A HYPERCOMPLEX METHOD OF CALCULATING STRESSES IN THREE-DIMENSIONAL BODIES

W. SPRÖSSIG /K. GÜRLEBECK

1. INTRODUCTION

In the plane linear theory of elasticity the application of complex functiontheory got an important mean. In the last 50 years was developed the analytical theory of quaternions.
Essential influence in this developement was given in papers by R. FUETER [7], A. W. BIZADSE [2], R. DELANGHE [5] [6], F. BRACKX/ R. DELANGHE/F. SOMMEN [3], P. LOUNESTO [14] and other authors from various countries. In [18] is made by W. SPRÖSSIG an experiment for using the analytical theory of quaternions to estimate thermal stresses in real bodies. The aim of this paper consist in finding out a connection between new results of numerical collocation in [9] by K. GÜRLEBECK and quaternionic representations of solutions of boundary-value-problems in three-dimensional linear elasticity.

2. PROBLEM

We consider an isotropic elastic body in the space R^3, $\partial G = \Gamma$ is a piecewise smooth Ljapunoff-surface.
In equilibrium state the displacements u fulfil the following system of equations

$$\Delta u + \frac{m}{m-2}\,\mathrm{grad}\,\mathrm{div}\,u = -f \quad \text{in } G \qquad (2.1)$$

where $u = (u_1, u_2, u_3)$, $f = (f_1, f_2, f_3)$ is the vector of outer forces, m is a real number with $m > 2$ or $m = 0$. For $m > 2$ this constant has a physical intèrpretation, m^{-1} is called POISSON-number. On the boundary Γ we have the condition (i) or (ii)

$$\text{(i)} \quad u = g \qquad (2.2)$$

$$\text{(ii)} \quad \frac{\partial u}{\partial n} + n\frac{\mathrm{div}\,u}{m-2} + \frac{1}{2}\, n \times \mathrm{rot}\, u = \tilde{g}. \qquad (2.3)$$

For thermal stresses the functions f and g can be choosen in

This paper is in final form and no version of it will be submitted for publication elsewhere.

the form

$$f = -2\,\frac{m+1}{m-2}\,\nabla\beta s, \quad \tilde{g} = \frac{m+1}{m-2}\,\beta s n,$$

where s is temperature and β thermal expansion number.

3. SOME LINEAR OPERATORS

Consider four-componentical real vectors $u = (u_0, \mathrm{u})$, $v = (v_0, \mathrm{v})$ $\mathrm{u} = (u_1, u_2, u_3)$, $\mathrm{v} = (v_1, v_2, v_3)$ and introduce the quaternionic product:

$$u \quad v = (u_0 v_0 - (\mathrm{u}, \mathrm{v}),\ \mathrm{u} \times \mathrm{v} + u_0 \mathrm{v} + v_0 \mathrm{u})$$

where $(\mathrm{u}, \mathrm{v}) = u_1 v_1 + u_2 v_2 + u_3 v_3$.
In setting

$$D(\nabla) = (0, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$$

we receive

$$D(\nabla)u = (-\operatorname{div} \mathrm{u},\ \operatorname{rot} \mathrm{u} + \operatorname{grad} u_0).$$

The Banach-spaces of quaternionic-valued or Q-valued functions are designed by C_Q^k, L_p^Q, H_Q^s a.s.o..
For the inner product of two functions f and g of the space $L_2^Q(G)$ we are writing

$$(f,g) = \int_G \bar{f}\ g\, dG_y .$$

We will recall the real vector $\bar{u} = (u_0, -\mathrm{u})$, $\mathrm{u} = (u_1, u_2, u_3)$ conjugated quaternion.
The class of all vectorfunctions $u \in C_Q^1(G)$ with

$$D(\nabla)u = 0 \tag{3.1}$$

we will call Q-analytical functions.
First investigations of this system were made by G. MOISIL/ N. THEODORESCU in 1941. It is necessary to introduce the weak singular integraloperator

$$(Tu)(x) = \frac{1}{4\pi} \int_G \frac{\theta \quad u}{|x-y|^2}\, dG_y, \quad x \in G, \tag{3.2}$$

where $\theta = (0, \frac{x-y}{|x-y|})$. The operator T fulfils the equation

$$D(\nabla)Tu = u$$

for every point in the domain G.
Furthermore if we are setting $n = (0, n_1, n_2, n_3)$, where (n_1, n_2, n_3) described the unit-vector of the outer normal on Γ in the point y,

the twodimensional operator of CAUCHY-typ

$$(Su)(x) = \frac{1}{4\pi} \int_{\Gamma} \frac{\theta \; n \; u}{|x - y|^2} d\Gamma_y, \quad x \notin \Gamma \qquad (3.3)$$

is connected with the operators T and $D(\nabla)$ by the formulas

$$\begin{aligned} u(x) &= (Su)(x) + (TD(\nabla)u)(x), \quad x \in G \\ 0 &= (Su)(x) + (TD(\nabla)u)(x), \quad x \in R^3 \bar{G}. \end{aligned} \qquad (3.4)$$

