1988 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 29. NO. 2

On a Classification of Hamiltonian Tournaments*)

MARCO BURZIO, DAVIDE CARLO DEMARIA**)

Torino, Italy

Received 1 March, 1988

A partition of the hamiltonian tournaments is given in order to find a characterization of tournaments which satisfy some extremal properties.

1. Introduction

In [3] we introduced the concept of non-coned 3-cycles in a tournament and we used these cycles to obtain a graphical characterization of the simply disconnected tournaments i.e. the tournaments whose fundamental group is not trivial (see [2]).

The natural generalization of the previous definition to a k-cycle let us clasify the collection $\mathscr{H}_n$ of the hamiltonian tournaments of order $n \geqq 5$ into $n - 4$ different classes. In detail, the first class of cyclic characteristic 3 is formed by the tournaments which contain a non-coned 3-cycle, the second one of cyclic characteristic 4 by the tournaments which contain a non-coned 4-cycle and whose 3-cycles are all coned, ..., the $(n - 4)$th class of cyclic characteristic $n - 2$ by the tournaments which contain a non-coned $(n - 2)$-cycle and whose cycles with lower length are all coned.

The Classification Theorem (see Theorem 11) states that the previous division is a proper partition of $\mathscr{H}_n$, and the cyclic characteristic is proved to be an invariant which is preserved by epimorphisms (see Proposition 5). Nevertheless, to study the classes and their relations, it is more interesting to consider, as an invariant, the difference between the order n of the tournament and the cyclic characteristic (called cyclic difference), since it does not increase for the hamiltonian subtournaments.

In particular (see Proposition 14), it follows that, for $n \geqq 7$, the class of cyclic difference 2 is a singleton and contains the only bineutral tournament A_n (see Definition 2), which plays a special role in this classification.

By considering the relations between the cyclic difference of a tournament and of its subtournaments, a local property of A_n is stated (see Theorem 17). Hence,

*) Work performed under the auspices of the *Consiglio Nazionale delle Ricerche* (*CNR, GNSAGA*) and of the *Gruppo Nazionale di Topologia* (Fondi M.P.I. 40%).

**) Universita di Torino, Dipartimento di Matematica, Via Principe Amedeo, 8, 10123 Torino, Italy.

we can immediately characterize the A_n as the tournaments with the least number of hamiltonian subtournaments. The analogous problem for the cycles was studied in [6], but using our methods, in another paper we shall also give a characterization of the tournaments with the least number of 3-cycles and those, also studied in [4], with only one hamiltonian cycle.

2. Some notations

A *tournament* T is a directed graph in which every pair of vertices is joined by exactly one arc. The vertex set of T is denoted by $V(T)$ and the arc set by $E(T)$. If the cardinality $|V(T)| = n$, T has order n and it is denoted by T_n. $T - v$ is the vertex-deleted subtournament and $\langle V(C)\rangle$ the subtournament induced by the vertices of a cycle C of T. $A \to B$ denotes that the vertices of a subtournament A are all predecessors of the vertices of a subtournament B.

A *hamiltonian* tournament (i.e. a tournament which contains a cycle through every vertex) is denoted by H_n.

A homomorphism of T_n into R_m is a mapping $p\colon V(T_n) \to V(R_m)$ such that if $(v, w) \in E(T_n)$ then either $(p(v), p(w)) \in E(R_m)$ or $p(v) = p(w)$. Since T_n can also be considered as a commutative groupoid (see [8]), a homomorphism is also an algebraic homomorphism between the two related groupoids and, if p is onto, R_m is isomorphic to the quotient groupoid T_n/p. In this case $V(T_n)$ can be partitioned into disjoint subtournaments $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ such that $S^{(i)} \to S^{(j)}$ if and only if $(v_i, v_j) \in E(R_m)$, where $w_i = p(V(S^{(i)}))$ and $w_j = p(V(S^{(j)}))$. Then T_n is the *composition* $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ of the *components* $S^{(i)}$ with the *quotient* R_m (see [2]). Moreover, T_n is *simple* if $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ implies that either $m = 1$ or $m = n$. We can recall that:

- R_m is isomorphic to a subtournament of T_n;
- $\forall T_n$, there exists exactly one non-simple quotient tournament;
- T_n is not hamiltonian if and only if its simple quotient is T_2.

3. Cyclic difference of a tournament

If we restrict the definition of neutral vertex to a hamiltonian tournament (see [5]), we obtain:

Definition 1. A vertex v of H_n is called a *neutral vertex* of H_n if $H_n - v$ is hamiltonian. The number of the neutral vertices of H_n is denoted by $\nu(H_n)$.

Remark. Since $\nu(H_n)$ is also the number of the hamiltonian subtournaments of order $n-1$ and in H_n $(n \geqq 4)$ there are at least two and at most n hamiltonian subtournaments of order $n-1$, it follows $2 \leqq \nu(H_n) \leqq n$ for $n \geqq 4$. For example, $\nu(H_n) = 2$ when H_4 is the hamiltonian tournament of order 4 and $\nu(H_4) = n$ for the tournaments which are the compositions of non-singleton components with a hamiltonian quotient.

