A. K. Kwaśniewski; E. Grądzka

Further remarks on extended umbral calculus

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 21st Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2002. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 69. pp. [117]--130.

Persistent URL: http://dml.cz/dmlcz/701693

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 69 (2002), pp. 117-130

FURTHER REMARKS ON EXTENDED UMBRAL CALCULUS

A. K. KWAŚNIEWSKI AND E. GRĄDZKA

ABSTRACT. ψ-calculus due to Viskov is an extension of classical operator calculus of Rota or equivalently - of umbral calculus of Roman and Rota. This ψ-extension relies on the notion of ∂_ψ-shift invariance of ∂_ψ-delta operators. Main results of Rotas' finite operator calculus have been already given their ψ-counterparts. By its nature - ψ-umbral calculus supplies a simple mathematical underpinning for infinitely many new ψ-deformed quantum-like oscillator algebras representations. For the $\psi_n(q) = [n_q!]^{-1}$ case [1]-[4] it provides also the natural underpinning for quantum group investigation. Moreover - the other way around - one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables $A, B : AB - qBA \equiv [A,B]_q = 0$. ψ-extension of calculus of Rota to be just announced here might be useful in an algebraic description of "ψ-quantum processes" - if any - with various parastatistics [5]. At the same time the reduced incidence algebra $R(L(S))$ is isomorphic to the algebra Φ_ψ of ψ-exponential formal power series. Therefore the ψ-extension Rota's operator calculus is a general representation of the algebra structure of $R(L(S))$. This paper is a self-contained extension of the recent note on extended finite operator calculus of Rota and quantum groups [6]. A systematic and detailed development is to be find in [7].

The article is suplemented by the short indicatory glossaries of terms and notation used by Ward, Viskov, Markowsky, Roman on one side and the Rota-oriented notation on the other side.

1. Extended Umbral Calculus in Brief

The very foundations of what we [6, 7] call *"ψ-extension of Rota finite operator calculus"* were led by Viskov [8, 9]. ψ-extended umbral calculus is arrived at [8, 9, 10, 11] by considering not only polynomial sequences of binomial type but also of $\{s_n\}_{n\geq 1}$-binomial type where $\{s_n\}_{n\geq 1}$-binomial coefficients are defined with help of the generalized factorial $n_S! = s_1 s_2 s_3 ... s_n$; $S = \{s_n\}_{n\geq 1}$ is an arbitrary sequence with the condition $s_n \neq 0$, $n \in N$.

Then the extension relies on the notion of ∂_ψ-shift invariance of ∂_ψ-delta operators. Here ∂_ψ denotes the ψ-derivative i.e. $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$; then ∂_ψ-linearly extended and n_ψ denotes the ψ-deformed number where in conformity with Viskov notation: $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$ hence $n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi (n-1)_\psi (n-2)_\psi (n-3)_\psi ....2_\psi 1_\psi$;

2000 *Mathematics Subject Classification*: 05A40, 81S99.
Key words and phrases: extended umbral calculus, deformed quantum oscillators.
The paper is in final form and no version of it will be submitted elsewhere.

$0_\psi! = 1$ and $n_\psi^{\underline{k}} = n_\psi (n-1)_\psi \dots (n-k+1)_\psi$. We choose to work with $\Im$ - the family of functions sequences (in conformity with Viskov notation) such that:
$\Im = \{\psi; R \supset [a,b]\ ;\ q \in [a,b]\ ;\ \psi(q) : Z \to F\ ;\ \psi_0(q) = 1\ ;\ \psi_n(q) \neq 0;\ \psi_{-n}(q) = 0;\ n \in N\}$.

With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known q-factorial $n_q! = n_q (n-1)_q!$; $1_q! = 0_q! = 1$ while the ψ-derivative ∂_ψ becomes now the Jackson's derivative (see [11]-[16]) $\partial_q : (\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}$.

A polynomial sequence $\{p_n\}_o^\infty$ is then of ψ-binomial type if it satisfies the recurrence
$E^y(\partial_\psi) p_n(x) \equiv p_n(x +_\psi y) \equiv \sum_{k \geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$; where $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$.