This formula is the so-called generalized POMPEIU-decomposition. If we calculate the limits

$$\lim_{\substack{x \to x' \in \Gamma \\ x \in G}} (Su)(x) = \frac{1}{2}(u + Su)(x') = (Pu)(x') \qquad (3.5)$$

$$\lim_{\substack{x \to x' \in \Gamma \\ x \in R^3 \bar{G}}} (Su)(x) = \frac{1}{2}(u - Su)(x') = (Qu)(x')$$

we have to define the operator S. It is given by

$$(Su)(x) = \frac{1}{2\pi} \int_{\Gamma} \frac{\theta \; n \; u}{|x - y|^2} d\Gamma_y, \quad x \in \Gamma. \qquad (3.6)$$

The integral is existing in sense of CAUCHY.

4. AN INTEGRAL REPRESENTATION

Let

$$D = \ker D(\nabla) \cap L_2^Q(G) + \mathbb{T}L_2^Q(G).$$

D describes the class of generalized (in sense of SOBOLJEW) differentiable functions, whose derivatives belong to $L_2^Q(G)$, see also W. SPRÖSSIG [19].

We consider the completed system

$$\begin{aligned} \Delta u_0 &= - f_0 \\ \Delta u + \frac{m}{m-2} \operatorname{grad} \operatorname{div} u &= - f. \end{aligned} \qquad (4.1)$$

By setting $f = (f_0, f)$, $u = (u_0, u)$ and $Mu = (\frac{2(m-1)}{m-2} u_0, u)$ and using the formula

$$D(\nabla)D(\nabla)u = - \Delta u$$

we obtain

$$D(\nabla)MD(\nabla)u = f. \qquad (4.2)$$

We remark, that for $m = 0$ follows $M = I$.

In generalization of the theory of I. N. VEKUA (1962, [20]) V. IFTIMIE got in 1965 a multidimensional analogue, see [11].

We find the following representation.

THEOREM 4.1. Let $f \in L_2^Q(G)$, then common solution in the class D is representing in the form

$$u = \Phi_0 + TM^{-1}\Phi_1 + TM^{-1}Tf, \qquad (4.3)$$

where $\Phi_0, \Phi_1 \in \ker D(\nabla) \cap L_2^Q(G)$.

REMARK 4.1. Formula (3.9) can be transformed into

$$u = \chi + \frac{m}{2(m-1)} T \operatorname{div} \chi + TM^{-1}Tf \qquad (4.4)$$

with $\chi \in \ker \Delta(G)$, $\chi : R^3 \to R^4$.

5. DIRICHLET-BOUNDARY-VALUE-PROBLEM

Now, we will solve the partial differential equation (4.2) with the boundary condition (2.2). It is true the following assertion:

THEOREM 5.1. Let $f \in L_r^Q(G)$, $r > 3$, $g \in H^s(\Gamma)$. The boundary value problem (4.2) - (2.2) has the unique solution

$$u = Sg - TM^{-1}Tf + TM^{-1}S(\operatorname{tr} TM^{-1}S)^{-1}(Qg+TM^{-1}Tf). \quad (5.1)$$

The operator $\operatorname{tr} TM^{-1}S$ is an isomorphism from $\operatorname{im} P \cap H^s(\Gamma)$ to $\operatorname{im} Q \cap H^s(\Gamma)$.

REMARK 5.1. If $(Vu)(x) = \frac{1}{4\pi} \int_\Gamma \frac{n \quad u}{|x-y|} d\Gamma_y$ then it's true the special representation for solution of the problem (4.2) - (2.2), if the parameter m is choosen zero

$$u = Sg - T^2f + TSV^{-1}(Qg + T^2f).$$

REMARK 5.2. If we use for describing these representation formulas, then we have to calculate the multidimensional integraloperators S, T, Q and the differential operator V^{-1}. Calculations of these operators cause annoyance. Therefore we developed in addition an other numerical approach.

