László Kozma

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1 Introduction

A (Riemannian) Weyl structure on a manifold M is given by a map $C: \mathcal{G} \to \Lambda^1 M$, where $\mathcal{G}$ means a conformal structure on M, i.e. an equivalence class of Riemannian metrics ($\tilde{g} \sim g$ iff $\tilde{g} = e^\sigma g$ for some $\sigma \in C^\infty(M)$) and for any $g, \tilde{g} \in \mathcal{G} : C(\tilde{g}) = C(g) + d\sigma$. Then a Weyl manifold is a pair (M, C) of a manifold and a Weyl structure C on it [3]. In this case there is no absolute length of tangent vectors, but there are e.g. angle and ratio. It was shown already by H. Weyl and in modern terminology in [3] that for any Weyl manifold (M, C) there is a unique torsion-free linear connection on M with the following property:

$$\forall\, g \in \mathcal{G} : \quad \nabla g = \omega \otimes g, \text{ where } \omega := C(g).$$

This condition is also called recurrency property and can be described locally as

$$\nabla_k g_{ij} = \omega_k g_{ij}.$$

We say also that this connection is compatibe with the given Weyl structure. The local components of the compatible connection are

$$\Gamma^a_{bc} = \gamma^a_{bc} - \frac{1}{2}(\delta^a_b \omega_c + \delta^a_c \omega_b - g^{ad} g_{bc} \omega_d)$$

where (γ^a_{bc}) denote the components of the Levi-Civita connection of g. Thus this compatible connection is not metric, i.e. the length of vectors may change at parallel transport along a curve. However, the ratio of vector lengths at the initial point and at the endpoint does not depend on the initial vector, only on the joining curve.

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In this paper we want to extent these notions and relationships the the Finslerian context. Then, instead of a Riemannian metric the basic notion is the Finsler fundamental function $L: TM \to R$ (see details below), which measures the length of vectors in this case. We give first the notions of Finslerian conformal structure and Finsler-Weyl manifolds. In the Finslerian case instead of linear connection we should use the notion of homogeneous connections. Their compatibility with the given Finsler-Weyl structure is considered and characterized, and then, the existence of a compatible homogeneous connection is proved. Finally, we make comments on some aspects of other possible generalizations to the Finslerian case given in [1].

2 Finsler spaces and homogeneous connections

A function $L: TM \to R$ is called a Finsler fundamental function on a manifold M if

1. $L(u) > 0 \ \forall u \in TM, \ \ u \neq 0$
2. $L(\lambda u) = \lambda L(u) \ \ \forall \lambda \in R^+, u \in TM$
3. L is smooth except on the zero section
4. $g_{ij} = \frac{\partial^2(\frac{1}{2}L^2)}{\partial y^i \partial y^j}(x, y)$ is not singular for any $(x, y) \neq 0$.

Regularly, some convexity property is also assumed (see [5, 6]), however, that does not play role in this investigation. Roughly speaking, a Finsler fundamental function L_p at $p \in M$ gives a norm in the tangent space T_pM. Thus a Finsler space (M, L) is a manifold M endowed with a Finsler fundamental function L.

It has a long history that in a Finsler space there is, in general, no possibility of introducing a metrical (length preserving) linear connection. There are, however, two ways for solving this defect. One of these is lifting all notions and investigations to the tangent manifold TM, traditionally saying, to the line element manifold (see [5, 6]). The motivation for this was that $g_{ij}(x, y)$ given in the assumption 4. could play the role of a Riemannian metric over TM. Then there is a series of metrical (with respect to $g_{ij}(x, y)$) linear connections of the line element manifold. Another outway proposes to find metrical — but not linear, only — homogeneous connections on the manifold itself. It was found by W. Barthel [2], implicitly already by L. Berwald and E. Cartan; but such a connection is not unique at all (see [5]). In this paper we follow this second line of approach.

