha,0}(\Omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_2(U_\alpha) \tag{2.1}$$

(we write the group operation in $\mathbb{S}^1$ additively, so 0 is the neutral element). We have[7]

Theorem 2.1 *If Γ is a connection on $P(M, \mathbb{S}^1)$ and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ is a bundle atlas for P, then for all $\alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ and for all $x \in U_{\alpha\beta}$:*

$$F^\alpha = dA^\alpha, \qquad dF^\alpha = 0, \tag{2.2}$$

$$A^\alpha|_{U_{\alpha\beta}} = A^\beta|_{U_{\alpha\beta}} + dg_{\beta\alpha} = A^\beta|_{U_{\alpha\beta}} - dg_{\alpha\beta}, \quad F^\alpha|_{U_{\alpha\beta}} = F^\beta|_{U_{\alpha\beta}} \tag{2.3}$$

Vice versa, if for a bundle atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on the principal bundle $P(M, \mathbb{S}^1)$ a family $\{A^\alpha \in \mathcal{A}_1(U_\alpha, \mathfrak{g})\}_{\alpha \in A}$ is given such that (2.3) holds, then there exists one unique connection Γ on $P(M, \mathbb{S}^1)$ such that $A^\alpha = \sigma^\star_{\alpha,0}(\omega^\Gamma|_{\pi^{-1}(U_\alpha)})$ for all $\alpha \in A$.

So the F^α constitute a global $F \in \mathcal{A}_2(M)$.

Every connection Γ on $P(M, G)$ induces horizontal and vertical projections h, v of vector fields and differential forms on every associated bundle $B(M, F, G) = P \times_G F$. For $\mathbb{S}^1$-invariant differential forms χ on F, i. e. $L^\star_g \chi = \chi$ for all $g \in \mathbb{S}^1$, we obtain:

Theorem 2.2 *Let Γ be a connection on $P(M, \mathbb{S}^1)$ and $B(M, F, \mathbb{S}^1)$ an associated bundle. For any $\mathbb{S}^1$-invariant $\chi \in \mathcal{A}_n(F)$ define $\nu \in \mathcal{A}_{n-1}(F)$ by*

$$\nu_f(F_f^{(1)}, \ldots, F_f^{(n-1)}) := \chi_f(F_f^{(1)}, \ldots, F_f^{(n-1)}, d\tau_f(1)) \quad \textit{for all } f \in F,\ F^{(i)} \in \mathcal{D}^1(F).$$

For any $U_\alpha \in \mathfrak{U}$ denote $\chi^\alpha := \pi^\star_\alpha \chi$, $\nu^\alpha := \pi^\star_\alpha \nu$. Then on all $U_{\alpha\beta} \neq \emptyset$

$$\begin{aligned} \chi^\alpha &= \chi^\beta + dg_{\alpha\beta} \wedge \nu, \qquad \chi^\alpha v = \chi^\alpha + A^\alpha \wedge \nu = \chi^\beta + A^\beta \wedge \nu = \chi^\beta v, \\ \nu^\alpha &= \nu^\alpha v = \nu^\beta = \nu^\beta v =: \nu. \end{aligned}$$

Thus χv and ν define global $\mathbb{S}^1$-invariant, vertical forms on B. $d\chi = 0$ yields $d\nu = 0$, too.

So vertical projection $v: \mathcal{A}(B) \to \mathcal{A}(B)$ on the bundle not only maps global forms to global forms but also these *locally* embedded $\mathbb{S}^1$-invariant forms on the fiber to *global* vertical forms on the bundle. This result remains true for any G-bundle, yet the projected forms do not look that simple any more, but require higher powers of A^α.

3. SPECTRAL SEQUENCES

For a countable ordered *good* cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M (which means that all finite intersections $U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$, $p \in \mathbb{N}_0$ are diffeomorphic to $\mathbb{R}^n$), let $C(\pi^{-1}\mathfrak{U}, \mathcal{A})$ denote the ČECH-DE-RHAM double complex[2] (Figure 1)

$$C(\pi^{-1}\mathfrak{U}, \mathcal{A}) := \bigoplus_{p,q \in \mathbb{N}_0} C^p(\pi^{-1}\mathfrak{U}, \mathcal{A}_q), \quad \text{where} \quad C^p(\pi^{-1}\mathfrak{U}, \mathcal{A}_q) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{A}_q(\pi^{-1}(U_{\alpha_0 \cdots \alpha_p})).$$

