Andreas Arvanitoyeorgos; Christina Beneki

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JACOBI VECTOR FIELDS AND GEODESIC TUBES IN CERTAIN KÄHLER MANIFOLDS

ANDREAS ARVANITOYEORGOS AND CHRISTINA BENEKI

ABSTRACT. In this paper we give a characterization of the Kähler manifolds of constant holomorphic sectional curvature by using Jacobi and Fermi vector fields as well as small geodesic tubes of the manifold.

1. INTRODUCTION

The characterization of a Riemannian manifold by using certain properties of a specific submanifold constitutes a central problem in differential geometry. The corresponding problem for the Riemannian manifolds of constant sectional curvature, has been investigated by L. Vanhecke and T. J. Willmore in [Va-Wi] and B. J. Papantoniou in [Pa3]. A similar problem by using the properties of small geodesic spheres has been treated in several papers (see for example [Bl-Le], [Bl-Va], [Le1], [Le2], [Dj-Va], [Pa1] and [Pa2]) and furthermore the same characterization by using geodesic tubes has been studied in [Pa3], [Bl-Pa], [Gr-Va]. M. Djorić [Dj] studied the characterization of complex space forms and locally Hermitian symmetric spaces by means of some extrinsic and intrinsic properties of tubular hypersurfaces relating to corresponding shape operator and the Ricci operator.

Recently J. Gillard [Gi] also gave another characterization of complex space forms by using the Ricci-semi-symmetry condition $\tilde{R}_{XY} \cdot \tilde{\varrho} = 0$ and the semi-parallel condition $\tilde{R}_{XY} \cdot \sigma = 0$ considering special choices of tangent vectors X,Y to small geodesic spheres or geodesic tubes, where $\tilde{R}$, $\tilde{\varrho}$ and σ denote the Riemann curvature tensor, the corresponding Ricci tensor and the second fundamental form of the spheres or tubes, even though his characterization is restricted to certain set of points called special points.

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In the present paper we characterize Kähler manifolds of constant holomorphic sectional curvature by using properties of the shape operator of small geodesic tubes about a geodesic in M.

Our main theorems are the following:

Theorem 1.1. *Let (M, g, J) be a connected Kähler manifold of constant holomorphic sectional curvature c and of real dimension $2n \geq 4$. Let $\sigma = \sigma(t)$ be a geodesic of M of finite length and $\gamma = \gamma(s)$ be a unit speed geodesic, meeting σ orthogonally at $m = \sigma(0)$. Suppose that $\{e^*\} = \{JN, \xi, e_4, \ldots, e_{2n}\}$ is an orthonormal frame field of parallel vectors along γ. Then every sufficiently small geodesic tube about σ of radius s in some neighbourhood U, is a quasi-umbilical hypersurface of M with respect to the plane spanned by JN and ξ.*

The basis $\{e^*\}$ mentioned in the theorem above will be constructed in section 3.

Theorem 1.2. *Let (M, g, J) be a connected Kähler manifold of real dimension $2n \geq 4$, $\sigma = \sigma(t)$ an arbitrary geodesic of M of finite length and $\gamma = \gamma(s)$ a unit speed geodesic meeting σ orthogonally at $m = \sigma(0)$. Suppose that every sufficiently small tube about σ of radius s is a quasi-umbilical hypersurface of M with respect to the plane spanned by $\{JN, \xi\}$ (i.e the corresponding shape operator has a parallel eigenspace along γ of dimension $2n - 3$). Then the manifold M is of constant holomorphic sectional curvature.*

2. Preliminaries

Let (M, g, J) be a connected Kähler manifold of dimension $2n \geq 4$. Denote by $\mathcal{X}(M)$ the Lie algebra of C^∞ vector fields on M. We will make use of the following results.

Theorem 2.1. *[Ko-No] A Kähler manifold has constant holomorphic sectional curvature c if and only if*

$$R(X,Y)Z = \frac{c}{4}[g(X,Z)Y - g(Y,Z)X + g(Y,JZ)JX - g(X,JZ)JY - 2g(X,JY)JZ]$$

for all $X, Y, Z \in \mathcal{X}(M)$.