6. COMPLETE SYSTEMS OF ANALYTIC FUNCTIONS

The proof of all numerical methods for approximate solution of boundary value problem (2.1) - (2.2) requires the construction of complete function systems in $\ker D(\nabla)\, MD(\nabla)(G) \cap H_Q^s(G)$. Since we will use formula (4.3) it will be sufficient, to find complete systems in $\ker D(\nabla)(G) \cap H^s(G)$ respectively $R(\ker D(\nabla)(G))$ . . The basic functions used in this paper result from fundamental solutions of generalized CAUCHY-RIEMANN-operator $D(\nabla)$, which singularities will be choosen in suitable manner in $R^3 \setminus \overline{G}$. Similar constructions were showed by F.E. BROWDER [4], V. D. KUPRADZE/M. A. ALEKSIDZE [1], C. MÜLLER and other authors ([10], [12], [16]) for special equations of mathematical physics.

DEFINITION 6.1. Let X be a normed right-vector-space. The system $\{f_i\}_{i=1}^{\infty} \subset X$ is called complete in X, if it is possible to approximate every element $f \in X$ arbitrary closely by finite right-linear-combinations of the elements $\{f_i\}_{i=1}^{\infty}$. $\{f_i\}_{i=1}^{\infty}$ is called closed in X, if for every bounded right-linear-functional L over X with values in Q $L(f_i) = 0 \quad i \in N$ implies $L = 0$.

LEMMA 6.1. Let X a normed right-vector-space over Q. The system $\{f_i\}_{i=1}^{\infty} \subset X$ is closed, if it is complete in X.

LEMMA 6.2. Every bounded right-linear Q-valued functional L over $L_2^Q(G) \cap \ker D(\nabla)(G)$ has a representation

$$L(f) = \int_G \overline{l}\, f\, dG \qquad f \in L_2^Q(G) \cap \ker D(\nabla)(G)$$

with $l \in L_2^Q(G) \cap \ker D(\nabla)(G)$.

Proof: See [3].

THEOREM 6.1. Let G and G_A bounded star-shaped domains of R^3 with smooth boundaries Γ resp. Γ_A, such that $\overline{G} \subset G_A$. Further let $\{x_i\}_{i=1}^{\infty} \subset \Gamma_A$ a dense subset of Γ_A and $\varphi_i(x) = \frac{\theta_i(x)}{|x-x_i|^2}$, where $\theta_i(x) = (0, \frac{x - x_i}{|x-x_i|})$. Then the system $\{\varphi_i\}_{i=1}^{\infty}$ is complete in $L_2^Q(G) \cap \ker D(\nabla)(G)$.

Proof: It is sufficient, to show the closure of $\{\varphi_i\}_{i=1}^{\infty}$. Let $L \in (L_2^Q(G) \cap \ker D(\nabla)(G))'$ and $L(\varphi_i) = 0 \quad i \in N$. Let G' bounded star-shaped domain with smooth boundary, such that $\overline{G} \subset G'$ and $\overline{G'} \subset G_A$, furthermore let $\eta \in C_0^{\infty}(G')$ with $\eta(x) = e_1 \quad x \in G$. The function L will be expanded from G to R^3 by zero-function. Conditions $L(\varphi_i) = 0 \quad i \in N$ involve

$$\int_G \overline{l}(y) \frac{\theta_i(y)}{|y-x_i|^2}\, dG_y = 0 \quad i \in N$$

intail $(T1)(x_i) = 0 \quad i \in N$ also

$$(T1)(x) = 0 \qquad x \in co\,\overline{G} = R^3\,\overline{G}. \qquad (6.1)$$

Let $\varphi \in L_2^Q(G') \cap \ker D(\nabla)(G) \subset L_2^Q(G) \cap \ker D(\nabla)(G)$ then

$$L(\varphi) = L(\varphi\eta) = (L * \delta)(\varphi\eta) = + \frac{1}{4\pi}[L*(0,\frac{y}{|y|^3})]D(\nabla)(\varphi\eta) =$$

$$= \int_G[\int_G T(y)\,\frac{1}{4\pi}\,\frac{\theta}{|y-x|^2}\,dG_y]\;D(\nabla)(\varphi\eta)(x)dG_x +$$

$$+ \int_{coG}[\int_G T(y)\,\frac{1}{4\pi}\,\frac{\theta}{|y-x|^2}\,dG_y]\;D(\nabla)(\varphi\eta)(x)dG_x = 0,$$

since $\varphi \in \ker D(\nabla)(G)$ and (6.1). Let $\{G_i\}_{i=1}^{\infty}$ a sequence of domains with following properties

$$\overline{G} \subset G_{n+1},\ \overline{G_{n+1}} \subset G_n,\ \overline{G_n} \subset G_A \qquad n \in N$$

$$\text{mes}\ (G_n\ G) \xrightarrow[n\to\infty]{} 0,\quad \text{mes}(\partial G_n) \xrightarrow[n\to\infty]{} \text{mes}\ \partial G.$$

Repetition of upper consideration shows

$$L(\varphi) = 0 \qquad \varphi \in \bigcup_{i=1}^{\infty}(L_2^Q(G_i) \cap \ker D(\nabla)(G_i)).$$

If we close $\bigcup_{i=1}^{\infty}(L_2^Q(G_i) \cap \ker D(\nabla)(G_i))$ in $L_2^Q(G)$ we get the solution [9]. $L(\varphi) = 0 \quad \varphi \in L_2^Q(G) \cap \ker D(\nabla)(G)$ involve $L = 0$.