Figure 1: The ČECH-DE-RHAM complex for a bundle B

q			
$0 \to \mathcal{A}_2(B) \xrightarrow{r}$	$\prod_{\alpha_0} \mathcal{A}_2(\pi^{-1}U_{\alpha_0})$	$\prod_{\alpha_0<\alpha_1} \mathcal{A}_2(\pi^{-1}U_{\alpha_0\alpha_1})$	$\prod_{\alpha_0<\alpha_1<\alpha_2} \mathcal{A}_2(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2})$
$0 \to \mathcal{A}_1(B) \xrightarrow{r}$	$\prod_{\alpha_0} \mathcal{A}_1(\pi^{-1}U_{\alpha_0})$	$\prod_{\alpha_0<\alpha_1} \mathcal{A}_1(\pi^{-1}U_{\alpha_0\alpha_1})$	$\prod_{\alpha_0<\alpha_1<\alpha_2} \mathcal{A}_1(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2})$
$0 \to \mathcal{A}_0(B) \xrightarrow{r}$	$\prod_{\alpha_0} \mathcal{A}_0(\pi^{-1}U_{\alpha_0})$	$\prod_{\alpha_0<\alpha_1} \mathcal{A}_0(\pi^{-1}U_{\alpha_0\alpha_1})$	$\prod_{\alpha_0<\alpha_1<\alpha_2} \mathcal{A}_0(\pi^{-1}U_{\alpha_0\alpha_1\alpha_2})$ $\to p$

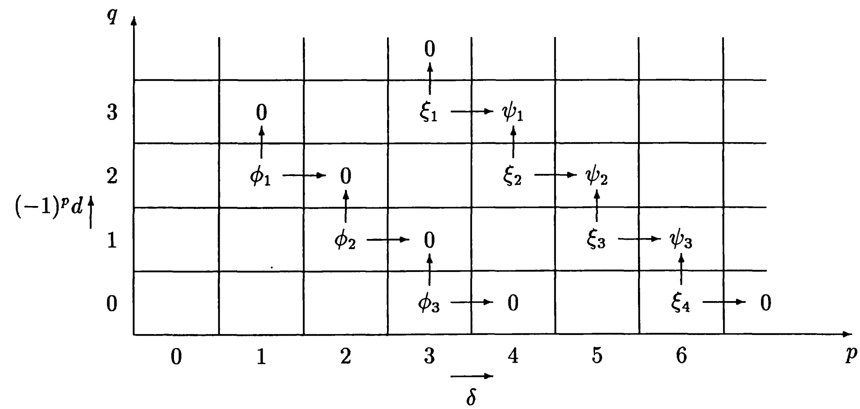
We have two commuting differential operators: on the vertical lines we have the exterior derivative d, and on the horizontal lines we have δ defined by ("$\hat{\ }$" denotes omission)

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} := \sum_{j=0}^{p+1}(-1)^j \omega_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_{p+1}}|_{\pi^{-1}(U_{\alpha_0\cdots\alpha_{p+1}})} \quad \forall\omega = \prod_{\alpha_0<\cdots<\alpha_p} \omega_{\alpha_0\cdots\alpha_p} \in \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{A}(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})).$$

$D := D' + D''$, with $D' := \delta$ and $D'' := (-1)^p d$, is the differential operator for the single (graded) complex with $C(\pi^{-1}\mathfrak{U}, \mathcal{A})^n = \bigoplus_{p+q=n} C^p(\pi^{-1}\mathfrak{U}, \mathcal{A}_q)$. Figure 2 shows a D-closed element $\Phi = \phi_1 + \phi_2 + \phi_3$ and a D-exact element $\Psi = \psi_1 + \psi_2 + \psi_3 = D\Xi$, where $\Xi = \xi_1 + \xi_2 + \xi_3 + \xi_4$:

$$\begin{aligned} D\Phi = 0 \quad &\Longleftrightarrow \quad d\phi_1 = 0,\ \delta\phi_1 + d\phi_2 = 0,\ \ \delta\phi_2 - d\phi_3 = 0,\ \ \delta\phi_3 = 0, \\ \Psi = D\Xi \quad &\Longleftrightarrow \quad d\xi_1 = 0,\ \delta\xi_1 + d\xi_2 = \psi_1,\ \delta\xi_2 - d\xi_3 = \psi_2,\ \delta\xi_3 + d\xi_4 = \psi_3,\ \delta\xi_4 = 0. \end{aligned}$$

Figure 2: D-closed and D-exact elements in a double complex



The possibility to compute $H^*(B)$ by spectral sequences relies on the fact that the D-cohomology of the ČECH-DE-RHAM complex and the DE-RHAM cohomology of B are isomorphic:[2]

Theorem 3.1 (Mayer-Vietoris Principle) *Let $\pi^{-1}\mathfrak{U}$ be an open cover of B, then the restriction map $r: \mathcal{A}(B) \to \prod_\alpha \mathcal{A}(\pi^{-1}(U_\alpha)) \subseteq C(\pi^{-1}\mathfrak{U}, \mathcal{A})$ induces an isomorphism:*

$$r^\star: H^*(B) \to H^*_D(C(\pi^{-1}\mathfrak{U}, \mathcal{A})), \quad H^n(B) \to H^n_D(C(\pi^{-1}\mathfrak{U}, \mathcal{A})).$$

