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ON TYPES OF NON-INTEGRABLE GEOMETRIES

THOMAS FRIEDRICH

ABSTRACT. We study the types of non-integrable G-structures on Riemannian manifolds. In particular, geometric types admitting a connection with totally skew-symmetric torsion are characterized. 8-dimensional manifolds equipped with a Spin(7)-structure play a special role. Any geometry of that type admits a unique connection with totally skew-symmetric torsion. Under weak conditions on the structure group we prove that this geometry is the only one with this property. Finally, we discuss the automorphism group of a Riemannian manifold with a fixed non-integrable G-structure.

CONTENTS

1. INTRODUCTION

Riemannian manifolds equipped with additional geometric structures occur in many situations and have interesting properties. The most important structures are almost complex structures and almost contact metric structures. Moreover, in special dimensions we have exceptional geometries resulting from the list of exceptional Lie

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groups, for example there is a 7-dimensional representation of the group G_2 and a 26-dimensional representation of the group F_4. In case the Riemannian geometry is compatible with the additional, geometric structure we call it integrable. The compatibility condition means that the geometric structure under consideration is parallel with respect to the Levi-Civita connection or, equivalently, the Riemannian holonomy group reduces to the subgroup preserving the geometric structure. Examples are Kähler manifolds, Calabi-Yau manifolds, parallel G_2-structures in dimension 7, parallel Spin(7)-structures in dimension 8 and symmetric spaces. However, there are many interesting Riemannian manifolds equipped with non-integrable geometric structures. This happens in any case for almost contact metric structures in odd dimensions, there are (non-Kähler) hermitian manifolds in even dimensions and non-symmetric, homogeneous spaces. Usually the Riemannian holonomy group of these manifolds is the full orthogonal group. Consequently, they are of general type in the sense of holonomy theory and cannot be distinguished from this point of view.

A lot of work has been done in order to understand special non-integrable geometries. In case the geometric structure can be defined by some tensor $\mathcal{T}$, one considers its Riemannian covariant derivative $\nabla^{LC}\mathcal{T}$. It is a 1-form with values in the representation space of the tensor. The decomposition of the corresponding tensor product under the action of the group G preserving the tensor $\mathcal{T}$ yields the different classes of non-integrable geometric structures. For any class of non-integrable geometries one derives a differential equation characterizing the class and involving the tensor $\mathcal{T}$. This program was developed, for example, for almost hermitian manifolds (Gray/Hervella [20]), for G_2-structures in dimension 7 (Fernandez/Gray [7]), for Spin(7)-structures in dimension 8 (Fernandez [6]) and for almost contact metric structures (Chinea/Gonzales [3]).

Some years ago I became interested in 16-dimensional Riemannian manifolds with a Spin(9)-structure (see [9], [10]). There the situation is slightly different, since a structure of that type is not defined by a single tensor. Therefore, I looked for another method in order to introduce a classification of non-integrable G-structures. The theory of principal fiber bundles and connections yields the idea that a classification of non-integrable G-structures can be based on the difference Γ between the Levi-Civita connection and the canonical G-connection induces on the G-structure. In some sense Γ measures the non-integrability of the G-structure in a natural way. It is a 1-form defined on the manifold with values in the subspace $\mathfrak{m}$ orthogonal to the Lie algebra $\mathfrak{g}$. At the same time A. Swann (see [24], [5]) and A. Fino (see [8]) considered this 1-form for different reasons, too (see also Chiossi/Salamon in [4]).

Let us *define* the different classes of non-integrable G-structures as the irreducible components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. If the geometric structure is given by a tensor, this point of view is completely equivalent to the approach described before. One of the aims of this note is to explain that one obtains all the known results in a unified way. The approach seems to be a kind of "folklore" for some people, but even in differential geometry it is not as popular as it should be. It will turn out that the reproduction of some classical results cited before becomes much less computational in our approach. It also has the advantage of being applicable to geometric structures not defined by a tensor. For example, we discuss irreducible SO(3)-structures on 5-dimensional manifolds, Spin(9)-structures on 16-dimensional manifolds as well as

F_4-structures on 26-dimensional manifolds. Some problems concerning non-integrable geometric structures can be solved immediately from this point of view. In string theory one wants to know which types of geometric structures admit affine connections ∇ with totally skew-symmetric torsion (see [16], [23]). It turns out that the answer depends mainly on the decomposition of two representations into irreducible components. An interesting example are 8-dimensional Riemannian manifolds with a Spin(7)-structure. It is well known (see [22]) that *any* Spin(7)-structure admits a unique connection with totally skew-symmetric torsion. In this paper we prove that under certain weak conditions on the structure group this is the only geometry with this property. Finally, we study the automorphism group of non-integrable geometric structures.

2. G-structures on Riemannian manifolds

Let $G \subset SO(n)$ be a closed subgroup of the orthogonal group and decompose the Lie algebra

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$$

into the subalgebra $\mathfrak{g}$ and its orthogonal complement $\mathfrak{m}$. We denote by $\mathrm{pr}_{\mathfrak{g}}$ and $\mathrm{pr}_{\mathfrak{m}}$ the projections of the Lie algebra $\mathfrak{so}(n)$ onto $\mathfrak{g}$ and $\mathfrak{m}$, respectively. Consider an oriented Riemannian manifold (M^n, g) and denote its frame bundle by $\mathcal{F}(M^n)$. It is a principal $SO(n)$-bundle over M^n. A *G-structure* of M^n is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi-Civita connection to $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(n)$:

$$Z\big|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a tensorial 1-form of type Ad, i.e., a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$. The triple $(M^n, g, \mathcal{R})$ is an *integrable* G-structure if the 1-form Γ vanishes, i.e., the Levi-Civita connection preserves the G-structure $\mathcal{R}$. Many interesting geometric structures are not of that type. In this paper we consider mainly *non-integrable* geometric structures, $\Gamma \neq 0$. We introduce a general classification of these structures using the G-type of the 1-form Γ. More precisely, the G-representation $\mathbb{R}^n \otimes \mathfrak{m}$ splits into irreducible components. The different *non-integrable types* of G-structures are defined — via the decomposition of Γ — as the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. Let us give a local formula for Γ. Fix an orthonormal frame $e_1, \ldots, e_n$ adapted to the reduction $\mathcal{R}$. The connection forms $\omega_{ij} := g(\nabla^{LC} e_i, e_j)$ of the Levi-Civita connection define a 1-form $\Omega := (\omega_{ij})$ with values in the Lie algebra $\mathfrak{so}(n)$ of all skew-symmetric matrices. The form Γ is the $\mathfrak{m}$-projection of Ω,

$$\Gamma = \mathrm{pr}_{\mathfrak{m}}(\Omega) = \mathrm{pr}_{\mathfrak{m}}(\omega_{ij}).$$