Here $E^y(\partial_\psi) \equiv \exp_\psi\{y\partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k \partial_\psi^k}{n_\psi!}$ is a generalized translation operator and ∂_ψ-shift invariance is defined accordingly. Namely we work with $\sum_\psi$ which is the algebra of F-linear operators acting on the algebra P of polynomials. We assume that char $F = 0$. These operators are ∂_ψ-shift invariant operators T_{∂_ψ} i.e.

$$\forall\, \alpha \in F; \quad [T; E^\alpha(\partial_\psi)] = 0 \quad ; \text{char } F = 0.$$

One then introduces the notion of ∂_ψ-*delta* operator according to:

Definition 1.1. Let $Q(\partial_\psi) : P \to P$; the linear operator $Q(\partial_\psi)$ is a ∂_ψ-*delta* operator iff

a) $Q(\partial_\psi)$ is ∂_ψ - shift invariant;
b) $Q(\partial_\psi)(id) = const \neq 0$

The strictly related notion is that of the ∂_ψ-basic polynomial sequence:

Definition 1.2. Let $Q(\partial_\psi) : P \to P$; be the ∂_ψ-*delta* operator. A polynomial sequence $\{p_n\}_{n \geq 0}$; deg $p_n = n$ such that:

1) $p_o(x) = 1$;
2) $p_n(0) = 0$; $n > 0$;
3) $Q(\partial_\psi) p_n = n_\psi p_{n-1}$ is called the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator$Q(\partial\psi)$.

After that and among many others the important Theorem 1.1 might be proved using the fact that $\forall \quad Q(\partial_\psi)\ \exists!$ invertible $S_{\partial_\psi} \in \quad \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}$.

Theorem 1.1. (ψ-Lagrange and ψ-Rodrigues formulas.) *Let $\{p_n(x)\}_{n=0}^\infty$ be ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$:*
$Q(\partial_\psi) \quad = \partial_\psi S_{\partial_\psi}$. *Then for $n > 0$:*

(1) $p_n(x) = Q(\partial_\psi)' S_{\partial_\psi}^{-n-1} x^n$;
(2) $p_n(x) = S_{\partial_\psi}^{-n} x^n - \frac{n_\psi}{n} (S_{\partial_\psi}^{-n})' x^{n-1}$;
(3) $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi S_{\partial_\psi}^{-n} x^{n-1}$;
(4) $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi (Q(\partial_\psi)')^{-1} p_{n-1}(x)$ ($\leftarrow$ Rodrigues ψ-formula).

For proof see [7].

Here we used the properties of the Pincherle ψ-derivative:

Definition 1.3. (compare with (17) in [9].) The Pincherle ψ-derivative i.e. the linear map ' : $\Sigma\psi \rightarrow \Sigma\psi$;
$T_{\partial_\psi}{}' = T_{\partial_\psi}\hat{x}_\psi - \hat{x}_\psi T_{\partial_\psi} \equiv [T_{\partial_\psi}, \hat{x}_\psi]$ where the linear map $\hat{x}_\psi : P \rightarrow P$; is defined in the basis $\{x^n\}_{n\geq 0}$ as follows

$$\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)}x^{n+1} = \frac{(n+1)}{(n+1)_\psi}x^{n+1}; \quad n \geq 0$$

Sheffer ∂_ψ-polynomials constitute the more general class. These are defined as follows:

Definition 1.4. A polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ is called the sequence $\{s_n(x)\}_{n=0}^{\infty}$ of Sheffer ∂_ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ iff
(1) $s_0(x) = c \neq 0$;
(2) $Q(\partial_\psi)s_n(x) = n_\psi s_{n-1}(x)$.

The following proposition relates Sheffer ∂_ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ to the unique ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$:

Proposition 1.1. *Let $Q(\partial_\psi)$ be a ∂_ψ-delta operator with ∂_ψ-basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$. Then $\{s_n(x)\}_{n=0}^{\infty}$ is a sequence of Sheffer q-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ iff there exists a ∂_ψ-shift invariant operator S_{∂_ψ} such that $s_n(x) = S_{\partial_\psi}^{-1}q_n(x)$. For proof see* [7].

The family of Sheffer ∂_ψ-polynomials' sequences $\{s_n(x)\}_{n=0}^{\infty}$ corresponding to the *fixed* ∂_ψ-delta operator $Q(\partial_\psi)$ is labeled by elements of the abelian group of all ∂_ψ-shift invariant invertible operators S_{∂_ψ}. It is an orbit of this group.

Let us also stress here again that q-deformed quantum oscillator algebra provides a natural setting for q-Laguerre polynomials and q-Hermite polynomials [24, 25, 26]. $sl_q(2)$ and the q-oscillator algebra give rise to basic geometric functions as matrix elements of certain operators in analogy with Lie theory [24, 25]. Also automorphisms of the q-oscillator algebra lead to Sheffer q-polynomials for example to q-generalization of the Charlier polynomials [24, 25].