The inverse map that collates together the components of an element in the Čech-de-Rham complex into a global form on B is less intuitive. For any partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\mathfrak{U}$ define $K: C^p(\pi^{-1}\mathfrak{U}, \mathcal{A}_q) \to C^{p-1}(\pi^{-1}\mathfrak{U}, \mathcal{A}_q)$ by

$$(K\omega)_{\alpha_0\cdots\alpha_{p-1}} := \sum_{\alpha\in A} \rho_\alpha \, \omega_{\alpha\alpha_0\cdots\alpha_{p-1}}, \tag{3.1}$$

then we have $K\delta + \delta K = \text{id}$ and the following Collating formula:[2]

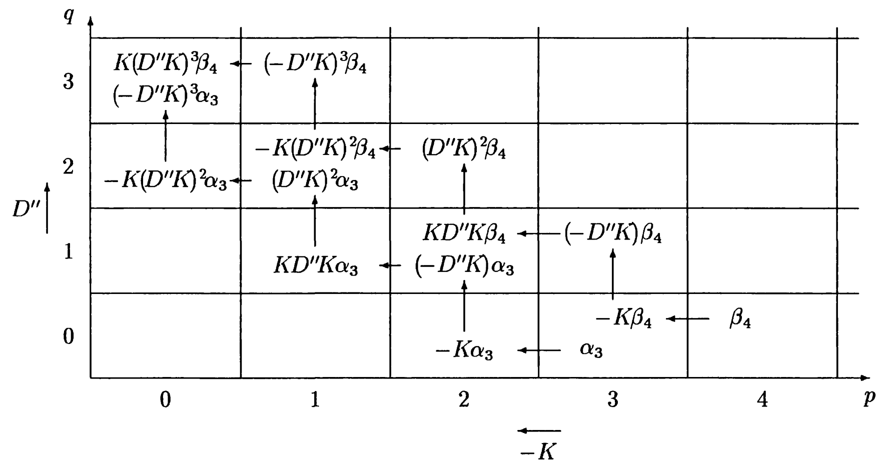
Theorem 3.2 *Let $\alpha = \sum_{i=0}^n \alpha_i \in C(\pi^{-1}\mathfrak{U}, \mathcal{A})^n$ with $\alpha_i \in C^i(\pi^{-1}\mathfrak{U}, \mathcal{A}_{n-i})$ and $D\alpha = \beta = \sum_{i=0}^{n+1} \beta_i$ with $\beta_i \in C^i(\pi^{-1}\mathfrak{U}, \mathcal{A}_{n+1-i})$, and define K by (3.1). Then*

$$f(\alpha) := \sum_{i=0}^n (-D''K)^i \alpha_i - \sum_{i=0}^n K(-D''K)^i \beta_{i+1} \in C^0(\pi^{-1}\mathfrak{U}, \mathcal{A}_n)$$

is a global form on B (resp., the restriction of such a form to the sets $\pi^{-1}(U_\alpha)$). The induced maps $f^\star$ and $r^\star$ on the cohomology level are inverse isomorphisms.

Figure 3 illustrates how the components α_3 and β_4 of α and β are mapped onto elements in $C^0(\pi^{-1}\mathfrak{U}, \mathcal{A}_3)$. For a global form on B, all components α_i and β_i must be mapped like this.

Figure 3: Illustration of the Collating formula



For any double complex the sequence $K_p := \bigoplus_{i\geq p} \bigoplus_{q\geq 0} K^{i,q}$, $p \in \mathbb{N}_0$ is a filtration by the columns of K with associated graded complex

$$GK = \bigoplus_{p\in\mathbb{N}_0} K_p/K_{p+1} = \bigoplus_{p\in\mathbb{N}_0} \left[\left(\bigoplus_{q\geq 0} K^{p,q} \right) + K_{p+1} \right].$$

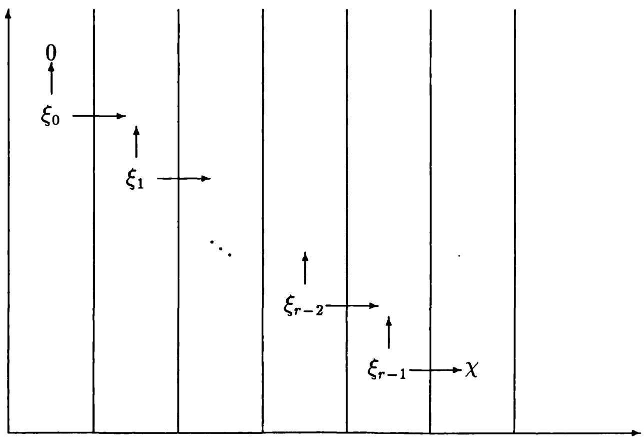
Obviously, the induced differential operator on $GC(\pi^{-1}\mathfrak{U}, \mathcal{A})$ is just $(-1)^p d$.