The case that the subgroup G is the isotropy group of some tensor $\mathcal{T}$ is of special interest. Suppose that there is a faithful representation $\varrho : SO(n) \to SO(V)$ and a tensor $\mathcal{T} \in V$ such that

$$G = \{ g \in SO(n) : \varrho(g)\mathcal{T} = \mathcal{T} \}.$$

Then a G-structure is a triple $(M^n, g, \mathcal{T})$ consisting of a Riemannian manifold equipped with an additional tensor field. The Riemannian covariant derivative is given by the formula

$$\nabla^{LC}\mathcal{T} \;=\; \varrho_*(\Gamma)(\mathcal{T}),$$

where $\varrho_* : \mathfrak{so}(n) \to \mathfrak{so}(V)$ is the differential of the representation. $\nabla^{LC}\mathcal{T}$ is an element of $\mathbb{R}^n \otimes V$. The algebraic G-types of $\nabla^{LC}\mathcal{T}$ define the algebraic G-types of Γ and vice versa. Indeed, we have

Proposition 2.1. *The* G*-map*

$$\mathbb{R}^n \otimes \mathfrak{m} \longrightarrow \mathbb{R}^n \otimes \mathrm{End}(V) \longrightarrow \mathbb{R}^n \otimes V$$

given by $\Gamma \to \rho_*(\Gamma)(\mathcal{T})$ *is injective.*

Proof. If $\rho_*(\Gamma)(\mathcal{T}) = 0$, then the endomorphism $\rho_*(\Gamma(X))$ stabilizes $\mathcal{T}$ for any vector $X \in \mathbb{R}^n$, i.e., $\rho_*(\Gamma(X)) \in \rho_*(\mathfrak{g})$. Since the representation is faithful, we conclude that $\Gamma(X) \in \mathfrak{g}$. On the other hand, we have $\Gamma(X) \in \mathfrak{m}$, i.e., $\Gamma \equiv 0$. □

The covariant derivative $\nabla^{LC}\mathcal{T}$ has been used for the classification of geometric structures — see the examples. The approach presented here uses the 1-form Γ and applies even in case that the geometric structure is not defined by a tensor. Moreover, in many situations it is simpler to handle the G-type of Γ then the G-type of the covariant derivative.

Proposition 2.2. *If the group* G *does not coincide with the full group* SO(n)*, then the* G*-representation* $\mathbb{R}^n$ *is always one of the components of the representation* $\mathbb{R}^n \otimes \mathfrak{m}$.

Proof. Indeed, consider the map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}\,, \qquad X \longrightarrow \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \wedge X),$$

where $e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{R}^n$. Suppose that a vector X belongs to its kernel. Then, for any vector Y, the exterior product $X \wedge Y$ is an element of the Lie algebra $\mathfrak{g}$. Since the commutator of two elements again belongs to the Lie algebra $\mathfrak{g}$, we conclude that the exterior product $Y \wedge Z$ of two vectors orthogonal to the vector X is in $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{so}(n)$. □

Geometrically this fact reflects the conformal transformation of a G-structure. Let $(M^n, g, \mathcal{R})$ be a Riemannian manifold with a fixed geometric structure and denote by $\hat{g} := e^{2f} \cdot g$ a conformal transformation of the metric. There is a natural identification of the frame bundles

$$\mathcal{F}(M^n, g) \;\cong \mathcal{F}(\hat{M}^n, \hat{g})$$

and a corresponding G-structure $\hat{\mathcal{R}}$. On the infinitesimal level, the conformal change is defined by the 1-form df.

2.1. **SO(3)-structures in dimension 5.** The group SO(3) has a unique, real, irreducible representation in dimension 5. We consider the corresponding non-standard embedding SO(3) ⊂ SO(5) as well as the decomposition

$$\mathfrak{so}(5) \;=\; \mathfrak{so}(3) \oplus \mathfrak{m}^7.$$

It is well known that the SO(3)-representation $\mathfrak{m}^7$ is the unique, real, irreducible representation in dimension 7. We decompose the tensor product into irreducible components

$$\mathbb{R}^5 \otimes \mathfrak{m}^7 \;=\; \mathbb{R}^3 \oplus \mathbb{R}^5 \oplus \mathfrak{m}^7 \oplus \mathrm{E}^9 \oplus \mathrm{E}^{11}.$$

There are five basic types of SO(3)-structures on 5-dimensional Riemannian manifolds. The symmetric space SU(3)/SO(3) is an example of a 5-dimensional Riemannian manifold with an integrable SO(3)-structure, ($\Gamma = 0$).