A homogeneous connection on M can be considered as a horizontal distribution, being supplement of the vertical subspaces

$$T_uTM = V_uTM \oplus H_u \quad \forall u \in TM, \ u \neq 0,$$

which are homogeneous in the following sense:

$$H_{tu} = d\mu_t(H_u) \ \ \forall \ t \in R^+, \ 0 \neq u \in TM,$$

where μ_t is the mapping $\mu_t: u \mapsto tu$. A homogeneous connection has 2-index parameters, by which the horizontal subspaces H_u can be generated:

$$\left(\frac{\partial}{\partial x^i}\right)_u - N_i^j(u) \left(\frac{\partial}{\partial y^j}\right)_u$$

These local parameters are positively 1-homogeneous: $N_i^j(x,ty) = tN_i^j(x,y)$. A homogenous connection determines also a covariant derivation, however, it has weaker properties than usual. Namely, the covariant derivation $\nabla_X Y$ is linear in X, satisfies the Leibniz rule in Y, but not additive in Y. In this case there exists parallel translation along curves in M. $Y \in \mathcal{X}(M)$ is called parallel along a curve $\gamma: I \to M$ if $\nabla_{\dot\gamma} Y = 0$, where $\dot\gamma$ denotes the tangent curve of γ.

The above mentioned — in some sense, canonical — homogeneous connection, called now as Berwald connection, is given as follows

$$G_j^i(x,y) = \gamma_{kj}^i(x,y)y^k - \frac{1}{2}g^{ir}(x,y)\frac{\partial g_{rj}}{\partial y^m}(x,y)\gamma_{kl}^m y^k y^l \tag{1}$$

where $(\gamma_{kj}^i(x,y))$ are the Christoffel symbols of $g_{ij}(x,y)$ with respect to x.

3 Finsler-Weyl manifolds

Two Finsler fundamental function L and $\tilde L$ on M are called *conformally equivalent* if there is a function $\sigma \in C^\infty$ such that $\tilde L = e^\sigma L$. *A Finslerian Weyl structure* is a map $C: \mathcal{L} \to \Lambda^1(M)$, where $\mathcal{L}$ is a conformal structure on M, i.e. an equivalence class of conformality with the property that

$$\text{for any } \tilde L,\ L \in \mathcal{L}:\ \ C(\tilde L) = C(L) + d\sigma.$$

So (M, C) is called a *Finsler-Weyl manifold* if C is a Finslerian Weyl structure on M.

A homogeneous connection ∇ is compatible with a F-W-structure $C: \mathcal{L} \to \Lambda^1(M)$ by definiton if

$$\forall\ L \in \mathcal{L}:\ \ \nabla L = C(L) \otimes L$$

i.e., using the notation $\omega = C(L)$

$$\forall\ X \in \mathcal{X}(M):\ \nabla_X L = \omega(X)L.$$

The covariant derivation refers originaly to vector fields. By extension we have

$$\nabla_X L = X^h(L) = X^i\left(\frac{\partial L}{\partial x^i} - N_i^j(x,y)\frac{\partial L}{\partial y^j}\right)$$

where X^h denotes the horizontal lift of X to TM.

Let us consider a curve $\gamma: I \to M$, and denote $P_{\gamma,u_0}: I \to TM$ the parallel translated vectors of $u_0 \in T_{\gamma(0)}M$ along γ. P_{γ,u_0} is in fact, a horizontal curve in TM over γ. Let us denote by $\dot\gamma$ the tangent of the curve γ at 0. Then

$$\nabla_{\dot\gamma} L(u_0) = X_{u_0}^h(L) = \dot P_{\gamma,u_0}(L) = (L \circ P_{\gamma,u_0})^{\cdot}. \tag{2}$$

First we investigate how the length of vector u_0 does change when u_0 is parallel translated along a curve γ to another point.