2. Remarks on Extended Umbral Calculus and ψ-deformed quantum oscillator

∂_q- delta operators and their duals and similarly ∂_ψ-delta operators with their duals provide us with pairs of generators of ψ -deformed quantum oscillator algebras (see Remark 2.2) - possible candidates to describe parastatistical behavior of some processes [5]. Namely as we shall see : $[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \mathrm{id}$. With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known q-deformed oscillator dual pair of operators leading to the corresponding C^* algebra description of q-Heisenberg-Weyl algebra. These oscillator-like algebras generators and q-oscillator-like algebras generators are encountered explicitly or implicitly in [1, 2] and in many other subsequent references - see [36, 4] and references therein. In many such references [24, 25] q-Laguerre and q-Hermite or q-Charlier polynomials appear which are just either Sheffer ψ-polynomials or just ∂_ψ-basic polynomial sequences of the ∂_ψ-delta operators $Q(\partial_\psi)$ for $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ and corresponding choice of $Q(\partial_\psi)$

functions of ∂_ψ (for example $Q = \mathrm{id}$). The case $\psi_n(q) = \frac{1}{R(q^n)!}$: $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ appears implicitly in [27] where advanced theory of general quantum coherent states is being developed. However there is no mension of $R(q^n)$-umbral calculus in [27]. In the q-case it was noticed among others also in [29] that commutation relations for the q-oscillator-like algebras generators from [1, 2] and others (see [38, 39, 4]) might be chosen in appropriate operator variables to be of the form [29]:

$$AA^+ - \mu A^+ A = 1 \quad ; \quad \mu = q^2 \tag{2.1}$$

As for the Fock space representation of normalized eigenstates $|n>$ of excitation number operator N various q-deformations of the natural number n are used in literature on quantum groups and at least some families of quantum groups may be constructed from q-analogues of Heisenberg algebra [1, 2, 29, 3]. In fact, these q-oscillator algebras generators are (see below) the ∂_q-delta operators $Q(\partial_q)$ and their duals i.e. basic objects of the q-extended finite operator calculus of Rota. (Of course $\partial_q\hat{x}$- q$\hat{x}\partial_q = \mathrm{id}$.)

The known important fact is that the *"q-Canonical Commutation Relations"* $AA^+ - qA^+A = 1$ lead [4] to the q-deformed spectrum of excitation number operator N and to various parastatistics [5]. More possibilities result from considerations of Wigner [37] extended by the authors of [5]. We therefore hope that the ψ-calculus of Rota to be developed here might be useful in a C^* algebraic [4] description of "ψ-quantum processes" - if any - with various parastatistics [5].

Here in below we shall propose a ψ-extension of the q-oscillator model algebra using basic concepts of Viskov's ψ-extension of calculus of Rota.

Definition 2.1. Let $\{p_n\}_{n\geq 0}$ be the ∂_q -basic polynomial sequence of the ∂_q-delta operator$Q(\partial_q)$. A linear map $\hat{x}_{Q(\partial_q)}$: $P \to P$; $\hat{x}_{Q(\partial_q)}p_n = p_{n+1}$; $n \geq 0$ is called the operator dual to $Q(\partial_q)$.

For $Q = \mathrm{id}$ we have : $\hat{x}_{Q(\partial_q)} \equiv \hat{x}_{\partial_q} \equiv \hat{x}$.

Comment: Dual in the above sense corresponds to adjoint in q-umbral calculus language of linear functionals' umbral algebra (see : Proposition 1.1.21 in Kirschenhofer [19]).

Definition 2.2. Let $\{p_n\}_{n\geq 0}$ be the ∂_ψ -basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi) = Q$. Then the $\hat{q}_{\psi,Q}$- operator is a liner map;

$$\hat{q}_{\psi,Q} : P \to P; \quad \hat{q}_{\psi,Q}p_n = \frac{(n+1)_\psi - 1}{n_\psi} p_n; \quad n \geq 0.$$

We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$*-mutator operator.*

Note: For $Q = \mathrm{id}$ $Q(\partial_\psi) = \partial_\psi$ the natural notation is $\hat{q}_{\psi,id} \equiv \hat{q}_\psi$. For $Q = \mathrm{id}$ and $\psi_n(q) = \frac{1}{R(q^n)!}$ and $R(x) = \frac{1-x}{1-q}$ $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q}x^n = q^n x^n$.