Let $\{E_r, D_r\}_{r\in\mathbb{N}_0}$ denote the spectral sequence for the ČECH-DE-RHAM complex: $E_0 = GC(\pi^{-1}\mathfrak{U}, \mathcal{A})$ and $E_{r+1} = H^*_{D_r}(E_r)$, where $D_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ is the differential operator induced by D on E_r. If E_R becomes stationary, i. e. $E_r = E_{r+1}$ for all $r \geq R$, we denote E_R by E_∞ and say that the spectral sequence converges to some filtered complex H if $E_\infty \cong GH$. $\beta \in C(\pi^{-1}\mathfrak{U}, \mathcal{A})$ "lives to" E_r iff it represents a cohomology class $[\beta]_r \in E_r$, i. e. if β is D_i-closed in $E_0, \ldots, E_{r-1}$. Then β is d-closed and we get a "zig-zag" $\Xi = \xi_0 + \ldots + \xi_{r-1}$ of elements $\xi_i \in C(\pi^{-1}\mathfrak{U}, \mathcal{A})$ with $\xi_0 := \beta$ and

$$D'\xi_i = \delta\xi_i = -D''\xi_{i+1}, \qquad i = 0, \ldots, r-2 \tag{3.2}$$

(cf. Figure 4). Since $D_r[\beta]_r = [\delta\xi_{r-1}]_r = [\chi]_r$, D_r is given by δ at the end of the zig-zag.

Figure 4: Illustration of the Differential operator D_r: $D_r[\xi_0]_r = [\delta\xi_{r-1}]_r = [\chi]_r$



Now LERAY's theorem states that $\{E_r, D_r\}_{r\in\mathbb{N}_0}$ converges to $H^*(B)$:[2, 9]

Theorem 3.3 (Leray's Theorem) *If $B(M, F, G)$ is a fiber bundle and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ is a good cover of M then there is a spectral sequence converging to $H^*(B)$ with E_1 term*

$$E_1^{p,q} = \prod_{\alpha_0 < \cdots < \alpha_p} H^q(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})) \cong \prod_{\alpha_0 < \cdots < \alpha_p} H^q(F).$$

If $H^*(F)$ is finitely generated and in addition M simply connected or $B \cong M \times F$, then

$$E_2^{p,q} = H^p(M, H^q(F)) \cong H^p(M) \otimes H^q(F) \oplus \mathrm{Tor}[H^{p+1}(M), H^q(F)].$$

Finally, we find the following result:[2]

Theorem 3.4 *For any closed form $\omega \in A^q(F)$, there exists a closed $\tilde{\omega} \in A^q(B)$, such that ω is the restriction of $\tilde{\omega}$, iff ω lives to E_{q+2}, i. e. iff a zig-zag $\Xi = \xi_0 + \ldots + \xi_q$ of elements $\xi_i \in C^i(\pi^{-1}\mathfrak{U}, \mathcal{A}_{q-i})$ exists, with $\xi_0 = \omega_q$ and $\delta\xi_q = 0$.*

4. Γ-ADAPTATION OF THE DIFFERENTIAL FORMS FOR $\mathbb{S}^1$-BUNDLES

Once a zig-zag for ω has been found, Theorem 3.2 exhibits representatives $\tilde{\omega}$ for the cohomology groups of the bundle. Yet we would like to obtain forms that are anyhow adapted to Γ. To this end, suppose $\Xi = \xi_0 + \cdots + \xi_r$ is a zig-zag for a closed $\mathbb{S}^1$-invariant $\omega_q \in \mathcal{A}_q(F)$, where every ξ_j is of the type ($\alpha_0/\cdots/\alpha_j$ indicates that one may use any trivialization)

$$(\xi_j)_{\alpha_0\cdots\alpha_j} = dg_{\alpha_j\alpha_{j-1}} \wedge \cdots \wedge dg_{\alpha_1\alpha_0} \wedge (\chi^j_{q-2j})^{\alpha_0/\cdots/\alpha_j}$$

with $\chi^j_{q-2j} \in \mathcal{A}_{q-2j}(F)$ and $d(\chi^{j+1}_{q-2(j+1)}) = \nu^j_{q-2j}$ from Theorem 2.2. Since ω_q is $\mathbb{S}^1$-invariant, all χ^j_{q-2j} can be chosen $\mathbb{S}^1$-invariant, otherwise use the HAAR measure $\int_{\mathbb{S}^1} L^*_g \chi^j_{q-2j}\, dg$ to achieve this. One checks that

$$(\delta\xi_j)_{\alpha_0\cdots\alpha_{j+1}} = dg_{\alpha_{j+1}\alpha_j} \wedge \cdots \wedge dg_{\alpha_1\alpha_0} \wedge \nu^j_{q-2j-1},$$

so Ξ indeed is a zig-zag for ω_q. ν^r_{q-2r-1} be zero such that χ^r_{q-2r} is global (and vertical) and $\delta\xi_r = D_{r+1}[\omega_q]_{r+1} = D\Xi = 0$. Then Theorem 2.1 and Theorem 2.2 yield that the following Γ-adapted differential form

$$(\omega^A_q)^\alpha := \sum_{j=0}^{r} \underbrace{F^\alpha \wedge \cdots \wedge F^\alpha}_{j} \wedge (\chi^j_{q-2j})^\alpha v \tag{4.1}$$

is the one we are looking for: it is global, it reproduces ω_q, when restricted to the fibers, and it is closed, because $d(\chi^j_{q-2j} v) = d\chi^j_{q-2j} + F \wedge \nu^j_{q-2j-1}$. Thus it represents $[\omega^A_q] \in H^q(B)$.