2.2. **Almost complex structures in dimension 6.** Let us consider 6-dimensional Riemannian manifolds $(M^6, g, \mathcal{J})$ with an almost complex structure $\mathcal{J}$. The subgroup $\mathrm{U}(3) \subset \mathrm{SO}(6)$ describes a geometric structure of that type. We decompose the Lie algebra

$$\mathfrak{so}(6) \;=\; \mathfrak{u}(3) \oplus \mathfrak{m}$$

and remark that the U(3)-representation in $\mathbb{R}^6$ is the real representation underlying $\Lambda^{1,0}$ and, similarly, $\mathfrak{m}$ is the real representation underlying $\Lambda^{2,0}$. We decompose the complexification under the action of U(3):

$$\Big(\mathbb{R}^6 \otimes \mathfrak{m}\Big)^{\mathbb{C}} \;=\; \Big(\Lambda^{1,0} \otimes \Lambda^{2,0} \oplus \Lambda^{1,0} \otimes \Lambda^{0,2}\Big)^{\mathbb{C}}_{\mathbb{R}}.$$

The symbol $(\ldots)^{\mathbb{C}}_{\mathbb{R}}$ means that we understand the complex representation as a real representation and complexify it. Next we split the complex U(3)-representations

$$\Lambda^{1,0} \otimes \Lambda^{2,0} \;=\; \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3) \;=\; \Lambda^{3,0} \oplus \mathrm{E}^8,$$

$$\Lambda^{1,0} \otimes \Lambda^{0,2} \;=\; \mathbb{C}^3 \otimes \Lambda^2(\overline{\mathbb{C}}^3) \;=\; \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3)^* \;=\; (\mathbb{C}^3)^* \oplus \mathrm{E}^6.$$

E^6 and E^8 are irreducible U(3)-representations of complex dimension 6 and 8, respectively. Finally we obtain

$$\mathbb{R}^6 \otimes \mathfrak{m} \;=\; \Lambda^{3,0} \oplus (\mathbb{C}^3)^* \oplus \mathrm{E}^6 \oplus \mathrm{E}^8.$$

Consequently, $\mathbb{R}^6 \otimes \mathfrak{m}$ splits into four irreducible representations of real dimensions 2, 6, 12 and 16, i.e., there are four basic types of U(3)-structures on 6-dimensional Riemannian manifolds (Gray/Hervella-classification — see [20]). In case we restrict the structure group to SU(3), we obtain two trivial summands in the decomposition of $\mathbb{R}^6 \otimes \mathfrak{su}(3)^{\perp}$ corresponding to *nearly Kähler* manifolds (see [18], [19], [24]). Almost hermitian manifolds of that type have special properties in real dimension $n = 6$. They are Einstein manifolds (see [19]), the differential equation describing the nearly Kähler manifolds is

$$(\nabla^{LC}_X \mathcal{J})(X) \;=\; 0, \qquad \nabla^{LC}\mathcal{J} \neq 0$$

and, finally, these are precisely the 6-dimensional manifolds with real Killing spinors (see [21]).

2.3. **G_2-structures in dimension 7.** We consider 7-dimensional Riemannian manifolds equipped with a G_2-structure. Since the group G_2 is the isotropy group of a 3-form ω^3 of general type, a G_2-structure is a triple (M^7, g, ω^3) consisting of a 7-dimensional Riemannian manifold and a 3-form ω^3 of general type at any point. We decompose the G_2-representation (see [11])

$$\mathbb{R}^7 \otimes \mathfrak{m} \;=\; \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27}$$

and, consequently, there are four basic types of non-integrable G_2-structure. In this way we obtain the Fernandez/Gray-classification of G_2-structures (see [7]). The different types of G_2-structures can be characterized by differential equations. For example, a G_2-structure is of type $\mathbb{R}^1$ (*nearly parallel* structures) if and only if there exists a number λ such that

$$d\omega^3 \;=\; \lambda \cdot (*\omega^3)$$

holds. Again, this condition is equivalent to the existence of a real Killing spinor (see [14]). The G_2-structures of type $\mathbb{R}^1 \oplus \Lambda^3_{27}$ (*cocalibrated* structures) are characterized by the condition that the 3-form is coclosed, $\delta\omega^3 = 0$. In general, the differential equations for any type of G_2-structure involving the 3-form ω^3 were derived in [7]. In the spirit of the approach of this paper one can find the computations in [11].

2.4. Spin(7)**-structures in dimension** 8. Let us consider Spin(7)-structures on 8-dimensional Riemannian manifolds. The subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ is the real Spin(7)-representation $\Delta_7 = \mathbb{R}^8$. The complement $\mathfrak{m} = \mathbb{R}^7$ is the standard 7-dimensional representation and the Spin(7)-structures on an 8-dimensional Riemannian manifold M^8 correspond to the irreducible components of the tensor product

$$\mathbb{R}^8 \otimes \mathfrak{m} \;=\; \mathbb{R}^8 \otimes \mathbb{R}^7 \;=\; \Delta_7 \otimes \mathbb{R}^7 \;=\; \Delta_7 \oplus \mathrm{K} \;=\; \mathbb{R}^8 \oplus \mathrm{K},$$

where K denotes the kernel of the Clifford multiplication $\Delta_7 \otimes \mathbb{R}^7 \longrightarrow \Delta_7$. It is well known that K is an irreducible Spin-representation, i.e., there are two basic types of Spin(7)-structures (the Fernandez-classification of Spin(7)-structures — see [6]).

2.5. Spin(9)**-structures in dimension** 16. The group Spin(9) is an interesting subgroup of SO(16). The representation in $\mathbb{R}^{16}$ is irreducible. We consider 16-dimensional Riemannian manifolds with Spin(9)-structures. Again, we split the Spin(9)-representation into four irreducible components

$$\mathbb{R}^{16} \otimes \mathfrak{m} \;=\; \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9) \oplus \mathcal{P}_3(\mathbb{R}^9).$$

Consequently, there are four basic types of non-integrable Spin(9)-structures on 16-dimensional Riemannian manifolds. A Spin(9)-structure on a 16-dimensional Riemannian manifold has an associated 8-form. Some differential equations for the different types of Spin(9)-structures involving the 8-form have been computed in the paper [9].

