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s-numbers of diagonal operators and Besov embeddings

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Abstract

In recent years, many papers have appeared in which the asymptotic behaviour of concrete s-numbers of the identity map from $l_p(m)$ into $l_q(m)$ had been calculated. We shall present a survey of these results for estimating the asymptotic behaviour of s-numbers of diagonal operators and Besov embeddings. For this end we shall prove a new finitization method.

Moreover we want to give some historical remarks.

0. Terminology

The class of all (bounded linear) operators between arbitrary Banach spaces is denoted by L , while $L(E,F)$ stands for the set of those operators acting from E into F . If M is a (closed linear) subspace of the Banach space E , then J_M^E denotes the natural injection from M into E and Q_M^E stands for the natural surjection from E onto E/M .

Let (ξ_k) and (η_k) be two sequences. Then $\xi_k \prec \eta_k$ means that there exists a positive constant c such that $\xi_k \leq c \eta_k$ for all $k=1,2,\dots$. The notation $\xi_k \asymp \eta_k$ is used, if simultaneously $\xi_k \prec \eta_k$ and $\eta_k \prec \xi_k$ hold.

Let $0 < r < \infty$ and $0 < w \leq \infty$. Then the Lorentz space $l_{r,w}$ consists of all bounded sequences $a=(\alpha_n)$ for which the

following expression is finite:

$$\|a\|_{r,w} := \begin{cases} \left(\sum_{n=1}^{\infty} (n^{1/r - 1/w} \alpha_n^{\bar{w}})^w \right)^{1/w} & \text{for } w < \infty \\ \sup_n n^{1/r} \alpha_n^{\bar{w}} & \text{for } w = \infty, \end{cases}$$

where $\alpha_n^{\bar{w}} = \inf \{ \zeta \geq 0 : \text{card}\{k : |\alpha_k| \geq \zeta\} \leq n \}$ denotes the so-called non-increasing rearrangement of (α_n) . In the case $|\alpha_1| \geq |\alpha_2| \geq \dots$ it turns out that $\alpha_n^{\bar{w}} = |\alpha_n|$.

1. s-Numbers

An *s-function* is a map *s* which assigns to every operator $S \in L$ a scalar sequence $(s_n(S))$ such that the following conditions are satisfied:

- (1) $\|S\| \geq s_1(S) \geq s_2(S) \geq \dots \geq 0$ for all $S \in L$.
- (2) $s_{n+m-1}(S+T) \leq s_n(S) + s_m(T)$ for $S, T \in L(E, F)$ and $n, m = 1, 2, \dots$
- (3) $s_n(BSX) \leq \|B\| s_n(S) \|X\|$ for $X \in L(E_0, E)$, $S \in L(E, F)$, $B \in L(F, F_0)$ and $n = 1, 2, \dots$.
- (4) $s_n(S) = 0$ for $\text{rank}(S) < n$.
- (5) $s_n(I: l_2(n) \rightarrow l_2(n)) = 1$, where *I* denotes the identical map.

We call $s_n(S)$ the *n*-th *s*-number of the operator *S*.

REMARK. In contrary to earlier definitions (cf. [62] 11.1.1 and 11.8.1) we assume in (2) already the additivity of the *s*-function.

Next we shall list some well-known facts about *s*-functions.

- (1) From $S \in L(E, F)$ and $s_n(S) = 0$ it follows $\text{rank}(S) < n$.
- (2) The *s*-numbers are continuous functions, namely

$$|s_n(S) - s_n(T)| \leq \|S - T\|$$

for all $S, T \in L(E, F)$.

(3) On the class of all compact operators acting between Hilbert spaces there is only one s -function.

We now want to define the most usual examples of s -functions.

Let $S \in L(E, F)$ and $n=1, 2, \dots$. Then we give the following definitions:

n -th approximation number

$$a_n(S) = \inf \{ \|S-L\| : \text{rank}(L) < n \},$$

n -th Gelfand number

$$c_n(S) = \inf \{ \|S|_M\| : M \subseteq E \text{ with } \text{codim } M < n \},$$

n -th Kolmogorov number

$$d_n(S) = \inf \{ \|Q_N^F S\| : N \subseteq F \text{ with } \text{dim } N < n \},$$

n -th Weyl number

$$x_n(S) = \sup \{ a_n(SX) : X \in L(l_2, E) \text{ with } \|X\| \leq 1 \},$$

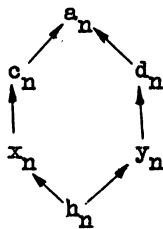
n -th Chang number

$$y_n(S) = \sup \{ a_n(SY) : Y \in L(F, l_2) \text{ with } \|Y\| \leq 1 \},$$

n -th Hilbert number

$$h_n(S) = \sup \{ a_n(SYX) : X \in L(l_2, E), Y \in L(F, l_2) \text{ and } \|X\|, \|Y\| \leq 1 \}.$$

The relationships between these s -functions will be indicated in the following diagram where the arrows point from the smaller s -numbers to the greater ones.