Proposition 1. *If* $\nabla L = \omega \otimes L$ *and* $u_1 \in T_{\gamma(1)}M$ *is the parallel translated vector of* $u_0 \in T_{\gamma(0)}M$ *along* γ, *then*

$$L(u_1) = e^{\int_0^1 \omega(\dot\gamma)} L(u_0).$$

Proof. Let $l: I \to R$ be the the length function of the parallel translated vectors: $l(t) = L(P_{\gamma,u_0}(t))$. Then using (2) we obtain

$$l'(0) = (L \circ P_{\gamma,u_0})^{\cdot} = \nabla \dot\gamma L(u_0) = \omega(\dot\gamma) l(0).$$

This is valid along the curve: $l'(t) = \omega(\dot\gamma) l(t)$. Thus the solution of this differential equation gives:

$$l(t) = e^{\int_0^t \omega(\dot\gamma)} L(u_0)$$

and especially,

$$L(u_1) = e^{\int_0^1 \omega(\dot\gamma)} L(u_0).$$

Q.e.d.

From this formula it follows that the length of the parallel translated vectors does not change if the tangents of the curve γ are annulated by the recurrency 1-form ω.

Now let us discuss another characterization of the compatibility of ∇ and a F-W-structure $C: \mathcal{L} \to \Lambda^1(M)$. Both induce a transportation of the norm L_p from a point $p \in M$ to point $q \in M$ along a joining curve γ. The compatibility of ∇ and C hold iff these transportations coincide.

First, let us take a F-W-structure $C: \mathcal{L} \to \Lambda^1(M)$ and a representative $L \in \mathcal{L}$. Let us define a new norm L'_q in the point $q \in M$, if γ is a curve joining p and q:

$$L'_q = e^{-\int_0^1 \omega(\dot\gamma)} L_q, \quad \text{where } \omega = C(L).$$

We show that this norm L'_q does not depend on the choice of the representative L, only on the curve γ and the norm L_p at p. Namely, if $\tilde L = e^{\sigma} L$ with $\tilde L_p = L_p$, i.e. $\sigma(p) = 0$, and therefore $\tilde\omega = \omega + d\sigma$, then

$$-\int_0^1 \tilde\omega(\dot\gamma) = -\int_0^1 (\omega + d\sigma)(\dot\gamma) = -\int_0^1 \omega(\dot\gamma) - \sigma(q).$$

So we have

$$e^{-\int_0^1 \tilde\omega(\dot\gamma)} \tilde L_q = e^{-\int_0^1 \omega(\dot\gamma) - \sigma(q)} e^{\sigma(q)} L_q = e^{-\int_0^1 \omega(\dot\gamma)} L_q = L'_q.$$

Secondly, the parallel transport, along γ, of the homogeneous connection ∇ also induces a transportation of the norm L_p at p to the point q:

$$L''_q(u) = L_p(P_{\bar\gamma,u}(1))$$

where $\bar\gamma$ denotes the reversed curve of γ.

Proposition 2. *A homogeneous connection ∇ is compatible with a Finslerian Weyl structure $C\colon \mathcal{L} \to \Lambda^1(M)$ if and only if the transportations L' and L'' of the norms by the F-W-structure and the connection resp. are equal.*

Proof. The necessity of the condition follows readily from Proposition 1:

$$L'_q(u) = e^{-\int_0^1 \omega(\dot\gamma)} L_q(u) = e^{-\int_0^1 \omega(\dot\gamma)} e^{\int_0^1 \omega(\dot\gamma)} L_p(P_{\bar\gamma,u}(1)) = L''_q(u)$$

On the other hand, from $L' = L''$ we have

$$L_q(u) = e^{\int_0^1 \omega(\dot\gamma)} L_p(u_0) \quad \text{with} \quad P_{\bar\gamma,u}(1) = u_0 \in T_pM,$$

or more generally,

$$L(P_{\gamma,u_0}(t)) = e^{\int_0^t \omega(\dot\gamma)} L_p(u_0).$$

Hence with $X(p) = \dot\gamma$

$$\nabla_X L(u_0) = \nabla_{\dot\gamma} L(u_0) = (L \circ P_{\gamma,u_0})^{\cdot} =$$

$$= \frac{d}{dt}\left(e^{\int_0^t \omega(\dot\gamma)}\right)|_{t=0} L(u_0) = \omega(\dot\gamma)L(u_0) = \omega(X)L(u_0).$$

Q.e.d.