2.6. $\mathbf{F_4}$**-structures in dimension** 26. We consider the subgroup $F_4 \subset \mathrm{SO}(26)$ and 26-dimensional Riemannian manifolds with a F_4-structure. The orthogonal complement

$$\mathfrak{so}(26) \;=\; \mathfrak{f}_4 \oplus \mathfrak{m}^{273}$$

is the unique, irreducible, 273-dimensional representation of F_4. We compute the decomposition

$$\mathbb{R}^{26} \otimes \mathfrak{m}^{273} \;=\; \mathbb{R}^{26} \oplus \mathfrak{f}_4 \oplus \mathfrak{m}^{273} \oplus \mathrm{E}^{324} \oplus \mathrm{E}^{1053} \oplus \mathrm{E}^{1274} \oplus \mathrm{E}^{4096}.$$

Consequently, there are seven basic types of F_4-structures in dimension 26.

3. G-CONNECTIONS WITH TOTALLY SKEW-SYMMETRIC TORSION

An interesting question in some models in string theory (see [11]) is to ask of which geometric structures admit a connection ∇ preserving the structure and with totally skew-symmetric torsion. In order to formulate the general condition, let us introduce the maps

$$\Theta_1 : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta_2 : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{g}$$

given by the formulas

$$\Theta_1(T) \ := \ \sum_i (\sigma_i \lrcorner T) \otimes \sigma_i, \quad \Theta_2(T) \ := \ \sum_j (\mu_j \lrcorner T) \otimes \mu_j,$$

where σ_i is an orthonormal basis in $\mathfrak{m}$ and μ_j is an orthonormal basis in $\mathfrak{g}$. Then we have

Theorem 3.1. *(see [10]) A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of a Riemannian manifold admits a connection ∇ with totally skew-symmetric torsion if and only if the 1-form Γ belongs to the image of Θ_1,*

$$2 \cdot \Gamma \ = \ -\Theta_1(T).$$

In this case the 3-form T is the torsion form of the connection.

Proof. Suppose there exists a connection ∇ with totally skew-symmetric torsion T. We compare it with the Levi-Civita connection and obtain the relation

$$\nabla_X Y \ = \ \nabla^{LC}_X Y + \frac{1}{2} \cdot T(X, Y, *).$$

Moreover, the definition of the 1-form Γ as well as the G-connection Z^* yield the equation

$$\nabla^{LC}_X Y \ = \ \nabla^{Z^*}_X Y + \Gamma(X) Y.$$

Finally, since ∇ preserves the G-structure, there exists a 1-form β with values in the Lie algebra $\mathfrak{g}$ such that

$$\nabla_X Y \ = \ \nabla^{Z^*}_X Y + \beta(X) Y.$$

Combining the three formulas we obtain, for any vector X, the equation

$$2 \cdot \beta(X) \ = \ 2 \cdot \Gamma(X) + T(X, *, *).$$

We project onto the subspace $\mathfrak{m}$. Since $\beta(X)$ belongs to the Lie algebra $\mathfrak{g}$, we conclude that Γ should be in the image of Θ_1, $\Theta_1(T) = -2 \cdot \Gamma$. □

Theorem 3.1 only decides which geometric types admit connections with totally skew-symmetric torsion. However, if the geometric structure is defined by a tensor $\mathcal{T}$, one prefers to express the torsion form T of the connection ∇ directly by this tensor $\mathcal{T}$. Formulas of that type were computed for almost complex structures, almost contact metric structures, G_2-structures and Spin(7)-structures (see [11], [12], [13], [22] and [1], [2]).

Example 3.1. We consider 5-dimensional Riemannian manifolds with an SO(3)-structure. Then we obtain

$$\mathbb{R}^5 \otimes \mathfrak{m}^7 \ = \ \mathbb{R}^3 \oplus \mathbb{R}^5 \oplus \mathfrak{m}^7 \oplus \mathrm{E}^9 \oplus \mathrm{E}^{11}, \quad \Lambda^3(\mathbb{R}^5) \ = \ \mathbb{R}^3 \oplus \mathfrak{m}^7.$$

In particular, a conformal change of an SO(3)-structure does *not* preserve the property that the structure admits a connection with totally skew-symmetric torsion.

Example 3.2. In the case of almost complex structures in dimension 6, we have

$$\mathbb{R}^6 \otimes \mathfrak{m} \;=\; \Lambda^{3,0} \oplus (\mathbb{C}^3)^* \oplus \mathrm{E}^6 \oplus \mathrm{E}^8, \quad \Lambda^3(\mathbb{R}^6) \;=\; \Lambda^{3,0} \oplus (\mathbb{C}^3)^* \oplus \mathrm{E}^6.$$

Consequently, an almost complex manifold $(M^6, g, \mathcal{J})$ admits a connection with totally skew-symmetric torsion if and only if the E^8-part of Γ vanishes. In case the connection exists, it is unique and the formula for its torsion has been derived in [11].

Example 3.3. In dimension 7 we decompose the G_2-representation

$$\Lambda^3(\mathbb{R}^7) \;=\; \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^3_{27}, \quad \mathbb{R}^7 \otimes \mathfrak{m} \;=\; \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27}.$$

Consequently, a G_2-structure admits a connection with totally skew-symmetric torsion if and only if it is of type $\mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^3_{27}$. These condition describes the conformal changes of cocalibrated G_2-structures. In case the connection exists, it is unique and the formula for its torsion has been derived in [11].

Example 3.4. Let us consider Spin(7)-structures on 8-dimensional Riemannian manifolds. Here we find

$$\mathbb{R}^8 \otimes \mathfrak{m} \;=\; \Delta_7 \oplus \mathrm{K}, \qquad \Lambda^3(\mathbb{R}^8) \;=\; \Delta_7 \oplus \mathrm{K},$$

i.e., $\Lambda^3(\mathbb{R}^8) \to \mathbb{R}^8 \otimes \mathfrak{m}$ is an isomorphism. Theorem 3.1 yields immediately that *any* Spin(7)*-structure on an* 8*-dimensional Riemannian manifold admits a unique connection with totally skew-symmetric torsion* (see [22]).