Furthermore there holds the inequality

$$h_n(S) \leq s_n(S) \leq a_n(S)$$

for any s -function s and all operators $S \in L$.

If we denote by S' the dual operator of S , then

$$x_n(S') = y_n(S), \quad y_n(S') = x_n(S), \quad h_n(S') = h_n(S)$$

for an arbitrary operator, while the analogous equalities

$$c_n(S') = d_n(S), \quad d_n(S') = c_n(S), \quad a_n(S') = a_n(S)$$

are true, in general, for compact operators only.

Using s -functions one can define quasi-normed operator ideals (for the definition of quasi-normed operator ideals we refer to [62] 6.1.1).

Let $0 < r < \infty$, $0 < w \leq \infty$ and s be any s -function. Then the operator ideal $L_{r,w}^{(s)}(E,F)$ consists of all operators $S \in L(E,F)$ such that the sequence $(s_n(S))$ belongs to the Lorentz space $l_{r,w}$.

In this case we have a quasi-norm given by

$$\|S\|_{L_{r,w}^{(s)}} := \|(s_n(S))\|_{l_{r,w}}.$$

REMARK. It is convenient to identify $L_{\infty,\infty}^{(s)}$ with L .

An important interpolation formula for the operator ideals $L_{r,w}^{(s)}$ is due to H. König [37] (cf. also [90]).

THEOREM 1. Let s be any s -function, $0 < r_0 < r_1 \leq \infty$, $0 < w, w_0, w_1 \leq \infty$ and $0 < \theta < 1$. If $1/r = (1-\theta)/r_0 + \theta/r_1$, then

$$(L_{r_0,w_0}^{(s)}, L_{r_1,w_1}^{(s)})_{\theta,w} \subseteq L_{r,w}^{(s)},$$

and there exists a constant $c > 0$, depending on the s -function and the numbers r, r_0, r_1, w, w_0, w_1 only such that

$$\|S\|_{L_{r,w}^{(s)}} \leq c \|S\|_{L_{r_0,w_0}^{(s)}}^{1-\theta} \|S\|_{L_{r_1,w_1}^{(s)}}^{\theta}$$

for all $S \in L(E,F)$.

It was shown by J. Peetre/G. Sparr [61] that in the case of approximation numbers the equality

$$(L_{r_0,w_0}^{(a)}, L_{r_1,w_1}^{(a)})_{\theta,w} = L_{r,w}^{(a)}$$

holds with equivalent quasi-norms.

HISTORICAL REMARKS. At the beginning of the century E. Schmidt [73] introduced the concept of singular numbers of integral operators between Hilbert function spaces. J. v. Neumann/R. Schatten [59] extended this concept to the setting of compact operators between Hilbert spaces. The n-th singular number $s_n(S)$ of such an operator S is defined to be the n-th eigenvalue $\lambda_n(|S|)$. Here $(\lambda_n(|S|))$ denotes the non-increasing sequence of all eigenvalues of the operator $|S| = (S^*S)^{1/2}$ counted according to their algebraic multiplicities. It turns out that the singular numbers yield the unique s-function on the class of all compact operators acting between Hilbert spaces. From attempts to classify compact operators between Hilbert spaces H. Triebel [88] gave the definition of the so-called Schatten classes

$$S_{r,w}(H,K) = \left\{ S \in L(H,K) : (s_n(S)) \in l_{r,w} \right\} .$$

For $w=r$ this goes back to J. v. Neumann/R. Schatten [59]. The following remarkable inequality is due to H. Weyl [92]. For any Hilbert space H and every operator $S \in S_r(H,H)$ it is

$$\left(\sum_{n=1}^{\infty} |\lambda_n(S)|^r \right)^{1/r} \leq \left(\sum_{n=1}^{\infty} s_n(S)^r \right)^{1/r} .$$

Therefore inequalities of this type nowadays are called also Weyl inequalities. The following results are known (cf. [37], [67]).

THEOREM 2. Let $0 < r < \infty$, $0 < w \leq \infty$ and s be one of the s-functions a, c, d, x, y. Then there exists a positive constant $c_{r,w}$ such that

$$\| (\lambda_n(S)) \|_{l_{r,w}} \leq c_{r,w} \| S \|_{L_{r,w}^{(s)}}$$

for all Banach spaces E and all $S \in L_{r,w}^{(s)}(E,E)$.

An important step in extending the concept of singular numbers of compact operators between Hilbert spaces to operators between

Banach spaces was the result of D. E. Alachverdiev [1] (cf. also [10]). He proved that the singular numbers of a compact operator between Hilbert spaces coincide with its approximation numbers. Later on A. Pietsch [63] defined the operator ideals $L_r^{(a)} := L_{r,r}^{(a)}$ as an extension of the Schatten classes.

The definition of Kolmogorov numbers was given by I. A. Novoselskij [60]. This concept is based on a paper of A. N. Kolmogorov [36] dealing with the so-called diameters or widths. An axiomatic approach to the theory of s -function is due to A. Pietsch [66]. He also gave the definition of the Gelfand numbers as a dual concept to the Kolmogorov numbers and as corresponding quantities to Gelfand diameters which were defined by B. S. Mitjagin/V. M. Tichomirov [57].