Definition 2.3. Let A and B be linear operators acting on P;
A : $P \to P$; B : $P \to P$. Then AB - $\hat{q}_{\psi,Q}$BA $\equiv$ [A,B] $_{\hat{q}_{\psi,Q}}$ is called $\hat{q}_{\psi,Q}$-mutator of A and B operators.

Note: $Q(\partial_\psi)\hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q}\hat{x}_{Q(\partial_\psi)}Q(\partial_\psi) \equiv [Q(\partial_\psi),\hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \mathrm{id}$.
This is easily verified in the ∂_ψ-basic $\{p_n\}_{n\geq 0}$ of the ∂_ψ-delta operator $Q(\partial_\psi)$.

Equipped with pair of operators ($Q(\partial_\psi)$, $\hat{x}_{Q(\partial_\psi)}$) and $\hat{q}_{\psi,Q}$-mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra. The meaning of the adjective: "canonical" includes also the content of the remark 2.2. For important historical reasons however here is at first [6, 7]:

Remark 2.1. The ψ-derivative is a particular example of a linear operator that reduces by one the degree of any polynomial. In 1901 it was proved [32] by Pincherle and Amaldi that every linear operator T mapping P into P may be represented as infinite series in operators $\hat{x}$ and D. In 1986 Kurbanov and Maximov [33] supplied the explicit expression for such series in most general case of polynomials in one variable; namely according to the Proposition 1 from [33] one has : "*Let $\mathcal{D}$ be a linear operator that reduces by one each polynomial.* Let $\{q_n(\hat{x})\}_{n\geq 0}$ *be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $T = \sum_{n\geq 0} q_n(\hat{x})\mathcal{D}^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$$T = \sum_{n\geq 0} q_n(\hat{x})\mathcal{D}^n ."$$

Note: In 1996 this was extended to algebra of many variables polynomials [34].

Remark 2.2. The importance of the pair of dual operators: $Q(\partial_\psi)$ and $\hat{x}_{Q(\partial_\psi)}$ is reflected by the facts:

a) $Q(\partial_\psi)\hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q}\hat{x}_{Q(\partial_\psi)}Q(\partial_\psi) \equiv [Q(\partial_\psi),\hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \mathrm{id}$.

b) *Let $\{q_n\left(\hat{x}_{Q(\partial_\psi)}\right)\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}_{Q(\partial_\psi)}$. Then $T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right)Q(\partial_\psi)^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$$T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right)Q(\partial_\psi)^n . \tag{2.2}$$

Equipped with pair of operators ($Q(\partial_\psi)$, $\hat{x}_{Q(\partial_\psi)}$) and $\hat{q}_{\psi,Q}$-mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra such that:

a) the above unique expansion $T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right)Q(\partial_\psi)^n$ holds

b) we have the structure of ψ-umbral or ψ-extended finite operator calculus - coworking.

3. DOES A ψ-ANALOGUE OF QUANTUM Q-PLANE FORMULATION EXISTS?

In [18, 19] Cigler and Kirchenhofer defined the polynomial sequence $\{p_n\}_o^\infty$ of q-binomial type equivalently by

$$p_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q p_k(A)\, p_{n-k}(B) \text{where} [B,A]_{q\equiv} BA - qAB = 0\,. \tag{3.1}$$

A and B might be interpreted then as coördinates on quantum q-plane. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. With this being adopted the following identification holds:

$$p_n(x +_q y) \equiv E^y(\partial_q)\, p_n(x) = \sum_{k\geq 0} \binom{n}{k}_q p_k(x)\, p_{n-k}(y) = p_n\left(\hat{x} + y\hat{Q}\right) \mathbf{1}$$

Also q-Sheffer polynomials $\{s_n(x)\}_{n=0}^\infty$ are defined equivalently (see : 2.1.1. Kirschenhofer in [19]) by

$$s_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q s_k(A)\, p_{n-k}(B) \tag{3.2}$$

where $[B,A]_q \equiv BA - qAB = 0$ and $\{p_n(x)\}_{n=0}^\infty$ of q-binomial type. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. Then the following identification takes place:

$$s_n(x +_q y) \equiv E^y(\partial_q)\, s_n(x) = \sum_{k\geq 0} \binom{n}{k}_q s_k(x)\, p_{n-k}(y) = s_n\left(\hat{x} + y\hat{Q}\right) \mathbf{1} \tag{3.3}$$

This means that one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables A, B :
$AB - qBA \equiv [A,B]_q = 0$.