Example 3.5. In case of $\mathrm{G} = \mathrm{Spin}(9)$, we have

$$\mathbb{R}^{16} \otimes \mathfrak{m} \;=\; \mathbb{R}^{16} \oplus \Lambda^3(\mathbb{R}^{16}) \oplus \mathcal{P}_3(\mathbb{R}^9),$$

and the $\mathbb{R}^{16}$-component is *not* contained in $\Lambda^3(\mathbb{R}^{16}) = \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9)$. A conformal change of a Spin(9)-structure does *not* preserve the property that the structure admits a connection with totally skew-symmetric torsion (see [10]).

Example 3.6. In dimension 26 and for the subgroup $\mathrm{F}_4 \subset \mathrm{SO}(26)$ we have

$$\begin{aligned} \mathbb{R}^{26} \otimes \mathfrak{m}^{273} &= \mathbb{R}^{26} \oplus \mathfrak{f}_4 \oplus \mathfrak{m}^{273} \oplus \mathrm{E}^{324} \oplus \mathrm{E}^{1053} \oplus \mathrm{E}^{1274} \oplus \mathrm{E}^{4096}, \\ \Lambda^3(\mathbb{R}^{26}) &= \mathfrak{m}^{273} \oplus \mathrm{E}^{1053} \oplus \mathrm{E}^{1274}. \end{aligned}$$

In particular, a conformal change of an F_4-structure does *not* preserve the property that the structure admits a connection with totally skew-symmetric torsion.

4. The automorphism group of non-integrable G-structures

We consider a Riemannian manifold $(M^n, g, \mathcal{R})$ with a fixed geometric structure. Since the Lie algebra $\Lambda^2(\mathbb{R}^n) = \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ splits, the bundle of 2-form $\Lambda^2(M^n)$ decomposes into two subbundles. They are associated with the reduction $\mathcal{R}$ of the frame bundle and we denote these two bundles again by $\mathfrak{g}$ and $\mathfrak{m}$, respectively,

$$\Lambda^2(M^n) \;=\; \mathfrak{g} \oplus \mathfrak{m}.$$

Let X be a Killing vector field. Then the covariant derivative $\nabla^{LC} X \in \Gamma(T \otimes T)$ is skew-symmetric. In fact, if we understand X as a 1-form on the manifold, the covariant derivative of X coincides with the exterior differential,

$$\nabla^{LC} X \;=\; \frac{1}{2} \cdot dX.$$

We suppose now that the G-structure $\mathcal{R}$ admits a unique connection ∇ with totally skew-symmetric torsion T. Then $\nabla X \in \Gamma(T \otimes T)$ is a skew-symmetric tensor, too. Moreover, we have

$$\nabla X = \nabla^{LC} X - \frac{1}{2} \cdot (X \lrcorner T).$$

Theorem 4.1. *Let $(M^n, g, \mathcal{R})$ be a G-structure and suppose that there exists a unique G-connection ∇ with totally skew-symmetric torsion T. If a Killing vector field X is an infinitesimal transformation of the G-structure, then*

$$\mathcal{L}_X T = 0, \quad [X, \nabla_Y Z] - \nabla_Y [X, Z] = \nabla_{[X,Y]} Z.$$

The 2-form $\nabla X \in \mathfrak{g}$ belongs to the subbundle $\mathfrak{g}$. In particular, we have

$$pr_{\mathfrak{m}}(dX) = pr_{\mathfrak{m}}(X \lrcorner T).$$

Proof. Since ∇ is the unique G-connection with totally skew-symmetric torsion, any transformation of $\mathcal{R}$ should preserve the connection and its torsion form, i.e., the first two conditions are necessary. In fact, the condition $\nabla X \in \mathfrak{g}$ characterizes the infinitesimal transformations preserving a G-structure. Let us — for completeness — give the argument. The covariant derivative of a vector field with respect to an affine metric connection can be computed via the formula

$$g(\nabla_Y X, Z)(p) := \frac{d}{dt} g(df_t(p)(Y), \tau_t^{\nabla}(Z)),$$

where $f_t : M^n \to M^n$ is the 1-parameter group generated by the vector field X and τ_t^{∇} denotes the parallel displacement along the curve $f_t(p)$. Fix a basis $e_1, \dots, e_n \in \mathcal{R}_p$ in the G-structure at the point $p \in M^n$ and denote by $A_{ij}(t)$ the matrix defined by

$$df_t(e_i) := \sum_{j=1}^{n} A_{ij}(t) \cdot \tau_t^{\nabla}(e_j).$$

The endomorphism $\nabla X(p)$ is given by the matrix $\big(A'_{ij}(0)\big)$. If the 1-parameter group f_t preserves the structure $\mathcal{R}$, then the matrix $\big(A_{ij}(t)\big)$ belongs to the subgroup G, i.e., ∇X is a 2-form in $\mathfrak{g}$. □

Remark 4.1. The formula $\mathrm{pr}_{\mathfrak{m}}(dX) = \mathrm{pr}_{\mathfrak{m}}(X \lrcorner T)$ was derived in case of a nearly parallel G_2-structure in [14, Theorem 6.2] (notice that there is a sign error). Indeed, a nearly parallel G_2-structure admits a unique connection with totally skew-symmetric torsion T, which was computed in [11, Example 5.2]. Using these expression for T we obtain from Theorem 4.1 the formula of Theorem 6.2 in [14].

The invariance of the torsion form restricts the dimension of the automorphism $\mathcal{G}(\mathcal{R})$. Denote by G_T the isotropy group of $T \in \Lambda^3(\mathbb{R}^n)$ and $dT \in \Lambda^4(\mathbb{R}^n)$ inside of G. Then we have

$$\dim\big(\mathcal{G}(\mathcal{R})\big) \leq n + \dim(G_T).$$

The group G_T preserves the Ricci tensor of the unique connection ∇ as well as the symmetric endomorphism $T_{imn} \cdot T_{jmn}$. These geometric objects have been computed in several cases and can be used in the computation of the isotropy group of the torsion form.