Theorem 2.2. *[Kos] A Kähler manifold of dimension $2n \geq 4$ has constant holomorphic sectional curvature, if and only if $R(X, JX)X$ is proportional to JX for every vector field X on M.*

Let $\sigma : (a, b) \to M$ be a curve of finite length in a Riemannian manifold M. To describe the geometry of M in a neighbourhood of the curve σ we use Fermi coordinates [Fe] which are the natural generalization of normal coordinates about a point $m \in M$, when one replaces m by a submanifold P. Next we apply the definition of Fermi coordinates [Gr] for the case of the curve σ [Gr-Va].

Definition 2.1. Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field along a curve σ and let $m = \sigma(0)$ be a point on σ. Assume that $\dot{\sigma}(t) = (E_1)_{\sigma(t)}$. Then the Fermi coordinates $(x_1, \ldots, x_n)$ of σ relative to $\{E_1, \ldots, E_n\}$ and m are given by

$$x_1(exp_{\sigma(t)}\sum_{j=2}^{n} t_j(E_j)_{\sigma(t)}) = t, \quad x_i(exp_{\sigma(t)}\sum_{j=2}^{n} t_j(E_j)_{\sigma(t)}) = t_i, \quad 2 \le i \le n$$

provided that the numbers $t_2, \dots, t_n$ are small enough so that the mapping $exp_{\sigma(t)}$ is a diffeomorphism.

The Fermi coordinates $(x_1, \dots, x_n)$ are defined on any open neighbourhood $\mathcal{U}$ of σ for which every point of $\mathcal{U}$ can be joined to σ by a shortest unit speed geodesic meeting σ orthogonally.

Let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ be the coordinate vector fields associated with the Fermi coordinates $(x_1, \dots, x_n)$. It is known [Gr, lemma 2.4] that the restrictions of the coordinate vector fields $\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$ to σ are orthonormal.

Proposition 2.1. *Let $\gamma = \gamma(s)$ be a unit speed geodesic of M normal to σ with $\gamma(0) = m = \sigma(0)$ and let $u = \gamma'(0)$. Then there exist a system of Fermi coordinates $(x_1, \dots, x_n)$ such that for small s we have [Gr, lemma 2.5]:*

$$\left.\frac{\partial}{\partial x_2}\right|_{\gamma(s)} = \gamma'(s)$$

$$\textit{and} \quad \left.\frac{\partial}{\partial x_1}\right|_m = \{\dot\sigma(t)\}_m, \quad \left.\frac{\partial}{\partial x_i}\right|_m \in \{\dot\sigma(t)\}_m^{\perp}$$

for $2 \le i \le n$.

Definition 2.2. We say that a vector field $X \in \mathcal{X}(\mathcal{U})$ is a Fermi vector field relative to $(x_1, \dots, x_n)$ if X can be expressed as $X = \sum_{i=2}^{n} c_i \frac{\partial}{\partial x_i}$ where the c_i's are constants.

We will need the following two simple objects.
For $r > 0$ we put

$$r^2 = \sum_{i=2}^{n} x_i^2 \qquad \text{and} \qquad N = \sum_{i=2}^{n} \frac{x_i}{r}\frac{\partial}{\partial x_i}.$$

For $m \in \sigma$ it is easily proved [Gr, lemma 2.6] that the definitions of r and N are independent of the choice of the Fermi coordinates at m. The geometric significance of these objects will be stated later on. Next proposition describes their most important properties.

Proposition 2.2. [Gr, lemma 2.8] *Let X be a Fermi vector field for $\mathcal{U} = \mathcal{U}(\sigma)$ and $A = \frac{\partial}{\partial x_1}$. Then we have*

(a) $\nabla_N N = 0$, *(b)* $||N|| = 1$, *(c)* $N(r) = 1$, *(d)* $A(r) = 0$, *(e)* $[X, A] = [N, A] = 0$,

(f) $[N, X] = -\frac{1}{r}X + \frac{1}{r}X(r)N$, *(g)* $[N, rX] = X(r)N$, *(h)* $\nabla_N\nabla_N U = R(N, U)N$

for all U of the form $U = A + rX$.