A survey of the essential properties of the approximation numbers and the Kolmogorov numbers was already given by B. S. Mitjagin/A. Pełczyński [56].

The Hilbert numbers were introduced by W. Bauhardt [3], while A. Pietsch [67] defined the Weyl and Chang numbers.

2. s -Numbers of diagonal operators

Let $a = (\alpha_n)$ be any sequence belonging to the Lorentz space $l_{t,w}$. Then we consider the corresponding diagonal operator D_a , which is given by $D_a(\xi_n) = (\alpha_n \xi_n)$. If $1 \leq p, q \leq \infty$ and $1/t > (1/q - 1/p)_+$, then D_a is an operator from l_p into l_q . In the special case where $\alpha_n = n^{-1/t}$ we shall denote the corresponding diagonal operator by D_t . Since the l_p spaces are symmetric, without loss of generality, we can assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$.

Let s be any s -function. Then we ask the question for which r the diagonal operator D_a belongs to $L_{r,w}^{(s)}(l_p, l_q)$. The following

theorem enables us to reduce this problem to the investigation of the asymptotic behaviour of the expression

$$\| I: l_p(m) \rightarrow l_q(m) | L_{r,w}^{(s)} \| .$$

THEOREM 3. Let s be any s -function, and let $1 \leq p, q \leq \infty$. Further assume that the exponents r and t are related via the linear equation

$$1/r = \mu/t + \nu$$

for $0 \leq t_0 < t < t_1 \leq (1/q - 1/p)_+^{-1}$, where the parameters μ and ν may depend upon p and q . Then the following are equivalent:

(1) If $0 < w \leq \infty$ and $t_0 < t < t_1$, then

$$a \in l_{t,w} \text{ implies } D_a \in L_{r,w}^{(s)}(l_p, l_q).$$

(2) If $t_0 < t < t_1$, then

$$s_n(D_t: l_p \rightarrow l_q) < n^{-1/r}$$

for $n=1, 2, \dots$.

(3) If $0 < w \leq \infty$ and $t_0 < t < t_1$, then

$$\| I: l_p(m) \rightarrow l_q(m) | L_{r,w}^{(s)} \| < m^{1/t}$$

for $m=1, 2, \dots$.

(4) The estimat (3) holds for $w=\infty$.

P r o o f: (1) \rightarrow (2): The assertion follows easily from the relation

$$n^{1/r} s_n(D_t: l_p \rightarrow l_q) \leq \| D_t: l_p \rightarrow l_q | L_{r,\infty}^{(s)} \|$$

for $n=1, 2, \dots$.

(2) \rightarrow (3): Let $a \in l_{t,\infty}$. Then

$$s_n(D_a: l_p \rightarrow l_q) \leq \sup_k |k^{1/t} \alpha_k| s_n(D_t: l_p \rightarrow l_q)$$

implies $D_a \in L_{r,\infty}^{(s)}(l_p, l_q)$. According to the closed graph theorem

there exists a constant $c > 0$ such that

$$\| D_a : l_p \rightarrow l_q | L_{r,\infty}^{(s)} \| \leq c \| a | l_{t,\infty} \|$$

for all $a \in l_{t,\infty}$. This yields

$$\| I : l_p(m) \rightarrow l_q(m) | L_{r,\infty}^{(s)} \| \leq c m^{1/t} .$$

Finally, the interpolation formula given in theorem 1 complete this step.

(3) \rightarrow (4): trivial.

(4) \rightarrow (1): Let t with $t_0 < t < t_1$. Since every quasi-norm is equivalent to an u -norm (cf. [62] 6.2.5) we may assume that $L_{r,\infty}^{(s)}$ is u -normed. Without loss of generality let $u \leq t$.

Further we define for $k=0,1,\dots$ the sets

$$N_k = \{ n \in \mathbb{N} : 2^k \leq n < 2^{k+1} \} .$$

If $a \in l_{t,u}$, then it follows

$$\begin{aligned} \| a | l_{t,u} \| ^u &= \sum_{n=1}^{\infty} (n^{1/t - 1/u} \alpha_n)^u = \sum_{k=0}^{\infty} \sum_{n \in N_k} (n^{1/t - 1/u} \alpha_n)^u \\ &\geq \sum_{k=0}^{\infty} 2^k (2^{(k+1)(1/t - 1/u)} \alpha_{2^{k+1}})^u = \frac{1}{2} \sum_{k=0}^{\infty} (2^{(k+1)/t} \alpha_{2^{k+1}})^u . \end{aligned}$$

Let D_k be the operator defined by

$$D_k(\xi_n) = \begin{cases} \alpha_n \xi_n & \text{for } n \in N_k \\ 0 & \text{for } n \notin N_k \end{cases} .$$