Let us now try to formulate - perhaps in vain - the basic notions of ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables A, B : $[A,B]_{\hat{q}_{\psi,Q}} = 0$ exactly in the same way as it was done by Cigler and Kirschenhofer in [18, 19].

For that to do let us consider appropriate generalization of $A = \hat{x}$ and $B = y\hat{Q}$ where this time the action of $\hat{Q}$ on $\{x^n\}_0^\infty$ is to be found from the condition

$$AB - \hat{q}_\psi BA \equiv [A,B]_{\hat{q}_\psi} = 0\,.$$

Acting with $[A,B]_{\hat{q}_\psi}$ on $\{x^n\}_0^\infty$ one easily sees that due to $\hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n$; $n \geq 0$ $\hat{Q}x^n = b_n x^n$ where $b_0 = 0$ and $b_n = \prod_{k=1}^{n} \frac{(k+1)_\psi - 1}{k_\psi}$ for $n > 0$ is the solution of the difference equation: $b_n - b_{n-1}\frac{(n+1)_\psi - 1}{n_\psi} = 0$; $n > 0$.

With all above taken into account one immediately verifies that for our A and B $\hat{q}_\psi$-commuting variables already

$$(A+B)^n \neq \sum_{k\geq 0} \binom{n}{k}_\psi A^k B^{n-k} \tag{3.4}$$

unless $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ hence $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q}x^n = q^n x^n$ i.e. unless we are back to the q-case.

In conclusion one sees that the above identifications of polynomial sequence $\{p_n\}_o^\infty$ of q-binomial type and Sheffer q-polynomials $\{s_n(x)\}_{n=0}^\infty$ fail to be extended to the more general ψ-case. This means that we can not formulate *that way* the ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : AB - \hat{q}_{\psi,Q}BA \equiv [A, B]_{\hat{q}_{\psi,Q}} = 0$ while considering algebra of polynomials P over the field F.

Nevertheless - already the q-case is already reach enough in abundant applications (see [1]-[5], [20]-[26], [28]-[30], [35]-[37] and hundreds of references therein) to various "q-quantum mechanical models" - q$\equiv \omega \equiv \exp\{\frac{2\pi i}{n}\}$ case included. One may expect the natural use of q-umbral calculus in these applications to be advantageous. Models using $\hat{q}_{\psi,Q}$-mutator $[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{R,Q}} = \mathrm{id}$ relations are suitable play-ground for ψ-umbral calculus (leading perhaps to ψ-lasers ? - see the q-footnote in [1] p. L887).
Indication: To this end we want to draw attention of the reader to new areas of possible applications of related extended binomial enumeration to combinatorics on one side and data structures organizations on the other side [38, 39].
Added: After completion of this work the autors became acquainted with the work of Ward [40] which is of primary importance for the whole direction of investigation as already seen from the indicatory Glossary of terms that now follow.

Morgan Ward: *Amer. J. Math.* **58**, 255 (1936).

Indicatory glossary of terms

Ward	**Rota - oriented** (this note)
$[n]$; $[n]!$ basic binomial coefficient $[n,r] = \frac{[n]!}{[r]![n-r]!}$	n_ψ; $n_\psi!$ ψ-binomial coefficient $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$
$D = D_x$ - the operator D $D\,x^n = [n]\,x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\,x^n = n_\psi\,x^{n-1}$
$(x+y)^n$ $(x+y)^n \equiv \sum\limits_{r=0}^{n}[n,r]\,x^{n-r}y^r$	$(x +_\psi y)^n$ $(x +_\psi y)^n = \sum\limits_{k=0}^{n}\binom{n}{k}_\psi x^k y^{n-k}$
basic displacement symbol E^t; $t \in \mathbf{Z}$ $E\varphi(x) = \varphi(x+1)$ $E^t\varphi(x) = \varphi(x+\bar{t})$	generalized shift operator $E^y(\partial_\psi) \equiv \exp_\psi\{y\partial_\psi\}$; $y \in \mathbf{F}$ $E^y(\partial_\psi)\varphi(x) = \varphi(x +_\psi 1)$ $E^y(\partial_\psi)x^n \equiv (x +_\psi y)^n$
basic difference operator $\Delta = E - id$ $\Delta = \varepsilon(D) - id = \sum\limits_{n=0}^{\infty}\frac{D^n}{[n]!} - id$	ψ-difference delta operator $\Delta_\psi = E^y(\partial_\psi) - id$

Steven Roman: *The umbral calculus* Academic Press, New York 1984.
Indicatory glossary of terms