Definition 2.3. A vector field Y along a geodesic γ is called a Jacobi field if it satisfies the following second order differential equation

$$Y'' = R(\gamma', Y)\gamma' .$$

Moreover if $\gamma = \gamma(s)$ is a geodesic normal to σ at $m = \sigma(0)$ and X a Fermi vector field on $\mathcal{U} = \mathcal{U}(\sigma)$, then the restrictions to γ of rX and $\frac{\partial}{\partial x_1}$ are Jacobi fields [Gr, corollary 2.9].

Definition 2.4. [Gh-Va] A solid tube of radius $r \geq 0$ about a curve σ is the set of points of M given by

$$T(\sigma, r) = \{exp_{\sigma(t)}X | X \in M_{\sigma(t)},\ ||X|| \leq r,\ g(X, \dot{\sigma}(t)) = 0,\ a < t < b\}$$

where $M_{\sigma(t)}$ denotes the tangent space of M at the point $\sigma(t)$. For small s, $0 < s \leq r$ we call the hypersurface of the form

$$P_s = \{m' \in T(\sigma, r) | d(m', \sigma) = s\}$$

the tubular hypersurface of radius s of M about σ, or just tube.

If σ is a geodesic of M, then the corresponding tubes are called geodesic tubes. At this point we can mention the geometric significance of the objects N and r defined earlier. The vector N is the unit outward normal from every tubular hypersurface and r is the distance of any point $m \in M$ from the curve σ. Furthermore, for any unit speed geodesic γ from σ to m meeting σ orthogonally, $N_{\gamma(s)} = \gamma'(s)$ $(s > 0)$ [Gr, lemma 2.7].

Let $R(s)$ be the endomorphism of $\{\gamma'(s)\}^{\perp} \subset T_{\gamma(s)}M$, given by $R(s)X = R(N, X)N$, and $S_N(s)$ be the shape operator of P_s with respect to N defined by $S_N(s) = -\nabla N$ [ON, Page 107].

We close this section with a brief discussion on quasi-umbilicity. Normally this refers to the shape operator S of a k-dimensional hypersurface as having at least $k-1$ equal eigenvalues. It is known [Ch-Ya] that if there exist, on an (n-1)-dimensional hypersurface M of an n-dimensional Riemannian manifold N, two functions p, q and a vector field v_a such that $h_{ab} = pg_{ab} + qv_a v_b$, $(a, b = 1, \ldots, n-1)$ where h_{ab}, g_{ab}, are the second fundamental form and the fundamental metric tensor of the hypesurface respectively, then M is said to be quasi-umbilical. We are interested on geodesic tubes as hypersurfaces in Kähler manifolds of dimension $2n$. At a point m of a tube P_s where two distinguished vectors JN and ξ (to be defined later on) are tangent vectors, we say that a hypersurface is quasi-umbilical with respect to the plane spanned by JN and ξ, if the shape operator at m has $2n-3$ eigenvalues equal, with eigenspace orthogonal to JN and ξ. (The remaining two eigenvalues may or may not be equal).

3. The shape operator on geodesic tubes

Let M be a Kähler manifold of constant holomorphic sectional curvature and let m be a point in M. Let $\sigma : (a, b) \to M$ be a geodesic with $\sigma(0) = m$ and P_s the geodesic tube of radius s about σ. Suppose that $\gamma = \gamma(s)$ is a unit speed geodesic in M meeting σ orthogonally at m, assuming that $\gamma(0) = m$ and $\gamma'(0) \in \{\dot\sigma(t)\}_m^\perp$. According to proposition 2.1 and the discussion after definition 2.4, there is a system of Fermi coordinates $(x_1, \ldots, x_n)$ with corresponding Fermi fields $X_i = \dfrac{\partial}{\partial x_i}$ such that for small s we have $\left.\dfrac{\partial}{\partial x_2}\right|_{\gamma(s)} = \gamma'(s) = N$. Let an orthonormal basis $\{E_1, \ldots, E_{2n}\}$ at m of the tangent space T_mM where $X_i = \left.\dfrac{\partial}{\partial x_i}\right|_m$ and $E_1, E_3, \ldots, E_{2n}$ being orthogonal to the geodesic γ at m, with $E_1 = \dot\sigma(0)$, $E_2 = \gamma'(0) = N$. We denote by the same letter N the tangent vector at m and the tangent vector field along γ.