Then $D_a = \sum_{k=0}^{\infty} D_k$. Hence the trivial fact $\| D_k : l_p \rightarrow l_p \| = \alpha_{2^k}$

and the assumption lead to

$$\begin{aligned} \| D_a | L_{r,\infty}^{(s)} \| &\leq \left(\sum_{k=0}^{\infty} \| D_k | L_{r,\infty}^{(s)} \| ^u \right)^{1/u} \leq c \left(\sum_{k=0}^{\infty} (2^{k/t} \alpha_{2^k})^u \right)^{1/u} \\ &\leq c \alpha_0 + c \left(\sum_{k=0}^{\infty} (2^{(k+1)/t} \alpha_{2^{k+1}})^u \right)^{1/u} \leq c \alpha_0 + 2^{1/u} c \| a | l_{t,u} \| < \infty . \end{aligned}$$

Since this is true for all $t \in (t_0, t_1)$, again using the interpolation formula, we obtain statement (1).

In the sequel we shall apply this theorem to that s-functions introduced in section 1. Because of the duality relations it is enough to consider the s-functions a, c, x and h .

As recently proved by E. D. Gluskin [13],

$$a_n(I:l_p(m) \rightarrow l_q(m)) \asymp \max\{c_n(I:l_p(m) \rightarrow l_q(m)), d_n(I:l_p(m) \rightarrow l_q(m))\}.$$

So we can restrict ourselves to the s-functions c, x and h .

First we shall review the known estimates of $s_n(I:l_p(m) \rightarrow l_q(m))$.

For abbreviation we shall denote the identity operator I from $l_p(m)$ into $l_q(m)$ by $I_{p,q}^m$.

3. Gelfand numbers

The following theorem is taken from E. D. Gluskin [12], [13] (cf. also [66], [78]).

THEOREM 4.

(1) Let $1 \leq q \leq p \leq \infty$. Then

$$c_n(I_{p,q}^m) = (m-n+1)^{1/q - 1/p} \quad \text{for } 1 \leq n \leq m.$$

(2.1) Let $2 \leq p \leq q \leq \infty$. Then

$$c_n(I_{p,q}^m) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{\frac{1/p - 1/q}{1 - 2/q}} & \text{for } 1 \leq n \leq m - m^{2/q} \\ m^{1/q - 1/p} & \text{for } m - m^{2/q} = n = m. \end{cases}$$

(2.2) Let $1 < p \leq 2 \leq q \leq \infty$. Then

$$c_n(I_{p,q}^m) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{1/2} & \text{for } 1 \leq n \leq m^{2/p'} \\ \left(\frac{m-n+1}{m}\right)^{1/2} m^{1/p'} n^{-1/2} & \text{for } m^{2/p'} \leq n \leq \frac{m}{m^{2/q} - 1 + 1} \\ m^{1/q - 1/p} & \text{for } \frac{m}{m^{2/q} - 1 + 1} \leq n \leq m. \end{cases}$$

(2.3) Let $1 \leq p \leq q \leq 2$. Then

$$c_n(I_{p,q}^m) \asymp \begin{cases} 1 & \text{for } 1 \leq n \leq m^{2/p'} \\ (m^{1/p'} n^{-1/2})^{1/p - 1/q} & \text{for } m^{2/p'} \leq n \leq m \end{cases} .$$

In order to apply theorem 3, we deduce from the previous theorem the following

LEMMA 1. Let $1 \leq p, q \leq \infty$ and $0 < r < \infty$. Then

$$\| I_{p,q}^m \{ L_{r,\infty}^{(c)} \} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \leq q \leq p \leq \infty$. Then

$$1/t = 1/r - 1/p + 1/q .$$

(2.1) Let $2 \leq p \leq q \leq \infty$. Then

$$1/t = 1/r .$$

(2.2) Let $1 < p \leq 2 \leq q \leq \infty$. Then

$$1/t = \begin{cases} 1/r - 1/p + 1/2 & \text{for } 0 < r \leq 2 \\ 2/(p'r) & \text{for } 2 < r < \infty . \end{cases}$$

(2.3) Let $1 \leq p \leq q \leq 2$. Then

$$1/t = \begin{cases} 1/r - 1/p + 1/q & \text{for } 0 < r \leq 2 \frac{1/p - 1/2}{1/p - 1/q} \\ 2/(p'r) & \text{for } 2 \frac{1/p - 1/2}{1/p - 1/q} < r < \infty . \end{cases}$$

To illustrate the following results, we shall indicate the algebraic expressions of $1/r$ as a function of t , p and q , in the unit square with the coordinates $1/p$ and $1/q$. On the plotted lines one, in general, only has $D_a \in L_{u,\infty}^{(s)}$ for all u with $u > r$.

THEOREM 5. If $1 \leq p, q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in L_{t,w}$, then D_a belongs to $L_{r,w}^{(c)}(1_p, 1_q)$, where $1/r$ takes values as indicated in the following diagrams.