<u>REMARK 6.1.</u> It is possible to prove Theorem 6.1 in $L_p^Q(G) \cap \ker D(\nabla)(G) \quad 1 < p < \infty$ [9].

Let $R_I = \{f \in C_Q^{0,\alpha}(\Gamma),\ \ 0 < \alpha \leq 1,\ \ Sf = f\}$

$R_A = \{f \in C_Q^{0,\alpha}(\Gamma),\ \ 0 < \alpha \leq 1,\ \ Sf = -f\}$

$I = \overline{R_I}^{L_2^Q(\Gamma)} \qquad A = \overline{R_A}^{L_2^Q(\Gamma)}.$

The extension of S from $C_Q^{0,\alpha}(\Gamma)$ to $L_2^Q(\Gamma)$ also will be denoted by S.

<u>THEOREM 6.2.</u> Let $\{x_i\}_{i=1}^{\infty}$, $\{\varphi_i\}_{i=1}^{\infty}$, G, G_A, Γ, Γ_A defined in analogy to Theorem 6.1 and γ_0 the trace operator [13]. Then the system $\{\gamma_0\varphi_i\}_{i=1}^{\infty}$ is complete in I.

<u>Proof:</u> We define for $f \in L_2^Q(\Gamma)$ the function $f_k = \overline{n\quad f}$ and show at first

$$(\bar{f}_k, \gamma_0 \varphi_i)_{L_2^Q(\Gamma)} = 0 \qquad i \in N$$

iff $\quad f \in {}_{I}$ (6.2).

$(\bar{f}_k, \gamma_0\varphi_i) = 0 \quad i \in N$ iff $\int_\Gamma \overline{n}\;\bar{f}\;\gamma_0\varphi_i d\Gamma = 0 \quad i \in N$ iff
$(Sf)(x_i) = 0 \quad i \in N$ iff $(Sf)(x) = 0 \quad x \in \text{co}\,\overline{G}$ iff [see [17])
$Sf = f$ iff $f \in {}_{I}$.
Further it will be shown, that for $f \in {}_{I}$

$$(\bar{f}_k, \varphi) = 0 \qquad \varphi \in {}_{I}. \tag{6.3}$$

Let $\{G_i\}_{i=1}^{\infty}$ the compact exhaustion of co $\overline{G}$ considered in proof of Theorem 6.1. From Theorem 6.1 follows the completeness of $\{\varphi_i\}_{i=1}^{\infty}$ in $L_2^Q(G) \cap \ker D(\nabla)(G_i) \quad i \in N$.
Consider at first $\varphi \in L_2^Q(G_i) \cap \ker D(\nabla)(G_i)$ for some $i \in N$.
The function φ can be represented by

$$\varphi = \lim_{j \to \infty} \sum_{i=1}^{j} \varphi_i \; c_{ij}$$

in $L_2^Q(G_i) \cap \ker D(\nabla)(G_i)$. It follows by using Theorem of HARNACK
$\gamma_0\varphi = \lim_{j \to \infty} \sum_{i=1}^{j} \gamma_0\varphi_i \; c_{ij}$ in $L_2^Q(\Gamma)$ also

$$(\bar{f}_k, \gamma_0\varphi) = \lim_{j \to \infty} \sum_{i=1}^{j} (\bar{f}_k, \gamma_0\varphi_i)c_{ij} = 0, \quad \text{since (6.2).}$$

Let $\varphi \in {}_{I}$, then exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset R_I$ with ψ_n tends to φ if n tends to infinity in $L_2^Q(\Gamma)$. $\psi_n \in {}_{I} \quad n \in N$ involves $\chi_n = S\psi_n \in C(\overline{G}) \cap \ker D(\nabla)(G)$ and $\gamma_0\chi_n = \psi_n$. So we receive the existence of a sequence $\{\eta_j^{(n)}\}_{j=1}^{\infty} \quad \eta_j^{(n)} \in C(\overline{G}_j) \cap \cap \ker D(\nabla)(G_j)$ and $\eta_j^{(n)}$ tends to χ_n if j tends to infinity in $C(\overline{G})$.