In order to compute the shape operator on a geodesic tube along the curve σ we need to find expressions of the Jacobi vector fields along γ. Because of the Kähler condition this basis can be moved by parallel translation along γ. Denote by $\{e_1, \ldots, e_{2n}\}$ the parallel orthonormal frame field along γ. For our purposes it will be convenient to make a special choice of the frame field.

Therefore, we may choose the frame field $\{e_1, \ldots, e_{2n}\}$ such that

$$JN = (-\kappa e_1 + \lambda e_3)_m \tag{3.1}$$

where $\kappa^2 + \lambda^2 = 1$, as in general the vector JN will have a component tangent to σ at m and also a component normal to it.

If we define a parallel unit vector field ξ along γ by

$$\xi = \lambda e_1 + \kappa e_3 \tag{3.2}$$

then one easily has that $g(\xi, N) = g(\xi, JN) = 0$.

Hence, the frame field $\{e^*\} = \{JN, \xi; e_4, \ldots, e_{2n}\}$ is an orthonormal frame field for the space $\{N\}^\perp$ at any point of $\gamma(s)$.

Proposition 3.1. *Let (M, g, J) be a connected Kähler manifold of dimension $2n \geq 4$ and of constant holomorphic sectional curvature $c > 0$. Suppose that $\sigma = \sigma(t)$ is a geodesic of M of finite length and let $\gamma = \gamma(s)$ be a unit speed geodesic of M meeting σ orthogonally at $m \in M$. Suppose that $\{e^*\}$ is an orthonormal frame field along γ as defined above. Then the Jacobi vector fields along γ which are uniquely determined by the initial conditions:*

$$\begin{aligned} &Y_1(0) = E_1, \quad Y_3(0) = 0, \quad Y_j(0) = 0, \\ &Y_1'(0) = 0, \quad Y_3'(0) = E_3, \quad Y_j'(0) = E_j, \quad j = 4, \ldots, 2n \end{aligned} \tag{3.3}$$

are given by

$$
\begin{aligned}
Y_1(s) &= -\kappa\cosh(\sqrt{c}s)JN &+& \quad \lambda\cosh(\frac{\sqrt{c}}{2}s)\xi \\
Y_3(s) &= \frac{\lambda}{\sqrt{c}}\sinh(\sqrt{c}s)JN &+& \quad \frac{2\kappa}{\sqrt{c}}\sinh(\frac{\sqrt{c}}{2}s)\xi \\
Y_j(s) &= \frac{2}{\sqrt{c}}\sinh(\sqrt{c}s)e_j(s) &,& \quad j=4,\dots,2n.
\end{aligned}
\tag{3.4}
$$

Proof. Since at any point of γ every Jacobi vector field Y_i is perpendicular to N, it may be written with respect to the basis $\{e^*\}$, as

$$
Y_i = Y_{i1}^* JN + Y_{i3}^*\xi + \sum_{j=4}^{2n} Y_{ij}e_j, \qquad i=1;3,\dots,4n. \tag{3.5}
$$

Now supposing that $c>0$, the relations (3.4) can be obtained combining Theorem 2.1, relation (3.5) and definition 2.3.

If $c<0$, then (3.4) are also valid by substituting all the hyperbolic functions by the corresponding trigonometric ones, and c by $|c|$.

If $c=0$ then we have

$$
\begin{aligned}
&Y_1(s) = -\kappa JN + \lambda\xi, \quad Y_3(s) = (\lambda s)JN + (\kappa s)\xi, \\
&Y_j(s) = se_j(s), \qquad\qquad j=4,\dots,2n.
\end{aligned}
\tag{3.6}
$$

and the proof is completed.

We can now prove Theorem 1.1.