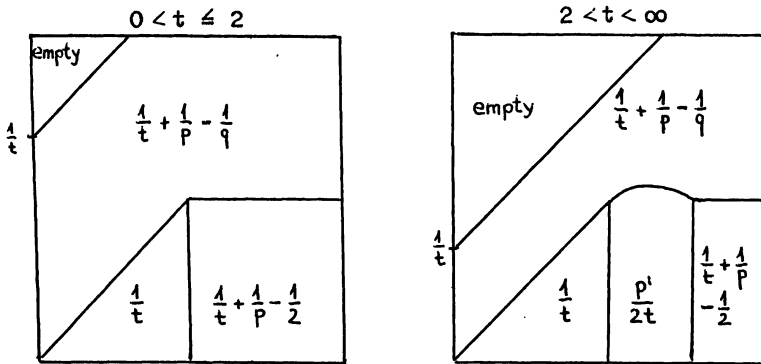


Figure 1

REMARK. The equation of the crooked line in the right diagram is given by $1/t = \frac{1/p - 1/q}{p'(1/p - 1/2)}$.

4. Approximation numbers

The next result follows immediately from the above mentioned relation between approximation numbers, Gelfand and Kolmogorov numbers.

THEOREM 6. If $1 \leq p, q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in l_{t,w}$, then D_a belongs to $L_{r,w}^{(a)}(l_p, l_q)$, where $1/r$ takes values as indicated in the following diagrams.

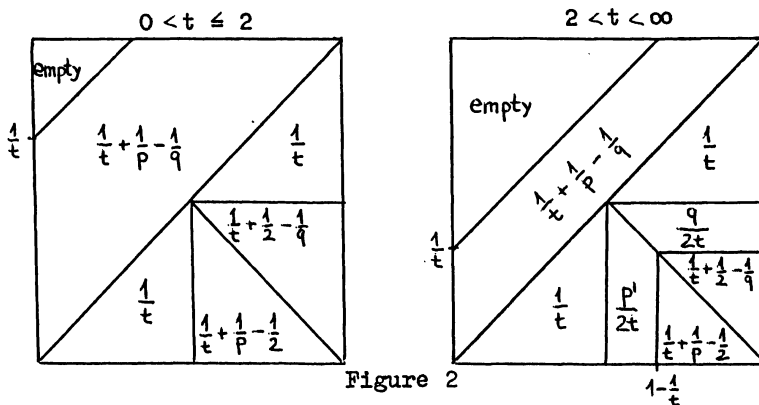


Figure 2

5. Weyl numbers

The following is known from [41], [68].

THEOREM 7.

(1) Let $1 \leq p < \max(2, q) \leq \infty$ and $1 \leq n \leq m/2$. Then

$$x_n(I_{p,q}^m) \asymp \begin{cases} 1 & \text{for } 2 \leq p \leq q \leq \infty \\ n^{1/q - 1/p} & \text{for } 1 \leq p \leq q \leq 2 \\ n^{1/2 - 1/p} & \text{for } 1 \leq p \leq 2 \leq q \leq \infty \\ m^{1/q - 1/p} & \text{for } 1 \leq q \leq p \leq 2 \end{cases} .$$

(2) Let $\max(2, q) \leq p$. Then

$$x_n(I_{p,q}^m) \asymp \begin{cases} m^{1/q - 1/p} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q} n^{-1/2} & \text{for } m^{2/p} \leq n \leq m \end{cases} .$$

From this we deduce the following behaviour of the corresponding quasi-norms.

LEMMA 2. Let $1 \leq p, q \leq \infty$ and $0 < r < \infty$. Then

$$\| I_{p,q}^m | L_{r,\infty}^{(x)} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \leq p < \max(2, q) \leq \infty$. Then

$$1/t = \begin{cases} 1/r & \text{for } 2 \leq p \leq q \leq \infty \\ (1/r - 1/p + 1/q)_+ & \text{for } 1 \leq p \leq q \leq 2 \\ (1/r - 1/p + 1/2)_+ & \text{for } 1 \leq p \leq 2 \leq q \leq \infty \\ 1/r - 1/p + 1/q & \text{for } 1 \leq q \leq p \leq 2 \end{cases} .$$

(2) Let $\max(2, q) \leq p$. Then

$$1/t = \begin{cases} 1/r - 1/p + 1/q & \text{for } 0 < r \leq 2 \\ 2/(pr) - 1/p + 1/q & \text{for } 2 < r < \infty. \end{cases}$$

Therefore we get

THEOREM 8. If $1 \leq p, q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in l_{t,w}$, then D_a belongs to $L_{r,w}^{(x)}(l_p, l_q)$, where $1/r$ takes values as indicated in the following diagrams.