Example 4.1. Denote by H^6 the 6-dimensional Heisenberg group. There exists a left-invariant, cocalibrated G_2-structure ω^3 on the 7-dimensional Lie group $H^6 \times \mathbb{R}^1$. In [11] we computed its torsion form T:

$$\omega^3 = e_{127}+e_{135}-e_{146}-e_{236}-e_{245}+e_{347}+e_{567}, \quad T = e_5\wedge(e_{13}-e_{67})+e_4\wedge(e_{37}+e_{16}).$$

Moreover, the Ricci tensor Ric^∇ of the unique connection with totally skew-symmetric torsion as well as the symmetric endomorphism $T_{imn}\cdot T_{jmn}$ are given by the formulas

$$\mathrm{Ric}^\nabla = \mathrm{diag}(-2, 0, -2, 0, 0, -2, -2), \quad T_{imn}\cdot T_{jmn} = \mathrm{diag}(4, 0, 4, 4, 4, 4, 4).$$

A transformation preserving the geometric structure preserves the Ricci tensor Ric^∇ and the symmetric form $T_{imn}\cdot T_{jmn}$, too. Consequently, for the Lie algebra $\mathfrak{g}_T$ of the group G_T we obtain the necessary conditions $\omega_{2\alpha} = 0$, $\omega_{4\beta} = 0$, $\omega_{5\gamma} = 0$ for any $1 \leq \alpha \leq 7$, $\beta \neq 5$ and $\gamma \neq 4$. Combining these 14 equations with the equations defining the Lie algebra $\mathfrak{g}_2$ inside of $\mathfrak{so}(7)$ (see [14] or [11]) we obtain seven nontrivial parameters $\omega_{13}, \omega_{16}, \omega_{17}, \omega_{36}, \omega_{37}, \omega_{67}, \omega_{45}$, related by three equations

$$\omega_{13} = -\omega_{67}, \quad \omega_{16} = \omega_{37}, \quad \omega_{17}+\omega_{36}+\omega_{45} = 0.$$

We understand the skew-symmetric matrix $\Omega := (\omega_{ij})$ as a vector field on $\mathbb{R}^7$ and compute the Lie derivative $\mathcal{L}_\Omega T$ of the torsion form,

$$\mathcal{L}_\Omega T = 2\cdot\omega_{13}\cdot(e_{147}-e_{346})+2\cdot\omega_{16}\cdot(e_{356}-e_{157})+2\cdot\omega_{17}\cdot(e_{357}+e_{156}+e_{134}-e_{467}).$$

Consequently, the Lie group G_T is one-dimensional and its Lie algebra is described by two parameters ω_{36}, ω_{45} and one equation:

$$\omega_{36}+\omega_{45} = 0.$$

Example 4.2. The product $N^6 \times \mathbb{R}^1$ of the 3-dimensional, complex, solvable Lie group N^6 by $\mathbb{R}^1$ admits a left invariant cocalibrated G_2-structure. In [11] we derived its torsion,

$$T = 2\cdot(e_{256}-e_{234}).$$

A straightforward calculation yields that the subgroup $G_T \subset G_2$ is a maximal torus of G_2. A basis of its Lie algebra is given by the following two matrices

$$\begin{bmatrix} 0&0&0&0&0&0&-2\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&-1&0&0&0&0\\ 2&0&0&0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}.$$

5. A characterization of Spin(7)-structures

Let us once again return to Example 3.4. Any 8-dimensional Riemannian manifold equipped with a Spin(7)-structure admits a unique connection preserving the Spin(7)-structure with totally skew-symmetric torsion (see [22]). In general, fix a compact, connected subgroup $G \subset SO(n)$ and consider G-geometries. *Any* Riemannian G-manifold

$(M^n, g, \mathcal{R})$ admits a unique G-connection with totally skew-symmetric torsion if and only if the G-representations

$$\Theta_1 : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}$$

are isomorphic (see Theorem 3.1). We will prove that under a certain condition on the group G only the case of $n = 8$ and $\mathrm{G} = \mathrm{Spin}(7)$ is possible.

Lemma 5.1. *Let* G *be a compact, connected Lie group and denote by* T *its maximal torus. Then the inequality*

$$\dim(\mathrm{G}) \;\leq\; 4 \cdot \big(\dim(\mathrm{T})\big)^2$$

holds. Moreover, if no exceptional Lie algebra occurs in the decomposition of the Lie algebra $\mathfrak{g}$*, then we have*

$$\dim(\mathrm{G}) \;\leq\; 3 \cdot \big(\dim(\mathrm{T})\big)^2.$$

Proof. Remark that the inequality holds for the product of two groups $\mathrm{T}_1 \subset \mathrm{G}_1$, $\mathrm{T}_2 \subset \mathrm{G}_2$ in case it holds for G_1 and G_2. Indeed, $\mathrm{T}_1 \times \mathrm{T}_2$ is a maximal torus in $\mathrm{G}_1 \times \mathrm{G}_2$ and we obtain

$$\dim(\mathrm{G}_1 \times \mathrm{G}_2) \;\leq\; 4 \cdot \big(\dim(\mathrm{T}_1)^2 + \dim(\mathrm{T}_2)^2\big) \;\leq\; 4 \cdot \big(\dim(\mathrm{T}_1) + \dim(\mathrm{T}_2)\big)^2.$$