Roman	**Rota - oriented** (this note)
$t;\ tx^n = nx^{n-1}$ $\langle t^k \| p(x)\rangle = [\partial_\psi^k p(x)]\|_{x=0}$	∂_ψ - the ψ-derivative $\partial_\psi x^n = n_\psi x^{n-1}$
evaluation functional $\epsilon_y(t) = \exp\{yt\}$ $\langle t^k \| x^n\rangle = n!\delta_{n,k}$ $\langle \epsilon_y(t) \| p(x)\rangle = p(y)$ $\epsilon_y(t)x^n = \sum_{k\geq 0}\binom{n}{k}x^k y^{n-k}$	generalized shift operator $E^y(\partial_\psi) = \exp_\psi\{y\partial_\psi\}$ $[E^y(\partial_\psi)p_n(x)]\|_{x=0} = p_n(y)$ $E^y(\partial_\psi)p_n(x) = \sum_{k\geq 0}\binom{n}{k}_\psi p_k(x)p_{n-k}(y)$
formal derivative $f'(t) \equiv \frac{d}{dt}f(t)$ $\overline{f}(t)$ compositional inverse of formal power series $f(t)$	Pincherle derivative $[Q(\partial_\psi)]' \equiv \frac{d}{d\partial_\psi}Q(\partial_\psi)$ $Q^{-1}(\partial_\psi)$ compositional inverse of formal power series $Q(\partial_\psi)$
$\theta_t;\ \theta_t x^n = \frac{n+1}{(n+1)_\psi}x^{n+1};\ n\geq 0$ $\theta_t t = \hat{x}D$	$\hat{x}_\psi;\ \hat{x}_\psi x^n = \frac{n+1}{(n+1)_\psi}x^{n+1};\ n\geq 0$ $\hat{x}_\psi\partial_\psi = \hat{x}D = \hat{N}$
$\sum_{k\geq 0}\frac{s_k(x)}{k!}t^k = [g(\overline{f}(z))]^{-1}\exp\{x\overline{f}(t)\}$ $\{s_n(x)\}_{n\geq 0}$ - Sheffer sequence for $(g(t), f(t))$	$\sum_{k\geq 0}\frac{s_k(x)}{k_\psi!}z^k = s(q^{-1}(z))\exp_\psi\{xq^{-1}(z)\}$ $q(t)$, $s(t)$ indicators of $Q(\partial_\psi)$ and S_{∂_ψ}

Roman	**Rota - oriented** (this note)
$g(t)\,s_n(x) = q_n(x)$ - sequence associated for $f(t)$	$s_n(x) = S_{\partial_\psi}^{-1}\,q_n(x)$ - ∂_ψ - basic sequence of $Q(\partial_\psi)$
The expansion theorem: $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)\vert p_k(x)\rangle}{k!} f(t)^k$ where $p_n(x)$ - sequence associated for $f(t)$	The First Expansion Theorem $T_{\partial_\psi} = \sum_{n\geq 0} \frac{[T_{\partial_\psi} p_n(z)]\vert_{z=0}}{n_\psi} Q(\partial_\psi)^n$ ∂_ψ - basic polynomial sequence $\{p_n\}_0^\infty$
$\exp y\overline{f}(t) = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k$	$\exp_\psi\{xQ^{-1}(x)\} = \sum_{k\geq 0} \frac{p_k(y)}{k!} z^k$
The Sheffer Identity: $s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_n(y) s_{n-k}(x)$	The Sheffer ψ-Binomial Theorem: $s_n(x +_\psi y) = \sum_{k\geq 0} \binom{n}{k}_\psi s_k(x) q_{n-k}(y)$

O. V. Viskov: *Soviet Math. Dokl.* **16**, 1521 (1975).
O. V. Viskov: *Soviet Math. Dokl.* **19**, 250 (1978).

Indicatory glossary of terms

Viskov	**Rota - oriented** (this note)
θ_ψ - the ψ-derivative $\theta_\psi\, x^n = \frac{\psi_{n-1}}{\psi_n} x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\, x^n = n_\psi\, x^{n-1}$
$A_p\ (p = \{p_n\}_0^\infty)$ $A_p\, p_n = p_{n-1}$	see A.K.K [6, 7]
$B_p\ (p = \{p_n\}_0^\infty)$ $B_p\, p_n = (n+1)\, p_{n+1}$	see A.K.K [6, 7]
$E_p^y\ (p = \{p_n\}_0^\infty)$ $E_p^y\, p_n(x) = \sum_{k=0}^{n} p_{n-k}(x) p_k(y)$	$E^y(\partial_\psi) = \exp_\psi\{y\partial_\psi\}$ $E^y(\partial_\psi)\ p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
$T - \varepsilon_p$-operator: $T\, A_p = A_p\, T$	E^y - shift operator: $E^y\varphi(x) = \varphi(x +_\psi y)$
$\forall_{y\in F}\ T E_p^y = E_p^y T$	T_{∂_ψ} - ∂_ψ-shift invariant operator: $\forall_{\alpha\in F}\ [T, E^\alpha(\partial_\psi)] = 0$