$0 < t \leq 1$

$\frac{1}{t} + \frac{1}{2} - \frac{1}{q}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{q}$
$\frac{1}{t}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{2}$

$1 < t \leq 2$

empty	
$\frac{1}{t}$	$\frac{p(\frac{1}{t} + \frac{1}{p} - \frac{1}{q})}{2}$
$\frac{1}{t} + \frac{1}{2} - \frac{1}{q}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{q}$
$\frac{1}{t}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{2}$

$2 < t < \infty$

empty	
$\frac{1}{t}$	$\frac{p(\frac{1}{t} + \frac{1}{p} - \frac{1}{q})}{2}$
$\frac{1}{t} + \frac{1}{2} - \frac{1}{q}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{q}$
$\frac{1}{t}$	$\frac{1}{t} + \frac{1}{p} - \frac{1}{2}$

6. Hilbert numbers

There are recent results concerning the asymptotic behaviour of Hilbert numbers of $l_p(m) \rightarrow l_q(m)$. Since $h_n(S') = h_n(S)$, we can restrict ourselves to the case $1 \leq p' \leq q \leq \infty$. Then we have the well-known estimates [5], [18].

THEOREM 9.

(1) Let $1 \leq p' \leq q \leq 2$. Then

$$h_n(I_{p,q}^m) \asymp \begin{cases} m^{1/q - 1/p} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q} n^{-1/2} & \text{for } m^{2/p} \leq n \leq m^{2/q'} \\ m n^{-1} & \text{for } m^{2/q'} \leq n \leq m \end{cases} .$$

(2) Let $2 \leq q \leq p \leq \infty$. Then

$$h_n(I_{p,q}^m) \asymp \begin{cases} m^{1/q - 1/p} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q} n^{-1/2} & \text{for } m^{2/p} \leq n \leq m \end{cases} .$$

(3) Let $2 \leq p' \leq q \leq \infty$. Then

$$h_n(I_{p,q}^m) \asymp n^{1/q - 1/p} \quad \text{for } 1 \leq n \leq m .$$

Using entropy numbers, for the open case $2 < p < q < \infty$, B. Carl (unpublished) could prove the following

PROPOSITION 1. Let $2 < p < q < \infty$. Then

$$h_n(I_{p,q}^m) < \begin{cases} \left(\frac{\log(m/n + 1)}{n} \right)^{p/2(1/p - 1/q)} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q} n^{-1/2} & \text{for } m^{2/p} \leq n \leq m \end{cases} .$$

It is also known that in the case $2 < p < q < \infty$ the inequality

$$h_n(I_{p,q}^m) > \begin{cases} m^{1/q - 1/p} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q} n^{-1/2} & \text{for } m^{2/p} \leq n \leq m \end{cases}$$

is true (cf. [18]). This yields

LEMMA 3. Let $1 \leq p, q \leq \infty$ and $0 < r < \infty$. Then

$$\| I_{p,q}^m | L_{r,\infty}^{(h)} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \leq p' \leq q \leq 2$. Then

$$1/t = \begin{cases} 1/r & \text{for } 0 < r \leq 1 \\ 2/(q'r) - 1 + 2/q & \text{for } 1 < r \leq 2 \\ 2/(pr) - 1/p + 1/q & \text{for } 2 < r < \infty. \end{cases}$$

(2) Let $2 \leq p, q \leq \infty$. Then

$$1/t = \begin{cases} 1/r - 1/2 + 1/q & \text{for } 0 < r \leq 2 \\ (2/(pr) - 1/p + 1/q)_+ & \text{for } 2 < r < \infty. \end{cases}$$

(3) Let $2 \leq p' \leq q \leq \infty$. Then

$$1/t = (1/r - 1/p + 1/q)_+ \quad \text{for } 0 < r < \infty.$$

Therefore we get

THEOREM 10. If $1 \leq p, q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in I_{t,w}$, then D_a belongs to $L_{r,w}^{(h)}(1_p, 1_q)$, where $1/r$ takes values as indicated in the following diagrams.

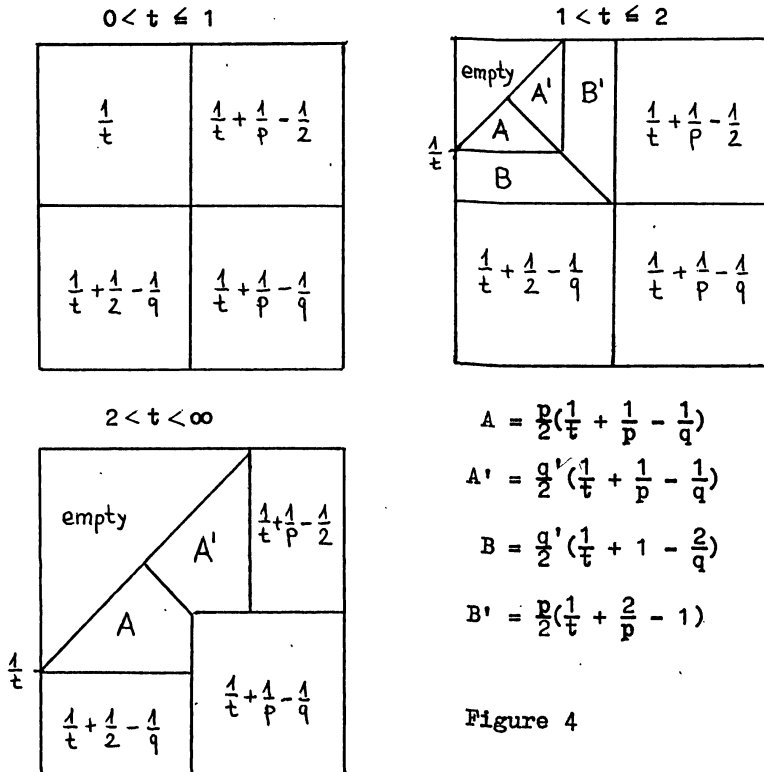


Figure 4

HISTORICAL REMARKS. The first results about s -numbers of diagonal operators are due to soviet mathematicians. These results are formulated in the language of diameters of subsets of normed spaces.