It follows $\gamma_0\eta_j^{(n)} \xrightarrow[j \to \infty]{} \gamma_0\chi_n = \psi_n$ in $L_2^Q(\Gamma)$, such that
$(\bar{f}_k, \varphi) = \lim_{n \to \infty} (\bar{f}_k, \psi_n) = \lim_{n \to \infty} \lim_{j \to \infty} (\bar{f}_k, \gamma_0\eta_j^{(n)}) = 0$, since (6.3).
Now the closure of $\{\gamma_0\varphi_i\}_{i=1}^{\infty}$ in ${}_{I}$ can be shown.

Let $f \in {}_{I}$, $(f, \gamma_0\varphi_i) = 0 \quad i \in N$ and $g = \overline{n} \quad f$ then we obtain

$$(\bar{g}_k, \gamma_0\varphi_i) = 0 \qquad i \in N.$$

These equations imply $g \in {}_{I}$ and also $(\bar{g}_k, f) = (f,f) = 0$ therefore $f = 0$.

<u>THEOREM 6.3.</u> Let G, G_I bounded star-shaped domains with smooth

boundaries Γ and Γ_I, $\{y_i\}_{i=1}^{\infty}$ a dense subset of Γ_I and $\psi_i(x) = \frac{\theta_i}{|x-y_i|^2}$. Then the system $\{\gamma_0\psi_i\}_{i=1}^{\infty}$ is complete in $_A$.

Proof: See Theorem 6.2.

THEOREM 6.4. Let G, G_I, G_A, Γ_I, Γ_A, ψ_i, φ_i such as in Theorem 6.1 - 6.3 defined. Then the system $\{\gamma_0\varphi_i\}_{i=1}^{\infty} \cup \{\gamma_0\psi_i\}_{i=1}^{\infty}$ is complete in $L_2^Q(\Gamma)$.

Proof: Every function $f \in L_2^Q(\Gamma)$ can be written in the form $f = \frac{1}{2}(I + S)f + \frac{1}{2}(I - S)f$, where $(I + S)f \in {}_I$ $(I - S)f \in {}_A$ and use of the Theorem 6.2 and 6.3.

REMARK 6.3. The condition at position of singularities of the functions φ_i resp. ψ_i can be formulated not so strongly by use of identity-theorem for generalized analytical functions [17].

REMARK 6.4. It is possible to get the appropriate results of Theorems 6.1 - 6.4 in $H_Q^s(G) \cap \ker D(\nabla)(G)$, $\mathbb{R}_I^{H_Q^s(\Gamma)}$, $\mathbb{R}_A^{H_Q^s(\Gamma)}$ and $H_Q^s(\Gamma)$.

7. DECOMPOSITION OF HOMOGENEOUS BOUNDARY VALUE PROBLEMS

Method of decomposition is described in general case as follows. Let A an elliptic differential operator of second order with constant coefficients. Consider the boundary value problem

$$\begin{aligned} Au &= 0 \quad \text{in } G \\ Ru &= g \quad \text{on } \Gamma. \end{aligned} \qquad (7.1)$$

Assume the correctness of (7.1) in suitable spaces. The problem (7.1) will transformed into two boundary value problems (of first order) of the equation $D(\nabla)u = 0$ in G.

$$\begin{aligned} D(\nabla)v &= 0 \quad \text{in } G \\ R_1 v &= Pg \text{ on } \Gamma \end{aligned} \quad (7.2) \qquad \begin{aligned} D(\nabla)w &= 0 \quad \text{in } G \\ R_2 w &= Qg \text{ on } \Gamma \end{aligned} \quad (7.3)$$

where $P^2 = P$, $Q^2 = Q$, $P + Q = I$ and a relation $u = \Phi(v,w)$ will be given between the solutions of (7.1) and (7.2), (7.3). We remark, that in applications to various boundary value problems the operators P and Q are generalizations of the projectors P and Q investigated by A. W. BIZADSE [2] and W. SPRÖSSIG [17] in detail. In this paper we will demonstrate the method for boundary value problems in three-dimensional linear elasticity.

<u>THEOREM 7.1.</u> Let $G \subset R^3$ bounded domain with smooth boundary Γ, γ_0 trace operator [13] and $g \in H_Q^s(\Gamma)$ $s > 3/2$. Then the boundary value problem

$$\begin{aligned} D(\nabla)MD(\nabla)u &= 0 \quad \text{in } G \\ \gamma_0 u &= g \quad \text{on } \Gamma \end{aligned} \tag{7.4}$$

can be decomposed into two unique solvable boundary value problems of first order.