Consider the tubular hypersurface P_s of radius s about the geodesic σ and let $S_N(s) = -\nabla N$ be its shape operator acting on the space $\{N\}^\perp$ where N is the unit tangent vector field along the geodesic γ. Let $Y_i = Y_i(s)$ be the Jacobi vector fields along γ, given by (3.4). Then an easy computation shows (e.g. see [BP] Lemma 2.2, page 69) that $[N, Y_i] = 0$ and by using that $Y_i'(s) = \nabla_N Y_i(s)$ we obtain

$$
Y_i'(s) = -S_N(s)Y_i(s), \quad i=1;3,\dots,2n. \tag{3.7}
$$

We denote by B the $(2n-1)\times(2n-1)$ matrix whose columns are the coordinates of the above Jacobi vector fields with respect to the basis $\{e^*\}$ of $\{N\}^\perp$. We write

$$
Y_i(s) = B(s)e_i(s), \quad i=1;3,\dots,2n. \tag{3.8}
$$

Differentiating this relation we obtain

$$Y_i'(s) = B'(s)e_i(s) \tag{3.9}$$

and therefore by using (3.7),(3.8) and (3.9) we easily get

$$S_N(s)B(s)e_i(s) = -B'(s)e_i(s)\,.$$

Hence, we obtain an expression of the shape operator of tubes in terms of the Jacobi vector fields [GV]

$$S_N(s) = -B'(s)B^{-1}(s)\,. \tag{3.10}$$

Suppose that $c > 0$. By substituting JN and ξ from (3.1) and (3.2) to (3.4) and by applying the relation (3.10), we conclude that the shape operator $S_N(s)$ of the tube P_s is given by the following $(2n-1)\times(2n-1)$ matrix:

$$S_N(s) = \begin{pmatrix} K & L & & & 0 \\ L & M & & & \\ & & -\frac{\sqrt{c}}{2}\coth(\frac{\sqrt{c}}{2}s) & & \\ & 0 & & \ddots & \\ & & & & -\frac{\sqrt{c}}{2}\coth(\frac{\sqrt{c}}{2}s) \end{pmatrix} \tag{3.11}$$

where

$$K = -A[(\lambda^4 + 4\kappa^4)\cosh(\frac{\sqrt{c}}{2}s)\sinh^2(\frac{\sqrt{c}}{2}s) - 2\kappa^2\lambda^2(1-\cosh(\sqrt{c}s)\cosh(\frac{\sqrt{c}}{2}s)],$$

$$L = A\kappa\lambda[(4\kappa^2 - \lambda^2)\cosh(\frac{\sqrt{c}}{2}s)\sinh^2(\frac{\sqrt{c}}{2}s) + (\kappa^2-\lambda^2)(1-\cosh(\sqrt{c}s)\cosh(\frac{\sqrt{c}}{2}s)],$$

$$M = -A[\lambda^2\kappa^2(2+5\cosh(\frac{\sqrt{c}}{2}s)\sinh^2(\frac{\sqrt{c}}{2}s) + (\kappa^4+\lambda^4)\cosh(\sqrt{c}s)\cosh(\frac{\sqrt{c}}{2}s)]$$

$$\text{and}\quad A = \frac{\sqrt{c}}{2\mu\sinh(\frac{\sqrt{c}}{2}s)},\quad \mu = \lambda^2\cosh^2(\frac{\sqrt{c}}{2}s) + \kappa^2\cosh(\sqrt{c}s). \tag{3.12}$$

Therefore, the shape operator $S_N(s)$ has the eigenfunction $\kappa_1(s) = -\frac{\sqrt{c}}{2}\coth(\frac{\sqrt{c}}{2}s)$ of multiplicity $2n-3$ and the corresponding eigenspace orthogonal to JN and ξ. Hence the tube P_s is a quasi-umbilical hypersurface of M with respect to the plane spanned by JN and ξ.

In order to compute the shape operator $S_N(s)$ for the negative curvature case ($c < 0$), it suffices to change the hyperbolic functions in (3.11) and (3.12) into the corresponding trigonometric functions and replace c by $|c|$. Hence, we obtain the same conclusion.

For the zero curvature case ($c = 0$), one can easily obtain that the shape operator $S_N(s)$ has the eigenfunction $\kappa_2(s) = -\frac{1}{s}$ of multiplicity $2n-2$ and therefore, we get the same result and the proof of the Theorem 1.1 has been completed.

Next we will prove Theorem 1.2.