Already in 1954 S. B. Stečkin [81] showed that

$$d_n(I:l_1(m) \rightarrow l_2(m)) = \left(\frac{m-n+1}{m}\right)^{1/2}$$

Later on M. I. Stesin [80] was able to prove

$$d_n(I:l_p(m) \rightarrow l_q(m)) = (m-n+1)^{1/q - 1/p}$$

for $1 \leq q \leq p \leq \infty$. After the axiomatic approach to the s -number theory in 1974, many further estimates of this kind were obtained. First A. Pietsch [66] extended the result of Stesin to approximation numbers and Gelfand numbers in the following way. If $1 \leq q \leq p \leq \infty$, then

$$a_n(I_{p,q}^m) = c_n(I_{p,q}^m) = d_n(I_{p,q}^m) = (m-n+1)^{1/q - 1/p}.$$

The behaviour of these three s -functions in the case $1 \leq p < q \leq \infty$ had been unknown for a long time. Using number theoretical methods R. S. Ismagilov [24] obtained the estimate

$$d_n(I:l_p(m) \rightarrow l_{\infty}(m)) < \left(\frac{m}{n}\right)^{1/2 + 1/p}$$

for $1 \leq p \leq \infty$. With the help of random matrices B. S. Kashin [32] and B. S. Mitjagin [55] improved this for $p=1$ as follows

$$a_n(I:l_1(m) \rightarrow l_{\infty}(m)) = d_n(I:l_1(m) \rightarrow l_{\infty}(m)) < n^{-1/2} (\ln \frac{em}{n})^{1/2}.$$

It turned out that random matrices are an useful tool for these investigations. So B. Carl/A. Pietsch [5] could prove the estimate

$$a_n(I:l_p(m) \rightarrow l_p(m)) < n^{-3/2 + 1/p} m^{2/p} (\ln(2m))^{1/p - 1/2}$$

for $1 \leq p \leq 2$. By deducing this problem to a combinatorial one, K. Höllig [19] was able to improve the result of Carl and Pietsch.

He obtained the estimate

$$a_n(I:l_p(m) \rightarrow l_p(m)) < n^{-3/2 + 1/p} m^{2/p} \left(\frac{\ln m}{\ln n}\right)^{2/p - 1}$$

for $1 \leq p \leq 2$. Recently E. D. Gluskin [13] succeeded in solving the whole problem. He determined the exact asymptotic behaviour of $s_n(I:l_p(m) \rightarrow l_q(m))$ for s equal to a , c or d .

Moreover his results imply the relation

$$a_n(I:l_p(m) \rightarrow l_q(m)) \asymp \max \left\{ c_n(I:l_p(m) \rightarrow l_q(m)), d_n(I:l_p(m) \rightarrow l_q(m)) \right\}$$

already mentioned above.

The concept of Weyl and Chang numbers was developed by A. Pietsch [67] in 1977. He also gave the first estimates for these numbers.

Then it was C. Lubitz [41] who determined all r such that the diagonal operator D_a with $a \in l_{t,w}$ belongs to $L_{r,w}^{(x)}(l_p, l_q)$. For this purpose he used the results of Pietsch and Gluskin.

The first estimates of Hilbert numbers go back to W. Bauhardt [3] and B. Carl/A. Pietsch [5]. Recently by results of S. Heinrich/R. Linde [18] and B. Carl, it was possible to calculate the asymptotic behaviour of Hilbert numbers of diagonal operators.

7. s-Numbers of Besov embeddings

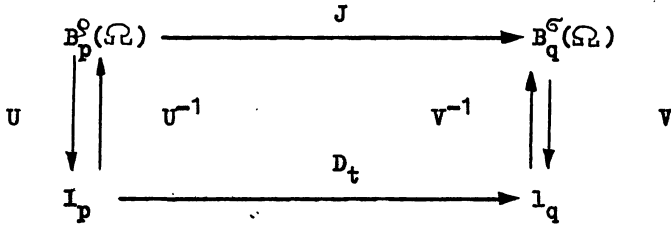
Let Ω be a sufficiently smooth region of \mathbb{R}^N . The main theorem in the theory of Besov spaces is the following.

THEOREM 11. Let $1 \leq p, q \leq \infty$ and $\varrho, \sigma \in \mathbb{R}$ such that $(\varrho - \sigma)/N > (1/p - 1/q)_+$. Then $B_p^{\varrho, \sigma}(\Omega)$ is continuously embedded in $B_q^{\sigma}(\Omega)$. Moreover the embedding operator is compact.