4 The existence of compatible connection

First we show that if a connection ∇ is compatible with a representative of a conformal class of Finsler fundamental functions, then its is compatible with the whole class. Namely, suppose that $\nabla_X L = \omega(X)L$ for some $L \in \mathcal{L}$, andd let $\tilde L = e^\sigma L \in \mathcal{L}$. This latter relationship means more precisely $\tilde L = (e^\sigma \circ \pi)L$. Then

$$\nabla_X \tilde L = \nabla_X((e^\sigma \circ \pi)L) =$$

$$X^h(e^\sigma \circ \pi)L + (e^\sigma \circ \pi)X^h(L) = X(e^\sigma)L + (e^\sigma \circ \pi)\nabla_X L = d\sigma(X)e^\sigma L + e^\sigma \omega(X)L =$$

$$= (d\sigma(X) + \omega(X))e^\sigma L = \tilde\omega(X)\tilde L.$$

So we have obtained

$$\nabla_X \tilde L = \tilde\omega(X)\tilde L.$$

Theorem. *For any Finsler-Weyl manifold (M, C) there is a homogeneous connection ∇ compatible with the F-W-structure C.*

Proof. Chosen a representative fundamental function $L \in \mathcal{L}$, one consider the Berwald connection ∇^B of L, given locally by (1). This is metrical with respect to L: $\nabla^B L = 0$. By the recurrency 1-form $\omega = C(L)$ one modifies the Berwald connection in the following way:

$$\nabla_X Y = \nabla^B_X Y - \omega(X)Y.$$

It is readily seen that ∇ is a homogeneous connection as well. Moreover, ∇ is compatible with (L, ω):

$$\nabla_X L = \nabla^B_X L + \omega(X)L = \omega(X)L.$$

As mentioned above ∇ is then compatible with the whole conformality structure C. Q.e.d.

Remarks.

1. Such a compatible connection is not unique. Torsion type restriction could not assumed, for there is no torsion tensor in the case of homogeneous connections.
2. The connection parameters of this compatible connection is given as follows:

$$N^i_j(x,y) = G^i_j(x,y) - \omega_j(x)y^i$$

where G^i_j is the parameters of the Berwald connection given by (1).

5 Comments

1. In [4] G.S. Hall gave another approach for (Riemannian) Weyl manifolds. There a torsion free linear connection ∇ is called a Weyl connection if there is a Riemannian metric on M and a 1-form $\omega \in \Lambda^1(M)$ such that $\nabla g = \omega \otimes g$. In this case the existence and uniqueness up to conformality are the first esssential questions. It was answered in [4] as follows: ∇ is a Weyl connection iff its holonomy group is isomorphic to a Lie subgroup of some conformal group $CO(n)$. However, the uniqueness up to conformality is not valid if the holonomy group of the connection ∇ is reducible. These results cannot transfer directly to the Finslerian case, for the holonomy group of a homogeneous connection is, in general, not a linear group.

2. As mentioned above, a Finsler fundamental function L determines a Riemannian metric g given in the assumption 4. in the so-called Finslerian vector bundle $\pi^*(TM)$. Moreover, if $\tilde{L} = e^{\sigma}L$, then also $\tilde{g} = e^{2\sigma}g$ holds. However, here g and $\tilde{g}$ are given in tangent vectors of M, while σ is defined in points of M. Thus the conformality of Finsler fundamental functions can be lifted to that of special Riemannian mentrics of $\pi^*(TM)$. This was done in [1] in the case when a homogeneous connection is fixed. The compatibility of a Finslerian pair connection (∇^F, ∇) and a pair $([g], \nabla)$ was considered and it was shown that for any Finslerian Weyl structure there is a special unique compatible Finslerian pair connection (∇^F, ∇). In this respect ∇^F means a linear connection in the Finslerian vector bundle $\pi^*(TM)$.

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Department of Mathematics
Lajos Kossuth University
H-4010 Debrecen Pf.12, Hungary