Since any principal fiber bundle over a (paracompact) manifold M admits a connection Γ,[7] we have proven the following theorem:

Theorem 4.1 *Let $L: \mathbb{S}^1 \times F \to F$ be a left* LIE *group action and $\omega_q \in \mathcal{A}_q(F)$ be $\mathbb{S}^1$-invariant. If $\chi^j_{q-2j} \in \mathcal{A}_{q-2j}(F)$, $j = 0, \ldots, r$, can be found, with $d(\chi^{j+1}_{q-2j-2}) = \nu^j_{q-2j-1}$ from Theorem 2.2, $\chi^0_q = \omega_q$ and $\nu^r_{q-2r-1} = 0$, then for any fiber bundle $B(M, F, \mathbb{S}^1)$, that comes along with L, independently of the base manifold M and its transition functions, ω lives to B and defines a cohomology class in $H^q(B)$. For any connection Γ on the associated principal bundle, $\omega^A_q \in \mathcal{A}_q(B)$ in (4.1) is a representative for that cohomology class.*

We conjecture that this condition for the existence of such an Γ-adapted representative is not only necessary but also sufficient. E. g., if ξ_1 with $(d\xi_1)_{\alpha\beta} = (\delta\xi_0)_{\alpha\beta} = dg_{\beta\alpha} \wedge \nu^0_{q-1}$ exists for *any* $\mathbb{S}^1$-bundle, then χ^1_{q-2} with $d\chi^1_{q-2} = \nu^0_{q-2j}$ exists and $(\xi_1)_{\alpha\beta} = dg_{\beta\alpha} \wedge (\chi^1_{q-2j})^{\alpha/\beta}$. Induction should show that this holds for any j.

5. APPLICATION TO THE SKYRMION BUNDLE

In order to apply our results to the skyrmion bundle in theoretical nuclear physics, let us briefly recall the main topological features of the SKYRME model[8] as an effective field theory related to quantum chromodynamics (QCD) by its underlying symmetry. In this ungauged SKYRME model, the meson fields π^a on space-time M generate differentiable functions $U: M \to \mathrm{SU}_{N_F}$ defined by (N_F denotes the number of flavors in QCD)

$$U = \exp(i \sum_{a=1}^{N_F^2-1} \pi^a \lambda_a) \qquad \text{with} \quad \lambda_a = (\lambda_a)^\dagger \in \mathbb{C}^{N_F \times N_F},\ \mathrm{Tr}(\lambda_a) = 0.$$

The vacuum is represented by the unit matrix $\mathbb{1} \in \mathrm{SU}_{N_F}$. Requiring $\pi^a(r) \to 0$ and thus $U(r) \to \mathbb{1}$ for $r \to \infty$ one can compactify euclidian space $\mathbb{R}^3$, resp., space-time $\mathbb{R}^4$, so that the meson fields constitute functions $U: \mathbb{R}_{(t)} \times \mathbb{S}^3 \to \mathrm{SU}_{N_F}$, resp., $U: \mathbb{S}^4 \to \mathrm{SU}_{N_F}$.

Let $L := U^{-1}\,dU = U^\dagger\,dU$ and $R := (dU)U^{-1} = (dU)U^\dagger \in \mathcal{A}_1(\mathrm{U}_m, \mathbb{C}^{m\times m})$ denote the left, resp., right invariant currents: $\mathbb{C}^{m\times m}$-valued 1-forms that are invariant under multiplication with constant elements of U_m from the left, resp., from the right and obey $L(\mathcal{X})(\mathbb{1}) = R(\mathcal{X})(\mathbb{1}) = X$ for all vector fields $\mathcal{X} \in \mathcal{D}^1(\mathrm{U}_m)$ with $\mathcal{X}(\mathbb{1}) = X \in \mathfrak{u}_m = L(\mathrm{U}_m)$. For any constant $Q \in \mathbb{C}^{m\times m}$, we define λ_k^Q and $\rho_k^Q \in \mathcal{A}_k(\mathrm{U}_m, \mathbb{C})$ by

$$\lambda_k^Q := \mathrm{Tr}(QL^k) := \mathrm{Tr}(Q\underbrace{L \wedge \cdots \wedge L}_{k}), \qquad \rho_k^Q := \mathrm{Tr}(QR^k) := \mathrm{Tr}(Q\underbrace{R \wedge \cdots \wedge R}_{k}),$$