Let $m = \sigma(0)$ be a point of M and denote by $\gamma = \gamma(s)$ the unit-speed geodesic of M which meets σ orthogonally at m with $\gamma(0) = m$.

From the hypothesis that every sufficiently small tube about σ is quasi-umbilical with respect to the plane spanned by $\{JN, \xi\}$, we obtain that the shape operator $S_N(s) = S(s)$ of each tube will have three eigenfunctions, say $\kappa_1 = \kappa_1(s), \kappa_2 = \kappa_2(s)$, of multiplicity one, and $\kappa_3 = \kappa_3(s)$ of multiplicity $2n-3$.

Let $\{e_1; e_3, \ldots, e_{2n}\}$ be a parallel orthonormal frame field along γ such that

$$Se_1 = \kappa_1 e_1, \quad Se_3 = \kappa_2 e_3, \quad Se_i = \kappa_3 e_i, \quad i = 4, \ldots, 2n \tag{3.13}$$

and let

$$e_2(s) = \gamma'(s) = N\,. \tag{3.14}$$

From equation $S = -B'B^{-1}$ we obtain $B' = -SB$, from which by differentiation we get eventually that

$$Y_i'' = (S^2 - S')Y_i, \quad i = 1; 3, \ldots, 2n \tag{3.15}$$

where Y_i are the Jacobi vector fields along γ perpendicular to N. On the other hand, we also have

$$Y_i'' = R(N, Y_i)N \tag{3.16}$$

along $\gamma \setminus \{m\}$.

Therefore, we get

$$R(N, Y_i)N = (S^2 - S')Y_i. \tag{3.17}$$

Since $R(N, -)N, S^2$ and S' are tensor fields, then, along $\gamma \setminus \{m\}$, (3.17) is equivalent to

$$R(N, Y)N = (S^2 - S')Y \tag{3.18}$$

for all vector fields Y perpendicular to N. By substituting relation (3.13) to (3.18) we obtain

$$
\begin{aligned}
R(N,e_1)N &= (\kappa_1^2-\kappa_1')e_1\\
R(N,e_3)N &= (\kappa_2^2-\kappa_2')e_3\\
R(N,e_i)N &= (\kappa_3^2-\kappa_3')e_i, \quad i=4,\dots,2n\,.
\end{aligned}
\tag{3.19}
$$

Thus e_1, e_3 and $e_i, i=4,\dots,2n$ are eigenvector fields of $R(N,-)N$ along $\gamma\setminus\{m\}$, corresponding to the eigenfunctions $\kappa_1^2-\kappa_1', \kappa_2^2-\kappa_2'$ of multiplicity 1, and $\kappa_3^2-\kappa_3'$ of multiplicity $2n-3$. By substituting at the point m we have

$$
\begin{aligned}
R(E_2,E_1)E_2 &= k(E_2,E_1)E_1\\
R(E_2,E_3)E_2 &= l(E_2,E_3)E_3\\
R(E_2,E_i)E_2 &= m(E_2,E_i)E_i, \quad i=4,\dots,2n\,.
\end{aligned}
\tag{3.20}
$$

Consider an arbitrary point $m\in M$ and let γ be a geodesic passing through m such that $N=E_2$ is its tangent vector at m. Let $JE_2=E_3$ and σ be the geodesic perpendicular to the plane spanned by JE_2 and E_2. Then from the relations (3.1) and (3.2) one concludes that $\xi=E_1$. Moreover, from the second equation of (3.20) one easily concludes that

$$R(E_2,JE_2)E_2=R(E_2,E_3)E_2=l(E_2,E_3)JE_2$$

and this relation holds for every tangent vector E_2 at m, as γ may be chosen in an arbitrary direction.

Hence by Theorem 2.2 the manifold M has constant holomorphic sectional curvature and the proof is completed.

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UNIVERSITY OF PATRAS
DEPARTMENT OF MATHEMATICS
GR-26500 PATRAS, GREECE

Current address:
THE AMERICAN COLLEGE OF GREECE
6 GRAVIAS STREET
GR-15342 AGHIA PARASKEVI, GREECE
E-mail: TAARV@HOL.GR, BENEKI@MATH.UPATRAS.GR