$$\begin{aligned} D(\nabla)v &= 0 \quad \text{in } G \\ \gamma_0 v &= \tfrac{1}{2}(I + S)g \quad \text{on } \Gamma \end{aligned} \tag{7.5}$$

$$\begin{aligned} D(\nabla)w &= 0 \quad \text{in } G \\ \gamma_0 TM^{-1}w &= \tfrac{1}{2}(I + S)g \quad \text{on } \Gamma \end{aligned} \tag{7.6}$$

and the relation between u, v. w is given by

$$u = v + TM^{-1}w.$$

<u>Proof:</u> From (3.4) follows for every $v \in H_Q^{s+1/2}(G)$

$$v = S\gamma_0 v + TD(\nabla)v, \tag{7.7}$$

from $D(\nabla)v = 0$ get $v = S\gamma_0 v$ and $\gamma_0 v = \frac{I + S}{2}\gamma_0 v = P\gamma_0 v$, that means $\gamma_0 v \in \operatorname{Im} P$. Let $h \in \operatorname{Im} P \cap H_Q^s(\Gamma)$. Then the problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } G \\ \gamma_0 v &= h \quad \text{on } \Gamma \end{aligned}$$

has a unique solution v with $\gamma_0 v = h \in \operatorname{Im} P$ and with (7.7) follows

$$\gamma_0 v = P\gamma_0 v + \gamma_0 TD(\nabla)v$$

and $\gamma_0 TD(\nabla)v = 0$, which implies $TD(\nabla)v = 0$, since

$$TD(\nabla)v \in \ker \Delta(G) \cap H_Q^{s+1/2}(G)$$

follows $D(\nabla)v = 0$ and v is also a solution of the boundary value problem

$$\begin{aligned} D(\nabla)v &= 0 \quad \text{in } G \\ \gamma_0 v &= h \quad \text{on } \Gamma. \end{aligned} \tag{7.8}$$

With that the condition $h \in \operatorname{Im} P \cap H_Q^s(\Gamma)$ is necessary and sufficient for solvability of (7.8). With $z = u - v$ we get from (7.4) - (7.5) the following boundary value problem

$$\begin{aligned} D(\nabla)MD(\nabla)z &= 0 \quad \text{in } G \\ \gamma_0 z &= Qg \text{ on } \Gamma. \end{aligned}$$

The substitution $w = MD(\nabla)z$ shows the solvability of (7.6). Uniqueness of solutions of (7.5) and (7.6) follows from

$$\ker D(\nabla)(G) \cap H_Q^{s+1/2}(G) \subset \ker \Delta(G) \cap H_Q^{s+1/2}(G),$$

$\ker T = \{0\}$ and uniqueness of (7.4).

REMARK 7.1. H is possible to get the result of Theorem 7.1 by use of weaker smoothness conditions. The invariance relations

$$Sf = f \quad f \in \gamma_0(\ker D(\nabla)(G) \cap H_Q^{s+1/2}(G)) \quad \text{and}$$
$$Sf = -f \quad f \in \gamma_0(\ker D(\nabla)(\mathrm{co}\,\bar{G}) \cap H_Q^{s+1/2}(\mathrm{co}\,\bar{G})$$

enable s simple numerical treatment of the singular integral operator S. These relations will be used now to construct an operator N with similar properties in spaces of boundary values which correspond to boundary conditions of second kind. Let F defined by

$$Fu = \mathrm{Im}[\frac{\partial u}{\partial n} + n\,\frac{\mathrm{div}\,u}{m-2} + \frac{1}{2}\,n \times \mathrm{rot}\,u].$$

From representation (4.3) get direct decomposition

$$F(\ker D(\nabla)MD(\nabla)(G) \cap H_Q^s(G)) = F(\ker D(\nabla)(G) \cap H_Q^s(G)) + $$
$$+ F(\ker D(\nabla)(\mathrm{co}\,\bar{G}) \cap H_Q^s(\mathrm{co}\,\bar{G})).$$

It is possible to prove the existence of complete orthonormal systems in

$$\mathcal{R}_I = F(\ker D(\nabla)(G) \cap H_Q^s(G)) \quad \text{resp.} \quad \mathcal{R}_A = F(\ker D(\nabla)(\mathrm{co}\,\bar{G}) \cap H_Q^s(\mathrm{co}\,\bar{G})).$$

With that the operator N can be defined correctly by the relations

$$Nf = f, \quad f \in \mathcal{R}_I, \quad Nf = -f, \quad f \in \mathcal{R}_A.$$

As a corrolary we get the property

$$N^2 f = f, \quad f \in H_Q^{s-3/2}(\Gamma)/_{\mathrm{span}\{f:\, \int_\Gamma f\,d\Gamma_x = 0,\, \int_\Gamma x \times f(x)\,d\Gamma_x = 0\}}.$$

If we define $P = \frac{1}{2}(I + N)$, $Q = \frac{1}{2}(I - N)$ and substitute γ_0 by F in Theorem 7.1 we get an analogous result in the case of boundary conditions of second kind.

