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INTEGRALS AND VARIATIONAL MULTIPLIERS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

LUBOMÍR KLAPKA

ABSTRACT. In this paper the relation between integrals and variational multipliers of a system of second-order ordinary differential equations is studied. A simple necessary and sufficient local condition for the existence of a multiplier is given.

1. INTRODUCTION

There are a few different approaches to the inverse problem of the variational calculus for second-order differential equations (see, e.g. [1, 2, 6, 7]), but all of them use the notion of variational multipliers. The purpose of this paper is to study the local relation between variational multipliers and integrals of these differential equations. An appropriate mathematical apparatus was developed in [3, 4]. For simplicity, all mappings which are used in this paper are supposed to be infinitely differentiable and all expressions which are used here are local expressions only.

2. INTEGRALS AND MULTIPLIERS

We say that a given system of differential equations of the form

$$\ddot{x}^k = f^k(t, x, \dot{x}) \,, \tag{1}$$

where $k = 1, 2, \dots, n$, has an *integral* F if and only if $F(t, x, \dot{x})$ is a function such that $dF/dt = 0$ on any solution of (1). In this case, $dF/dt \equiv (\partial F/\partial \dot{x}^k)(\ddot{x}^k - f^k)$. Integrals are said to be *independent* if they are independent as functions.

We say that (1) has a *variational multiplier* if and only if there exists a non-singular matrix $(g_{kl}(t, x, \dot{x}))$, such that for some function $L(t, x, \dot{x})$

$$g_{kl}\,(\ddot{x}^l - f^l) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k} \,. \tag{2}$$

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Theorem. *A system of differential equations* (1) *has a variational multiplier if and only if it has independent integrals* $P_1, P_2, \dots, P_n, Q^1, Q^2, \dots, Q^n$ *satisfying conditions*

$$\frac{\partial P_\alpha}{\partial \dot{x}^k}\frac{\partial Q^\alpha}{\partial \dot{x}^l} - \frac{\partial P_\alpha}{\partial \dot{x}^l}\frac{\partial Q^\alpha}{\partial \dot{x}^k} = 0\,. \tag{3}$$

Proof. Suppose that P_α, Q^α are independent integrals of (1) satisfying the condition (3). This condition implies the existence of a function $S(t, x, \dot{x})$, such that

$$\frac{\partial S}{\partial \dot{x}^k} = -P_\alpha \frac{\partial Q^\alpha}{\partial \dot{x}^k}\,. \tag{4}$$

Then the function $L \equiv P_\alpha(dQ^\alpha/dt) + dS/dt$ does not depend on $\ddot{x}^k$ and so by straightforward calculation we have (2), where

$$g_{kl} \equiv \frac{\partial P_\alpha}{\partial \dot{x}^l}\frac{\partial Q^\alpha}{\partial x^k} - \frac{\partial P_\alpha}{\partial x^k}\frac{\partial Q^\alpha}{\partial \dot{x}^l}\,. \tag{5}$$

From (3) and (5) we get $dP_1 \wedge \cdots \wedge dP_n \wedge dQ^1 \cdots \wedge dQ^n = (-\det|g_{kl}|)^n\, dx^1 \wedge \cdots \wedge dx^n \wedge d\dot{x}^1 \wedge \cdots \wedge d\dot{x}^n$. Since the integrals are independent, the matrix (g_{kl}) is non-singular. Hence, the system (1) has a variational multiplier.

Conversely, suppose that a system of differential equations (1) has a variational multiplier. Then from (2) one gets $\partial^2 L/\partial \dot{x}^k \partial \dot{x}^l = g_{kl}$. Since (g_{kl}) is a non-singular matrix, we can use the Hamiltonian formalism. Denote the Hamilton function by H. There exists a canonical transformation $(p_k, x^k, t) \to (P_\alpha, Q^\alpha, t)$ reducing H to zero (see, e.g. [4, 5]). In this case P_α, Q^α are independent integrals of motion and $p_k\, dx^k - P_\alpha\, dQ^\alpha - H dt = dS$, where S is a corresponding generating function. If we return to Lagrangean coordinates $x^k, \dot{x}^k, t$ we get, among others, the formula (4). The integrability condition of (4) with respect to S gives (3). This completes the proof.

3. Remark

The above Theorem is a direct consequence of Theorem 1. in [3], p. 3785, as well as Proposition 9.2.11 in [4], p. 180. However, they lead to another proof, based on the concept of a Lepagean 2-form.

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