These are left, resp., right invariant complex-valued k-forms on U_m; for $Q = \mathbb{1}$ we have

$$\omega_k := \lambda_k^{\mathbb{1}} = \rho_k^{\mathbb{1}} = \mathrm{Tr}(L^k) = \mathrm{Tr}(R^k) \in \mathcal{A}_k(\mathrm{U}_m, \mathbb{C}),$$

which are invariant under all multiplications. Obviously $\omega_{2l} = 0$. The forms ω_{2l+1} are closed since the MAURER-CARTAN identities $dL = -L \wedge L$, $dR = R \wedge R$ yield

$$\begin{aligned} dL^{2l+1} &= -L^{2l+2}, \quad dR^{2l+1} = R^{2l+2}, \qquad dL^{2l+2} = dR^{2l+2} = 0, &(5.1)\\ d(UL^{2l}) &= UL^{2l+1}, \quad d(L^{2l}U^\dagger) = -L^{2l+1}U^\dagger, \quad d(UL^{2l+1}) = d(L^{2l+1}U^\dagger) = 0. &(5.2)\end{aligned}$$

Moreover, ω_{2l+1} generate the DE-RHAM cohomology $H^*(\mathrm{SU}_m, \mathbb{C})$, resp., $H^*(\mathrm{U}_m, \mathbb{C})$.

In the SKYRME model, baryons appear as topological soliton solutions of the meson fields. Their number B can be computed by an integration over the space manifold:

$$B(U) = \int_{\mathbb{S}^3} -\frac{1}{24\pi^2} U^\star\omega_3 = \int_{\mathbb{R}^3} -\frac{1}{24\pi^2} \sum_{i,j,k=1}^{3} \mathrm{Tr}(L_iL_jL_k)\, dx^i \wedge dx^j \wedge dx^k. \qquad (5.3)$$

Compactification of space-time is crucial: normally there is no guarantee that the integral in (5.3) is an integer, but for spheres we have the following Index theorem:[1]

Theorem 5.1 *For every map $U: \mathbb{S}^{2n-1} \to \mathrm{U}_m$ the integral*

$$n(U) = \int_{\mathbb{S}^{2n-1}} \left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} U^\star\omega_{2n-1} \quad \textit{is an integer.}$$

The assignment $[U] \mapsto n(U): \pi_{2n-1}(\mathrm{U}_m) \to \mathbb{Z}$ is an isomorphism for $m \geq n$.

We are thus able to identify $\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!}\,\omega_{2n-1}$ with the generators of the integer valued cohomology of the unitary groups. At any time the meson fields respresent elements of the homotopy groups $\pi_3(\mathrm{U}_{N_F}) \cong \mathbb{Z}$ for $N_F \geq 2$; the integer characterizing the homotopy class is a topological invariant, the "topological charge" $B(U)$.

The vacuum map represents the zero element, and so $B(U \equiv \mathbb{1}) = 0$. For proton and neutron we have $B = 1$, for their antiparticles $B = -1$. Annihilation of proton and antiproton corresponds to the "addition" of their maps within the homotopy group and generates a mesonic field of topological charge $B = 0$.

The meson fields obey the field equations derived as EULER-LAGRANGE equations from a lagrangian $\mathcal{L}(U, dU)$ by variation of the action integral $\Gamma(U) = \int_{\mathbb{S}^4} \mathcal{L}\, dV$. The latter splits into two parts: one of them (N_C denotes the number of colors in QCD),

$$\Gamma_{AN}(U) = \lambda \int_{D^5} (U')^\star\omega_5 \qquad \text{with} \qquad \lambda = \frac{i\,N_C}{240\pi^2}, \qquad (5.4)$$

describes the anomalous processes of QCD: one uses $\pi_4(\mathrm{SU}_{N_F}) = 0$ and extends U to a differentiable map $U'\colon D^5 \to \mathrm{SU}_3$ from a five-dimensional disc D^5 whose boundary ∂D^5 is space-time $\mathbb{S}^4$. The topological quantization of the coupling constant λ in (5.4) is again a consequence of Theorem 5.1, and of the requirement that for any extension U' the result has to be unique.[10]

It is well known that electromagnetic fields can conveniently be described by a MAXWELL connection Γ on a principal U_1-bundle over space-time M. Since $P(M, \mathrm{U}_1) \cong P(M, \mathbb{S}^1)$, we can apply our results. Yet for compatibility reasons we will extract the electromagnetic charge e such that $ie\,\mathrm{U}_1 = \mathbb{S}^1$, resp., $ie\,\mathrm{u}_1 = \mathbb{R}$. Then the gauge potentials A^α and the gauge field F are related to the connection 1-form and the curvature 2-form as in (2.1) (apart from this additional factor ie). The homogeneous MAXWELL equations then simply take the form $dF = 0$, cf. (2.2). Non-triviality of the bundle is always related to the appearance of magnetic monopoles. In fact, observe that (2.3) yields that a global gauge potential A exists iff the bundle is trivial.