Definition 2. The tournament A_n ($n \geqq 4$) with vertex set $V(A_n) = \{x_1, x_2, \ldots, x_n\}$ and arc set $E(A_n) = \{(x_i, x_j)/j < i - 1 \text{ or } j = i + 1\}$ contains the only two neutral vertices x_1, x_n and it is called the *bineutral* tournament of order n (see Example a)).

Remark. Las Vergnas proved in [6] that A_n is the only tournament with two neutral vertices for each $n \geqq 4$.

Definition 3. A subtournament T' of a tournament T is said to be *coned* by a vertex v (i.e. v cones T') if there exists a vertex v of $T - T'$ such that either $v \to T'$ or $T' \to v$. If no vertex of $T - T'$ cones T', T' is said to be *non-coned*. If C is a cycle of T, C is said to be *coned* by v (resp. *non-coned*) if $\langle V(C) \rangle$ is coned by v (resp. non-coned).

Proposition 1. A tournament T_n ($n \geqq 5$) is hamiltonian if, and only if, there exists a non-coned m-cycle C, where $3 \leqq m \leqq n - 2$.

Proof. Let v be a neutral vertex of H_n and w_1, w_2 two neutral vertices of $H_n - v$. Suppose that the two hamiltonian tournaments $H_n - \{v, w_1\}$ and $H_n - \{v, w_2\}$ are coned. Since w_1 cannot cone $H_n - \{v, w_1\}$, otherwise $H_n - \{v, w_2\}$ is not hamiltonian, and w_2 cannot cone $H_n - \{v, w_2\}$, otherwise $H_n - \{v, w_1\}$ is not hamiltonian, both $H_n - \{v, w_1\}$ and $H_n - \{v, w_2\}$ are coned by v. Hence v cones $H_n - v$, which is a contradiction since H_n is hamiltonian. Then at least one of the two tournaments $H_n - \{v, w_1\}$ and $H_n - \{v, w_2\}$ is non-coned (i..e in H_n there exists at least one non-coned $(n - 2)$-cycle).

Conversely, if T_n is not hamiltonian, i.e. its simple quotient is T_2, each cycle of T_n is included in a component and, therefore, is coned. ∎

Remark. Also H_3 and H_4 contain non-coned m-cycles, but now the condition $m \leqq n - 2$ is not satisfied.

Given a non-coned cycle C of H_n and a vertex $v \notin V(C)$ it is possible to extend C to a cycle through all the vertices of $H_n - v$. Then we can give the following:

Definition 4. If C is a non-coned cycle of H_n, the set $P_C = V(H_n) - V(C)$ consists of neutral vertices of H_n, which are called *poles* of C.

Definition 5. A non-coned cycle C of H_n is said to be *minimal* if each cycle C', such that $V(C') \subset V(C)$, is coned by at least one vertex of H_n.

A minimal cycle is said to be *characteristic* if it possesses the shortest length of the minimal cycles.

The length of a characteristic cycle is called the *cyclic characteristic* of H_n and is denoted by $cc(H_n)$. The difference $n - cc(H_n)$ is called the *cyclic difference* of H_n and is denoted by $cd(H_n)$.

Remark. If C is a characteristic cycle in H_n, then $cd(H_n) = |P_C|$.

4. Examples

a) The bineutral tournament A_7 of order 7:
– the neutral vertices are x_1 and x_7;
– the cycle $x_2, x_3, x_4, x_5, x_6, x_2$ is characteristic;
– $cc(A_7) = 5$ and $cd(A_7) = 2$.

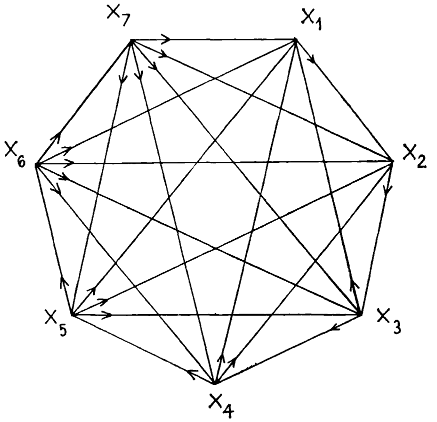

Fig. 1

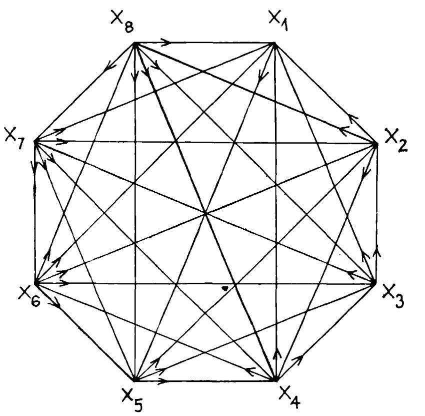

Fig. 2

b) A tournament H_8 which contains the subtournament A_7.