We split the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l \oplus \mathfrak{z}$ into the simple ideals $\mathfrak{g}_i$ and its center $\mathfrak{z}$. Unless $\mathfrak{g}_i$ is a classical simple Lie algebra, we know that its dimension is bounded by $3 \cdot (\dim(\mathrm{T}_i))^2$ (see [15]). For the exceptional Lie algebras we obtain

$$\dim(\mathrm{G}_2) = 14, \quad 4 \cdot (\dim(\mathrm{T}))^2 = 16; \qquad \dim(\mathrm{F}_4) = 52, \quad 4 \cdot (\dim(\mathrm{T}))^2 = 64;$$
$$\dim(\mathrm{E}_6) = 72, \quad 4 \cdot (\dim(\mathrm{T}))^2 = 144; \qquad \dim(\mathrm{E}_7) = 133, \quad 4 \cdot (\dim(\mathrm{T}))^2 = 196;$$

and $\dim(\mathrm{E}_8) = 248$, $4 \cdot (\dim(\mathrm{T}))^2 = 256$. □

Let $h \in \mathrm{G}$ be an element of the Lie group and denote by

$$\mathrm{Z}(h) \;:=\; \big\{g \in \mathrm{G} : g \cdot h = h \cdot g\big\}, \quad \mathrm{z} \;:=\; \dim \mathrm{Z}(h)$$

its centralizer as well as its dimension. We agree to say that a subgroup $G \subset \mathrm{SO}(n)$ of dimension $\mathrm{g} := \dim(G)$ has the *involution property* if one of the following conditions is satisfied:

(1) $n^2 \neq 3 \cdot \mathrm{g} + 1$;
(2) $n^2 = 3 \cdot \mathrm{g} + 1$, but there does not exist a pair (h, p) consisting of an involution $h \in \mathrm{G}$ and an even integer $0 < p < n$ such that

$$3 \cdot (\mathrm{g} - \mathrm{z}) \;=\; 2 \cdot p \cdot \big(\sqrt{3 \cdot \mathrm{g} + 1} - p\big).$$

Solving the latter equation with respect to p we obtain

$$p \;=\; \frac{1}{2}\big(\sqrt{1 + 3 \cdot \mathrm{g}} \pm \sqrt{6 \cdot \mathrm{z} - 3 \cdot \mathrm{g} + 1}\big).$$

Remark 5.1. Using the representation of G in $\mathbb{R}^n$ we can formulate this condition in a more geometric way. Fix an involution $h \in \mathrm{G} \subset \mathrm{SO}(n)$ and consider the two symmetric spaces

$$\frac{\mathrm{G}}{\mathrm{Z}(h)} \subset \frac{\mathrm{SO}(n)}{\mathrm{Z}_{\mathrm{SO}(n)}(h)} = \mathrm{G}_{n,p},$$

where $\mathrm{G}_{n,p}$ denotes the Grassmannian manifold of all oriented p-planes in $\mathbb{R}^n$ (p even). By the involution property we want to exclude the case that $n = \sqrt{3 \cdot \mathrm{g} + 1}$ and the ratio of the dimensions of these two symmetric spaces is $\frac{2}{3}$.

Example 5.1. Consider the group SU(3). Then $\sqrt{3 \cdot \mathrm{g}+1} = 5$ and the dimension of the centralizer of the involution $h = \mathrm{diag}(1,-1,-1)$ equals $\mathrm{z} = 4$. In particular, $6 \cdot \mathrm{z} - 3 \cdot \mathrm{g} + 1 = 1$ and $p = 2, 3$. Nevertheless there does not exist a subgroup of SO(5) that is isomorphic to SU(3), i.e., SU(3) has the involution property.

Example 5.2. In case of Spin(7), we have $\sqrt{3 \cdot \mathrm{g} + 1} = 8$, but there is no involution $h \in \mathrm{Spin}(7)$ such that

$$3 \cdot \big(21 - \dim \mathrm{Z}(h)\big) = 2 \cdot p \cdot \big(8 - p\big)$$

for an even number p. More generally, we have

Proposition 5.1. *Any compact simple Lie group* G *has the involution property.*

Proof. For the exceptional Lie groups G_2, F_4, E_6, E_7, E_8 the number $\sqrt{3 \cdot \mathrm{g} + 1}$ is not an integer. For the classical groups the irreducible symmetric spaces $\mathrm{G}/\mathrm{Z}(h)$ given by an involution $h \in \mathrm{G}$ are well known (see [17, Chapter 11.2.4])

$$\mathrm{SU}(m)/\mathrm{S}\big(\mathrm{U}(r) \times \mathrm{U}(m-r)\big), \quad \mathrm{SO}(m)/\mathrm{SO}(r) \times \mathrm{SO}(m-r), \quad \mathrm{Sp}(m)/\mathrm{Sp}(r) \times \mathrm{Sp}(m-r).$$

The dimension of the group $\mathrm{G} = \mathrm{SU}(m)$ equals $(m^2 - 1)$ and we obtain the restriction $n = \sqrt{3 \cdot m^2 - 2} \approx \sqrt{3} \cdot m$. On the other hand, the lowest *real* dimension of an $\mathrm{SU}(m)$-representation is $2 \cdot m$, $(m \geq 3)$. This means that even in case $n = \sqrt{3 \cdot m^2 - 2}$ is an integer, there is no subgroup of $\mathrm{SO}(n)$ that is isomorphic to $\mathrm{SU}(m)$. A similar argument applies to the group $\mathrm{Sp}(m)$. Finally, we discuss the case that G is locally isomorphic to $\mathrm{SO}(m)$. Then $n = \sqrt{\frac{3}{2} \cdot m \cdot (m-1) + 1} \approx \sqrt{\frac{3}{2}} \cdot m$. Taking into account the dimensions of all irreducible real $\mathrm{SO}(m)$-representations we conclude that the embedding $\mathrm{SO}(m) \to \mathrm{SO}(n)$ is the standard inclusion

$$h \longrightarrow \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix},$$

and the embedding of the symmetric spaces is the usual inclusion of two Grassmannian manifolds

$$\frac{\mathrm{G}}{\mathrm{Z}(h)} = \mathrm{G}_{m,p} \longrightarrow \frac{\mathrm{SO}(n)}{\mathrm{Z}_{\mathrm{SO}(n)}(h)} = \mathrm{G}_{n,p}.$$

In particular, p is bounded by m, $p < m$. The dimension condition yields $3 \cdot m - p = 2 \cdot n$ and together with $2 \cdot n^2 = 3 \cdot m \cdot (m-1) + 2$ we see that there is no solution (p, m, n) of these two equations satisfying the condition $p < m$. □

Theorem 5.1. *Let* $G \subset SO(n)$ *be a compact, connected group with the involution property. Decompose the Lie algebra into* $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ *and suppose that the* G-*representations*

$$\Theta_1 : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}$$

are isomorphic. Then $n = 8$, $G = \mathrm{Spin}(7)$ *and the representation is the unique irreducible representation of* Spin(7) *in* $\mathbb{R}^8$.