Viskov	**Rota - oriented** (this note)
Q - δ_ψ-operator: Q - ϵ_p-operator and $Qx = \text{const} \neq 0$	$Q(\partial_\psi)$ - ∂_ψ-delta-operator: $Q(\partial_\psi)$ - ∂_ψ-shift-invariant and $Q(\partial_\psi)(id) = \text{const} \neq 0$
$\{p_n(x), n \geq 0\}$ - (Q, ψ)-basic polynomial sequence of the δ_ψ-operator Q	$\{p_n\}_{n\geq 0}$ -∂_ψ-basic polynomial sequence of the ∂_ψ-delta-operator $Q(\partial_\psi)$
ψ-binomiality property $\Psi_y s_n(x) = = \sum_{m=0}^{n} \frac{\psi_n \psi_{n-m}}{\psi_n} s_m(x) p_{n-m}(y)$	ψ-binomiality property $E^y(\partial_\psi) p_n(x) = = \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
$T = \sum_{n\geq 0} \psi_n [VTp_n(x)] Q^n$ $T\Psi_y p(x) = = \sum_{n\geq 0} \psi_n s_n(y) Q^n STp(x)$	$T_{\partial_\psi} = \sum_{n\geq 0} \frac{[T_{\partial_\psi} p_n(z)]\|_{z=0}}{n_\psi} Q(\partial_\psi)^n$ $T_{\partial_\psi} p(x +_\psi y) = \sum_{k\geq 0} \frac{s_k(y)}{k_\psi!} Q(\partial_\psi)^k S_{\partial_\psi} T_{\partial_\psi} p(x)$

George Markovsky: *J. of Math. Anal. and Appl.* **63**, 145 (1978).
Indicatory glossary of terms

Markovsky	**Rota - oriented** (this note)
L - the differential operator $L\,p_n = p_{n-1}$	see A.K.K [6, 7]
M $M\,p_n = p_{n+1}$	see A.K.K [6, 7]
L_y $L_y\,p_n(x) =$ $= \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$	$E^y(\partial_\psi) = \exp_\psi\{y\partial_\psi\}$ $E^y(\partial_\psi)\,p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y)$
E^a - shift-operator: $E^a\,f(x) = f(x+a)$	E^y - shift operator: $E^y\varphi(x) = \varphi(x +_\psi y)$
G - shift-invariant operator: $EG = GE$	T_{∂_ψ} - ∂_ψ-shift invariant operator: $\forall_{\alpha\in F}\,[T, E^\alpha(\partial_\psi)] = 0$
G - delta-operator: G - shift-invariant and $Gx = const \neq 0$	$Q(\partial_\psi)$ - ∂_ψ-delta-operator: $Q(\partial_\psi)$ - ∂_ψ-shift-invariant and $Q(\partial_\psi)(id) = const \neq 0$

Markovsky	**Rota - oriented** (this note)
$\{Q_0, Q_1, \ldots\}$ - basic family for differential operator L	$\{p_n\}_{n\geq 0}$ -∂_ψ-basic polynomial sequence of the ∂_ψ-delta-operator $Q(\partial_\psi)$
binomiality property $P_n(x+y) =$ $= \sum_{i=0}^{n} \binom{n}{i} P_i(x) P_{n-i}(y)$	ψ-binomiality property $E^y(\partial_\psi)p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$

References

[1] Biedenharn, L. C., *J. Phys. A: Math. Gen.* **22**, L873 (1989).

[2] Macfarlane, A., *J. Phys. A: Math. Gen.* **22**, 4581 (1989).

[3] Haruo Ui and Aizawa, N., *Modern Physics Letters* **5**, 237 (1990).

[4] Jorgensen, P. E. T., *Pacific Journal of Math.* **165**, 131 (1994).

[5] Odaka, K., Kishi, T. and Kamefuchi, S., *J. Phys. A: Math. Gen.* **24**, L591 (1991).