For the definition of $B_p^{\varrho, \sigma}(\Omega) := B_{p,p}^{\varrho, \sigma}(\Omega)$ we refer to the monograph of H. Triebel [91].

Using the results of Z. Ciesielski [6], S. Ropela [71] and Z. Ciesielski/T. Figiel [7] about spline bases, it turns out

that up to finite dimensional perturbation there exist two invertible operators U and V such that the following diagram becomes commutative:



Here the number t is determined by the relation

$$1/t = (\sigma - \sigma')/N - 1/p + 1/q .$$

So it is possible to carry over all results concerning diagonal operators to Besov embeddings.

HISTORICAL REMARKS. As already mentioned, A. N. Kolmogorov [36] introduced in 1936 the concept of the n -th (Kolmogorov) diameter $d_n(K, F)$ of a subset K of a normed space F ,

$$d_n(K, F) = \inf_{\substack{N \subseteq F \\ \dim N < n}} \sup_{x \in K} \inf_{y \in N} \|x - y\|_F .$$

To compute Kolmogorov diameters of the unit ball of Sobolev spaces V. E. Majorov [42] used a discretization method (cf. also E. D. Gluskin [11]).

In this paper the method of discretization (finitization) with the help of quasi-norms is presented for the first time.

It holds the following relation between the n -th Kolmogorov number and the n -th Kolmogorov diameter. If $S \in L(E, F)$, then

$$d_n(S) = d_n(S(U_n), F) .$$

In this notation the first results are due to Kolmogorov himself. He proved that

$$d_n(J: W_2^{\sigma} [0, 1] \rightarrow L_2 [0, 1]) \asymp n^{-\sigma} .$$

In the sequel the main interest was concentrated on the so-called Sobolev embedding $J:W_p^{\varrho}(\Omega) \rightarrow L_q(\Omega)$, which exists for $\varrho/N > (1/p - 1/q)_+$. The following table contains the asymptotic behaviour of $d_n(J:W_p^{\varrho}[0,1] \rightarrow L_q[0,1])$.

Simultaneously it gives a historical order of these results. Let $1 \leq p, q \leq \infty$ and $\varrho > (1/p - 1/q)_+$. Then

$$d_n(J:W_p^{\varrho}[0,1] \rightarrow L_q[0,1]) \asymp n^{-f(\varrho, p, q)}$$

p, q	ϱ	$f(\varrho, p, q)$	author
$p=q=2$	$0 < \varrho < \infty$	ϱ	Kolmogorov [36]
$p=\infty, q=2$ $q=2$	$0 < \varrho < \infty$	ϱ	Stečkin [81]
$1 \leq p=q \leq \infty$	$0 < \varrho < \infty$	ϱ	Babáňanov/ Tichomirov [2]
$1 \leq p \leq 2, q=2$	$\frac{1}{p} - \frac{1}{2} < \varrho < \infty$	$\varrho - \frac{1}{p} + \frac{1}{2}$	Ismagilov [23]
$p=1, q=\infty$	$1 < \varrho < \infty$	$\varrho - \frac{1}{2}$	Joffe/ Tichomirov [28]
$p=\infty, 1 \leq q \leq \infty$	$0 < \varrho < \infty$	ϱ	Makovoz [47]
$1 \leq q \leq p \leq \infty$	$0 < \varrho < \infty$	ϱ	Tichomirov [86]
$1 \leq p \leq q \leq 2$	$\frac{1}{p} - \frac{1}{q} < \varrho < \infty$	$\varrho - \frac{1}{p} + \frac{1}{q}$	Ismagilov [24]
$1 \leq p \leq 2 \leq q \leq \infty$	$1 < \varrho < \infty$	$\varrho - \frac{1}{p} + \frac{1}{2}$	Kashin [32]
$2 \leq p \leq q \leq \infty$	$\frac{1}{2} < \varrho < \infty$	ϱ	
$p=1, 2 \leq q \leq \infty$	$1 - \frac{1}{q} < \varrho < 1$	$\frac{q}{2}(\varrho - 1 + \frac{1}{q})$	Kashin [35]
$1 \leq p \leq 2 \leq q \leq \infty$	$\frac{1}{p} < \varrho \leq 1$	$\varrho - \frac{1}{p} + \frac{1}{2}$	Gluskin [12]
	$\frac{1}{p} - \frac{1}{q} < \varrho \leq \frac{1}{p}$	$\frac{q}{2}(\varrho - \frac{1}{p} + \frac{1}{q})$	Lubitz [41]
$2 \leq p \leq q \leq \infty$	$\frac{1}{p} - \frac{1}{q} < \varrho \leq \frac{1}{2}$	ϱ	
	$\frac{1}{p} - \frac{1}{q} < \varrho \leq \frac{1}{p} - \frac{1}{2q}$	$\frac{q}{2}(\varrho - \frac{1}{p} + \frac{1}{q})$	

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