Proof. Denote by $\chi, \chi^* : G \to \mathbb{R}^1$ the characters of the G-representation in $\mathbb{R}^n$ and of the adjoint representation $\mathfrak{g}$. Since $\Lambda^3(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^n \otimes \mathfrak{m}$ by assumption, an easy computation yields the following functional equation between these two characters

$$3 \cdot \chi(h) \cdot \chi^*(h) = \chi^3(h) - \chi(h^3), \qquad h \in G$$

(see [15, p. 381]). Evaluating the characters at the element $h = e$ we obtain a formula relating the dimensions of the group G to the dimension of the representation $\mathbb{R}^n$,

$$n^2 = 3 \cdot \dim(G) + 1.$$

Fix a maximal torus $T^t \subset G$ of dimension t and denote by $h_1 := e, h_2, \cdots, h_{2^t}$ its elements of order two,

$$h_i^2 = e, \quad h_i \cdot h_j = h_j \cdot h_i.$$

The character equation simplifies for each of these elements,

$$3 \cdot \chi(h_i) \cdot \chi^*(h_i) = \chi^3(h_i) - \chi(h_i) = \chi(h_i) \cdot (\chi^2(h_i) - 1).$$

Suppose that $\chi(h_{i_0}) \neq 0$ for some index $2 \leq i_0 \leq 2^t$. Then we obtain $3 \cdot \chi^*(h_{i_0}) = \chi^2(h_{i_0}) - 1$. Using the equations

$$\chi^*(h_{i_0}) = 2 \cdot \dim Z(h_{i_0}) - \dim(G), \quad n^2 = 3 \cdot \dim(G) + 1, \quad \chi(h_{i_0}) = n - 4 \cdot q$$

we see that the latter equation contradicts the involution property of the group G except for $q = n/2$ and $h_{i_0} = -\mathrm{Id}_{\mathbb{R}^n}$. Consequently, the character $\chi(h_i) = 0$ vanishes for any element $h_i \neq \pm \mathrm{Id}_{\mathbb{R}^n}$. Then the number

$$k := \frac{n}{2^{t-1}} \in \mathbb{Z}$$

must be an integer. Indeed, denote by H the finite group consisting of all involutions h_i. Consider the space $(\mathbb{R}^n)^H$ of all H-invariant vectors and its dimension (see [15, p. 16]),

$$\dim (\mathbb{R}^n)^H = \frac{1}{2^t} \sum_{i=1}^{2^t} \chi(h_i).$$

If the involution $(-\mathrm{Id}_{\mathbb{R}^n}) \notin H$ does not belong to the subgroup, we obtain

$$\dim (\mathbb{R}^n)^H = \frac{n}{2^t}.$$

If $(-\mathrm{Id}_{\mathbb{R}^n})$ is an element of H, we choose a subgroup $H_0 \subset H$ of order two not containing this involution. In this case we obtain

$$\dim (\mathbb{R}^n)^{H_0} = \frac{n}{2^{t-1}}.$$

By the previous Lemma 5.1 we have the inequality

$$4 \cdot t^2 \geq \dim(G) = \frac{1}{3}(4^{t-1} \cdot k^2 - 1) \geq \frac{1}{3}(4^{t-1} - 1).$$

In particular, the rank of the compact group G is bounded by five, $t \leq 5$. The cases $t = 1, 2$ or 4 can be directly excluded by the conditions

$$3 \cdot \dim(\mathrm{G}) + 1 \;=\; 4^{t-1} \cdot k^2, \quad \dim(\mathrm{G}) \;\leq 4 \cdot t^2.$$

Let us discuss the case of $t = 3$. Then we obtain the conditions

$$\dim(\mathrm{G}) \;\leq\; 36, \quad 3 \cdot \dim(\mathrm{G}) + 1 \;=\; 16 \cdot k^2.$$

If $k = 1$, the dimension of the group equals five, $\dim(\mathrm{G}) = 5$, and the dimension n of the real representation is given by the formula $n^2 = 3 \cdot \dim(\mathrm{G}) + 1 = 16$. Therefore, the group G is a compact subgroup of rank three in SO(4), a contradiction. In case of $k = 2$ we obtain $\dim(\mathrm{G}) = 21$ and $n = 8$. The group is a 21-dimensional subgroup of rank 3 in SO(8), i.e., $\mathrm{T}^3 \subset \mathrm{G} = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The cases that $k \geq 3$ are impossible $(t = 3)$.

Finally, we discuss the case of $t = 5$. Then we obtain the conditions

$$\dim(\mathrm{G}) \;\leq\; 100, \quad 3 \cdot \dim(\mathrm{G}) + 1 \;=\; 256 \cdot k^2.$$

The parameters $k \geq 2$ are impossible and $k = 1$ yields an 85-dimensional subgroup $\mathrm{G} \subset \mathrm{SO}(16)$ of rank five. Since $85 = \dim(\mathrm{G}) \not\leq 3 \cdot t^2$, by Lemma 5.1 the decomposition of the Lie algebra $\mathfrak{g}$ into simple Lie algebras must contain one of the exceptional algebras $\mathfrak{g}_2$ or $\mathfrak{f}_4$. Again, this is impossible. Suppose, for example, that $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}^*$. Then the compact Lie algebra $\mathfrak{g}^*$ has dimension 71 and rank 3 and these parameters contradict Lemma 5.1. A similar argument excludes the second exceptional summand. □

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