[6] Kwaśniewski, A. K., *On extended finite operator calculus of Rota and quantum groups*, Białystok Univ. Inst. Comp. Sci. UwB/Preprint#04/July/2000 Integral Transforms and Special Functions - in press.

[7] Kwaśniewski, A. K., *Towards ψ-extension of Finite Operator Calculus of Rota*, Rep. Białystok Univ. Inst. Comp. Sci. UwB/Preprint#05/July/2000; Rep. Math. Phys., in press.

[8] Viskov, O. V., *Soviet Math. Dokl.* **16**, 1521 (1975).

[9] Viskov, O. V., *Soviet Math. Dokl.* **19**, 250 (1978).

[10] Roman, S. R., *J. Math. Anal. Appl.* **107**, 222 (1985).

[11] Roman, S. M., *The umbral calculus*, Academic Press, New York 1984.

[12] Jackson, F.H., *Amer. J. Math.* **32**, 305 (1910).

[13] Jackson, F.H., *Quart. J. Pure and Appl. Math.* **41**, 193 (1910).

[14] Hahn, W., *Math. Nachr.* **2**, 4 (1949).

[15] Hahn, W., *Math. Nachr.* **2**, 263 (1949).

[16] Hahn, W., *Math. Nachr.* **2**, 340 (1949).

[17] Fairlie, D.B. and Zachos, C.K., *Phys. Lett. B* **256**, 43 (1991).

[18] Cigler, J., *Indag. Math.* **40**, 27 (1978).

[19] Kirschenhofer, P., *Sitzunber. Abt. II Oster. Ackad. Wiss. Math. Naturw. Kl.* **188**, 263 (1979).

[20] Floreanini, R. and Vinet, L., *Letters in Mathematical Physics* **22**, 45 (1991).

[21] Floreanini, R. and Vinet, L., *Letters in Mathematical Physics* **27**, 179 (1991).

[22] Floreanini, R., LeTourneux, J. and Vinet, L., *J. Phys. A: Math. Gen.* **28**, L287 (1995).

[23] Kalnis, E. G., *J. Math. Phys.* **35**, 1951 (1994).

[24] Floreanini, R. and Vinet, L., *Annals of Physics* **221**, 53 (1993).

[25] Floreanini, R. and Vinet, L., *Physics Letters A* **180**, 393 (1993).

[26] Burban, I. M. and Klimyk, A. U., *Letters in Math. Phys.* **29**, 13 (1993).

[27] Odzijewicz, A., *Commun. Math. Phys.* **192**, 183 (1998).

[28] Kulish, P. and Damaskinky, E., *J. Phys. A: Math. Gen.* **23**, L415 (1990).

[29] Chaichian, M. and Kulish, P., *Physics Letters B* **234**, 72 (1990).

[30] Chang-Pu Sun and Hong-Chen Fu, *J. Phys. A: Math. Gen.* **22**, L983 (1989).

[31] Wigner, E. P., *Phys. Rev.* **77**, 711 (1950).

[32] Pincherle, S. and Amaldi, U., *Le operazioni distributive e le loro applicazioni all'analisi*, N. Zanichelli, Bologna 1901.

[33] Kurbanov, S. G. and Maximov, V. M., *Dokl. Akad. Nauk Uz. SSSR* **4**, 8 (1986).

[34] Di Bucchianico, A. and Loeb, D., *Integral Transforms and Special Functions* **4**, 49 (1996).

[35] Chaichian, M. and Ellinas, D., *J. Phys. A: Math. Gen.* **23**, L291 (1990).

[36] Kwaśniewski, A. K., *Advances in Applied Clifford Algebras* **8**, 417 (1998).

[37] Rauch de Traubenberg, M., *Advances in Applied Clifford Algebras* **4**, 131 (1994).

[38] Jaroszewski, I. and Kwaśniewski, A. K., *Integral Transforms and Special Functions* **11**, 1 (2001) No 1

[39] Kwaśniewski, A. K., *More on the Principal Recurrence of Data Structures Organization*, Biaystok Univ. Inst. Comp. Sci. UwB/ Preprint # 23 /December / 2000

[40] Ward, M., *Amer. J. Math.* **58**, 255 (1936).

[41] Markovsky, G., *J. of Math. Anal. and Appl.* **63**, 145 (1978).

INSTITUTE OF COMPUTER SCIENCE
BIALYSTOK UNIVERSITY
PL-15-887 BIAŁYSTOK, UL.SOSNOWA 64
POLAND
E-mail: `kwandr@uoc.uwb.edu.pl`