REMARK 7.2. In the case $m = 0$ we get in Theorem 7.1 corresponding results for the DIRICHLET-problem of LAPLACE equation.

8. TRANSFORMATION FORMULA

To use decompositon results for numerical methods it is necessary to describe the range of the weak singular integral operator T simple computation checks following

THEOREM 8.1. Let $G \subset R^3$ bounded domain and $x_1 \notin \bar{G}$. Then

$$T\left(-\frac{\theta_i}{|x-x_i|^3}\right) = -\frac{1}{|x-x_i|}e_1 + \Phi_G, \quad \Phi_G \in \ker D(\nabla)(G) \qquad (8.1)$$

$$T\left(\frac{x_i^k-x^k}{|x-x_i|^3}e_k\right) = \frac{1}{2|x-x_i|}e_1 - \frac{1}{2}\frac{\theta_i}{|x-x_i|^3}(x_i^k-x^k)e_k + \Psi_{G,k}, \qquad (8.2)$$

where $\Psi_{G,k} \in \ker D(\nabla)(G)$ and $e_1 = (1,0,0,0),\ldots,e_4 = (0,0,0,1)$.

REMARK 8.1. The essential (non-analytical) parts of (8.1) and (8.2) are independent of the domain if $x_i \notin G$ and so are the following systems constructed in section 8.

9. COMPLETE SYSTEMS OF SOLUTIONS OF DIFFERENTIAL EQUATION

THEOREM 9.1. Let $G \subset R^3$ bounded star-shaped domain with smooth boundary Γ, $G_A \subset R^3$ with $\overline{G} \subset G_A$, $\Gamma_A = \partial G_A$ and $\{x_i\}_{i=1}^{\infty} \subset \Gamma_A$ a dense subset of Γ_A. Then every element $u \in \ker D(\nabla)MD(\nabla)(G) \cap H_Q^s(G)$ can be approximated in $H^s(G)$ arbitrarily closely by expressions

$$u_{n_1,n_2}(x) = \sum_{i=1}^{n_1}\frac{\theta_i}{|x_i-x|^3}a_i + \sum_{i=1}^{n_2}\Big[\frac{1}{|x_i-x|}e_1 b_{1i} + \sum_{k=2}^{4}\Big(\frac{3m-4}{4m-4}\frac{1}{|x_i-x|}e_k + \frac{m}{4m-4}(x_i^k-x^k)\frac{\overline{\theta}_i}{|x_i-x|^3}\Big)b_{ki} \qquad (9.1)$$

with suitable choosen coefficients $a_i \in Q$, $b_{ki} \in R^1$.

Proof: Use (4.3), Theorem 6.1, Remark 6.4, Theorem 8.1 and continuity of $T : H_Q^s(G) \to H_Q^s(G)$.

REMARK 9.1. We get complete systems of harmonic functions in the case $m = 0$.

REMARK 9.2. It is possible to construct complete systems in appropriate subspaces of $H^s(\Gamma)$ by using theorems about traces ([13], [9]).

Presented principle of construction of complete systems in kernel of various elliptical differential operators by means of generalized analytical functions summarizes other approaches (see [1], [10, [12], [16]). Existing relations between other concepts and our construction will be demonstrated at example of PAPKOVIC-

NEUBER-statements in linear elasticity.
From (4.3) follows by simple computation there presentation

$$u = \chi - \frac{m}{2m-2} \operatorname{TRe} D(\nabla)\chi \quad \text{with} \quad \chi \in \ker \Delta(G), \qquad (9.2)$$
$$\chi : R^3 \to R^4.$$

The PAPKOVIC-NEUBER-statements

$$u = \psi - \frac{m}{4m-4} \operatorname{grad}(x \cdot \psi) - \frac{m}{4m-4} \operatorname{grad} \varphi, \qquad (9.3)$$

where $\psi : R^3 \to R^3$, $\varphi : R^3 \to R^1$, $\varphi, \psi \in \ker \Delta(G)$, can be transformed by $\psi = (\varphi, \psi)$ and $u = (u_0, u)$ into

$$\operatorname{Im} u = \operatorname{Im} \psi - \frac{m}{4m-4} \operatorname{Im} D(\nabla)(x \cdot \operatorname{Im} \psi) - \frac{m}{4m-4} \operatorname{Im} D(\nabla)\varphi.$$

Then is valid

$$D(\nabla)[\operatorname{Im} \psi - \frac{m}{2m-2} \operatorname{TRe} D(\nabla)\operatorname{Im}\psi] = D(\nabla)[\operatorname{Im} \psi - \frac{m}{4m-4} D(\nabla)(x \cdot \operatorname{Im} \psi) - \frac{4}{4m-4} D(\nabla) \operatorname{Re} \psi].$$