For the purpose of treating interactions between electromagnetic fields on the one hand and mesons, resp., baryons on the other hand, we have to gauge the SKYRME model and introduce the skyrmion bundle as follows:[4, 5] instead of considering maps $U\colon M \to \mathrm{SU}_n$ we now think of the meson fields as of global sections in a bundle $B(M, \mathrm{SU}_n, \mathrm{U}_1)$ associated to $P(M, \mathrm{U}_1)$. The left action of U_1 on SU_n is given by the inner automorphisms

$$L_g(U) \;=\; \tau_U(g) \;=\; e^{-iegQ}Ue^{+iegQ},$$

which do not effect the vacuum being diagonal symmetry operations. Q is the hermitian $n \times n$-matrix containing the quark charges in units of e (again $n = 2$, resp., 3)

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}, \qquad \text{resp.,} \qquad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

Thus the transition functions are $U^\alpha(x) = e^{-ieg_{\alpha\beta}(x)Q}\; U^\beta(x)\; e^{+ieg_{\alpha\beta}(x)Q}$. So not only vacuum $U \equiv \mathbb{1}$ is a global section but every $U(x) = e^{i\chi(x)Q}$ with a differentiable map $\chi\colon M \to \mathbb{S}^1$. Since ω_{2l+1}, ρ_l^Q and λ_l^Q are U_1-invariant and $d\tau_U(X) = -ieX\,[Q, U]$ for all $X \in \mathrm{u}_1$, Theorem 2.2 yields:

Lemma 5.2 *$\omega_{2l+1}v$, $\rho_l^Q v$ and $\lambda_l^Q v$ for $l \in \mathrm{N}_0$ are global forms on B and we have:*

$$\begin{aligned}
\omega^\alpha_{2l+1}v &= \omega^\alpha_{2l+1} - (2l+1)ie\, A^\alpha \wedge (\rho^Q_{2l} - \lambda^Q_{2l}),\\
(\rho^Q_{2l})^\alpha v &= (\rho^Q_{2l})^\alpha - ie\, A^\alpha \wedge \sum_{j=1}^{2l}(-1)^j\, \mathrm{Tr}(QUL^{j-1}QL^{2l-j}U^\dagger)^\alpha,\\
(\lambda^Q_{2l})^\alpha v &= (\lambda^Q_{2l})^\alpha - ie\, A^\alpha \wedge \sum_{j=1}^{2l}(-1)^j\, \mathrm{Tr}(QUL^{j-1}QL^{2l-j}U^\dagger)^\alpha,\\
(\rho^Q_{2l+1})^\alpha v &= (\rho^Q_{2l+1})^\alpha - ie\, A^\alpha \wedge \sum_{j=1}^{2l+1} \mathrm{Tr}(QR^{j-1}QR^{2l+1-j} - QL^{j-1}U^\dagger QUL^{2l+1-j})^\alpha,\\
(\lambda^Q_{2l+1})^\alpha v &= (\lambda^Q_{2l+1})^\alpha - ie\, A^\alpha \wedge \sum_{j=1}^{2l+1} \mathrm{Tr}(QUL^{j-1}QL^{2l+1-j}U^\dagger - QL^{j-1}QL^{2l+1-j})^\alpha,\\
(\rho^Q_{2l} - \lambda^Q_{2l})^\alpha v &= (\rho^Q_{2l} - \lambda^Q_{2l})^\alpha, \qquad\qquad (\rho^Q_1 + \lambda^Q_1)^\alpha v = (\rho^Q_1 + \lambda^Q_1)^\alpha,\\
(\rho^Q_3 + \lambda^Q_3)^\alpha v &= (\rho^Q_3 + \lambda^Q_3)^\alpha - 2ie\, A^\alpha \wedge \mathrm{Tr}\left[Q^2(R^2 - L^2) + Q\, dU^\dagger \wedge Q\, dU\right]^\alpha,\\
(\rho^Q_{2l+1} + \lambda^Q_{2l+1})^\alpha v &= (\rho^Q_{2l+1} + \lambda^Q_{2l+1})^\alpha - 2ie\, A^\alpha \wedge \sum_{j=1}^{l} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j+1}U^\dagger)^\alpha\\
&\quad - ie\, A^\alpha \wedge \sum_{j=0}^{l} \mathrm{Tr}(QR^{2j}QR^{2l-2j} - QL^{2j}QL^{2l-2j})^\alpha.
\end{aligned}$$

Analogous relations — with $dg_{\alpha\beta}$ instead of A^α — hold for the transformation rules, proving that $\rho_{2l}^Q - \lambda_{2l}^Q$ and $\rho_1^Q + \lambda_1^Q$ are global forms on B.

For calculations we need the action integral and the topological charge for the skyrmion bundle. Thus ω_3 and ω_5 have to be lifted to the bundle in order to get forms ω_3^A, resp., ω_5^A on B, whose pullbacks by the mesonic sections $U: M \to B$ can be integrated over space-time, resp., the space manifold only. The topological charge $B^A(U)$ for the skyrmion bundle is then defined analogously to (5.3). In view of Theorem 4.1, we can read the following differential forms $\chi^j_{2(l-j)+1} \in \mathcal{A}_{2(l-j)+1}(\mathrm{SU}_n)$ from (5.1), (5.2) and Lemma 5.2:

$$\begin{aligned}\chi^1_{2l-1} &= (2l+1)(\rho^Q_{2l-1} + \lambda^Q_{2l-1}),\\ \chi^2_{2l-3} &= (2l+1)\Big[2(\rho^{Q^2}_{2l-3} + \lambda^{Q^2}_{2l-3}) + \sum_{j=1}^{l-2} \mathrm{Tr}(QR^{2j-1}QR^{2l-2j-2} + QL^{2j-1}QL^{2l-2j-2})\\ &\quad + \sum_{j=1}^{l-1} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j-2}U^\dagger + QUL^{2j-2}QL^{2l-2j-1}U^\dagger)\Big].\end{aligned}$$