H involves the existence of a function $\Phi \in \ker D(\nabla)(G)$ with

$$\operatorname{Im} \psi - \frac{m}{2m-2} \operatorname{TRe} D(\nabla)\operatorname{Im} \psi + \Phi =$$
$$= \operatorname{Im} \psi - \frac{m}{4m-4} D(\nabla)(x \cdot \operatorname{Im} \psi) - \frac{m}{4m-4} D(\nabla)\operatorname{Re} \psi$$

and at last

$$\operatorname{Im} \psi + \Phi - \frac{m}{2m-2} \operatorname{TRe} D(\nabla)(\operatorname{Im} \psi + \Phi) = \operatorname{Im} \psi - \frac{m}{4m-4} D(\nabla)(x \cdot \operatorname{Im} \psi) - \frac{m}{4m-4} D(\nabla)\operatorname{Re} \psi. \qquad (9.4)$$

From (9.4) follows the relation

$$\chi = \operatorname{Im} \psi + \Phi, \quad \Phi \in \ker(\nabla)(G)$$

between χ from (9.2) and ψ from (9.3).

<u>REMARK 9.3.</u> An essential advantage of (4.3) for numerical treatment contains in uniqueness of analytical parts Φ_1 and Φ_2.

<u>REMARK 9.4.</u> For numerical solution of elliptic boundary value problems by means of collocation methods based on presented decomposition theorems and constructed function systems will be refered to [9].

<u>REFERENCEC</u>

[1] Алексидзе, М.А.: Решение граничных задач методом разложения по неортогональным функциям, Изд. наука, Москва (1978)

[2] Bizadse, A.W.: Grundlagen der Theorie der analytischen Funktionen, Berlin (1977), Akademieverlag.

[3] Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis, Research Notes in Mathematics 76, Pitman Advanced Publ. Progr., London (198).

[4] Browder, F.E.: Approximation by solutions of partial differential equations, Americ. Journ. of Math., Vol LXXXIV, 137 - 160.

[5] Delanghe, R.: On regular-analytic functions with values in a Clifford algebra, Math. Ann. 185, 91 - 111 (1970).

[6] Delanghe, R.: On the singularities of functions with values in a Clifford algebra, Math. Ann. 196 (293 - 319) (1972).

[7] Fueter, R.: Ueber die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen, Comm. Math. Helv. vol 8 g 371.

[8] Goldschmidt, B.: Regularity Properties of Generalized Analytical Vectors in R^n, MLU Halle-Wittenberg, prepr. Nr. 41 (1980).

[9] Gürlebeck, K.: Dissertation (A), TH Karl-Marx-Stadt (1984).

[10] Herrmann, P., Kersten, H.: Die Lösung der Prae-Maxwellschen Gleichungen mit Hilfe vollständiger Lösungssysteme, Math. Meth. in the appl. Sc. 2, No 4, 410 - 418.

[11] Iftimie, V.: Functions hypercomplexes, Bull. Math. Soc. SCi. Math. R. S. Roumanie 9 (57) 279 - 337.

[12] Kersten, H.: Ueber die gleichmässige Approximierbarkeit harmonischer Funktionen in regulären Gebieten einschliess-

lich des Randes, Diss. RWTH Aachen (1977).

[13] Лионс, Ж.-Л. / Мадженес Э.: Неоднородные граничные задачи и их приложения, Изд. Мир, Москва (1971)

[14] Lounesto, P.: Spinor valued regular functions in hypercomplex analysis, Report HTKK - Mat. A 154 (1979).

[15] Moisil. G./Theodorescu, N.: Mathematica V 5 141 (1931).

[16] Müller, Cl.: Neue Verfahren zur Lösung der elliptischen Randwertprobleme der Mathematischen Physik, Vorträge der Rheinisch-Westfälischen Akademie d. Wiss., No. 288, 27 - 68, Westdeutscher Verlag Opladen (1978).

[17] Spröβig, W.: Dissertation (B), TH Karl-Marx-Stadt (1979).

[18] Spröβig, W.: Anwendung der analytischen Theorie der Quaternionen zur Lösung räumlicher Probleme der linearen Elastizität, ZAMM 59 (1979) 741 - 743.

[19] Spröβig, W.: Räumliches Analogon zum komplexen T-Operator, Beiträge zur Analysis 12 (1978) 113 - 126.

[20] Vekua, I.N.: Verallgemeinerte analytische Funktionen, Akademieverlag, Berlin (1977).

WOLFGANG SPRÖSSIG, KLAUS GÜRLEBECK
TECHNISCHE HOCHSCHULE KARL-MARX-STADT
SEKTION MATHEMATIK
DDR 9010 KARL-MARX-STADT
PSF 964