This yields the following corollary to Theorem 4.1:

Corollary 5.3 *Gauge invariant generalizations of ω_3 and ω_5 adapted to Γ and generating cohomology groups isomorphic to $\mathbb{R}$ on every bundle $B(M, \mathrm{SU}_n, \mathrm{U}_1)$, where U_1 acts on SU_n by inner automorphisms, are*

$$\begin{aligned}(\omega_3^A)^\alpha &= \omega_3 v + ie\, F \wedge \chi_1^1 v \;=\; [\omega_3^\alpha - 3ie\, A^\alpha \wedge (\rho_2^Q - \lambda_2^Q)] + 3ie\, F \wedge (\rho_1^Q + \lambda_1^Q),\\ (\omega_5^A)^\alpha &= \omega_5 v + ie\, F \wedge \chi_3^1 v + (ie)^2 F \wedge F \wedge \chi_1^2 v \;=\; [\omega_5^\alpha - 5ie\, A^\alpha \wedge (\rho_4^Q - \lambda_4^Q)]\\ &\quad + 5ie\, F \wedge \{(\rho_3^Q + \lambda_3^Q)^\alpha - 2ie\, A^\alpha \wedge Tr[Q^2(R^2 - L^2) + Q\, dU^\dagger \wedge Q\, dU]^\alpha\}\\ &\quad + 5(ie)^2\, F \wedge F \wedge [2(\rho_1^{Q^2} + \lambda_1^{Q^2})^\alpha + Tr(Q\, dU\, QU^\dagger - QUQ\, dU^\dagger)^\alpha].\end{aligned}$$

These forms coincide with the ones found by "trial-and-error" in the literature.[10, 6] The integral over $U^\star\omega_3^A$ gives the topological charge, and the integral over $U^\star\omega_5^A$ is the anomalous action for the skyrmion bundle. Nevertheless these forms are not unique in the sense that they are the only possible generalizations of type (4.1). An additional term

$$r\,(ie)^l F^l \wedge d\,\mathrm{Tr}(QU^\dagger QU), \qquad r \in \mathbb{R},$$

may be added to ω_{2l+1}^A, and this is still of the given type, because $d\,\mathrm{Tr}(QU^\dagger QU)$ is global, U_1-invariant and vertical.[6] One could even add any $F^l \wedge d\alpha$ with $\alpha = L_g^\star \alpha \in \mathcal{A}_0(\mathrm{SU}_n)$. In order to exclude these, one needs further physical requirements like parity invariance, equality of the numbers of F's and Q's, etc.

These forms now allow for the treatment of the monopole-induced proton decay within the skyrmion bundle. In fact, although we have proven that ω_3^A is a correct closed differential form for the topological charge, and although the integer valued cohomology group induced by $\frac{-1}{24\pi^2}\omega_3^A$ is isomorphc to $\mathbb{Z}$, the number of baryons $B^A(U)$ is not topologically conserved any more, whenever magnetic monopoles are present. This is due to the fact, that in contrast to the ungauged SKYRME model, the Index theorem 5.1 does not apply any more. There is no possibility to compactify space to an $\mathbb{S}^3$, so the topological charge can vanish through the monopole singularities.[3]

REFERENCES

1. Bott R. and Seeley R. «Some Remarks on the Paper of Callias», Commun. math. Phys. **62** (1978), 235 - 245

2. Bott R. and Tu L. W. «Differential Forms in Algebraic Topology», Springer Verlag, New York, 1982

3. Callan C. E. and Witten E. «Monopole Catalysis of Skyrmion Decay», Nucl. Phys. **B 239** (1984), 161 - 176

4. Chemtob M. «Cross Section of monopole-induced Skyrmion Decay», Phys. Rev. D **39**, No. 7 (1989), 2013 - 2031

5. Gross C. «Topology of the skyrmion bundle», J. Math. Phys., to appear

6. Kaymakcalan Ö., Rajeev S. and Schechter J. «Non-Abelian Anomaly and Vector-meson Decays», Phys. Rev. D **30**, No. 3 (1984), 594 - 602

7. Kobayashi S. and Nomizu K. «Foundations of Differential Geometry», Vol. I, John Wiley & Sons, 1963

8. Skyrme T. H. R. «A unified field theory of mesons and baryons», Nucl. Phys. **31** (1962), 556 - 569

9. Spanier E. H. «Algebraic Topology», McGraw-Hill, 1966

10. Witten E. «Global Aspects of Current Algebra», Nucl. Phys. **B 223** (1983), 422 - 432

CHRISTIAN GROSS
FACHBEREICH MATHEMATIK
TECHNISCHE HOCHSCHULE DARMSTADT
64289 DARMSTADT
GERMANY