- x_1, x_8, x_7, x_1 is a characteristic cycle, whereas $x_2, x_3, x_4, x_5, x_6, x_2$ is a minimal cycle;
- $cc(H_8) = 3$ and $cd(H_8) = 5$.

c) A tournament H_6' with four characteristic cycles.

- x_1, x_2, x_5, x_3, x_1; x_1, x_2, x_5, x_6, x_1; x_2, x_5, x_3, x_4, x_2; x_2, x_5, x_6, x_4, x_2 are charactristic 4-cycles.
- $cd(H_6') = 2$.

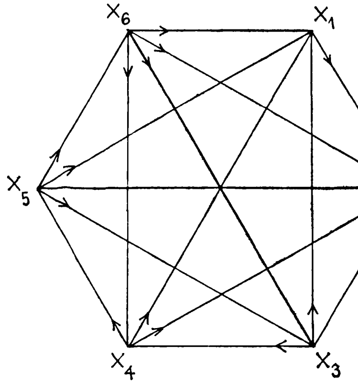

Fig. 3

d) tournament H_8 with $cd(H_8) = 3$, whose subtournaments H_7 have all the same cyclic difference $cd(H_7) = 3$.

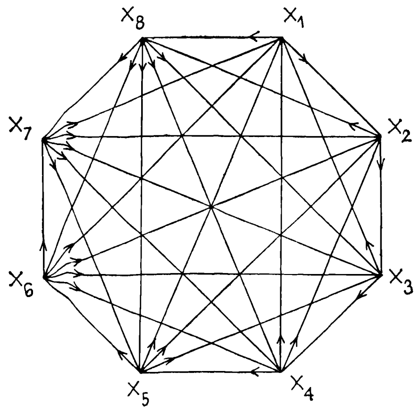

Fig. 4

Proposition 2. A vertex v of H_n is neutral if and only if in H_n there exists a minimal cycle C such that $v \in P_C$.

Proof. If v is neutral, $H_n - v$ is hamiltonian (and non-coned). Then there exists a minimal cycle C included in $H_n - v$ and $v \in P_C$. The converse follows directly from Definition 4. ■

Remark. If v is a neutral vertex of H_n, in general there is no characteristic cycle C such that $v \in P_C$ (see Example b)).

Corollary 3. The set of the neutral vertices of H_n is the union of the sets of the poles of the minimal cycles of H_n. Hence it follows $\mathrm{cd}(H_n) \leqq \nu(H_n)$. ■

Proposition 4. $\mathrm{cd}(H_n) = \nu(H_n)$ if and only if in H_n there is only one minimal cycle (the characteristic one).

Proof. Suppose that in H_n there exactly m $(m > 1)$ minimal cycles. Then at least one, e.g. C, is characteristic. Moreover, if C' is a minimal cycle different from C, there exists a neutral vertex $v \in P_{C'}$ which is not an element of P_C, i.e. $|P_C| < \nu(H_n)$. Then also $\mathrm{cd}(H_n) < \nu(H_n)$ by Remark to Definition 5.

Conversely, if C is a characteristic cycle and $\mathrm{cd}(H_n) < \nu(H_n)$, there exists a neutral vertex v such that $v \notin P_C$. Then a new minimal cycle C' different from C can be found by Proposition 2. ■

Proposition 5. If H_n is the composition $H_n = H'_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ of m tournaments $S^{(i)}$ with quotient tournament H'_m, then $\mathrm{cc}(H_n) = \mathrm{cc}(H'_m)$.

Proof. If C is a non-coned r-cycle in H_n, the r vertices of C are elements of r different components, i.e. $p(C)$ is a non-coned r-cycle in H'_m, where $p\colon V(H_n) \to V(H'_m)$ is the canonical projection.

Conversely, let H''_m be a subtournament of H_n isomorphic to H'_m. The image in H''_m of a non-coned m-cycle of H'_m is a non-coned m-cycle of H_n. ■

Proposition 6. Let $H_n = H'_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$, then v is a neutral vertex of H_n if, and only if, either v is included in a non-singleton component or $p(v)$ is a neutral vertex of H'_m, where $p\colon V(H_n) \to V(H'_m)$ is the canonical projection. ■

Proposition 7. Let H'_m a hamiltonian subtournament of H_n. Then $\mathrm{cd}(H'_m) \leq \mathrm{cd}(H_n)$.

Proof. We must similarly prove that $\mathrm{cc}(H'_m) \geqq \mathrm{cc}(H_n) - n + m$.

i) At first we can see that the last inequality is true for $m = n - 1$, i.e. $\mathrm{cc}(H'_{n-1}) \geqq \geqq \mathrm{cc}(H_n) - 1$.

Let $\mathrm{cc}(H'_{n-1}) = h$ and consider a characteristic cycle C_h in $H'_{n-1} = H_n - v$.
– If v does not cone C_h, then C_h is a non-coned cycle in H_n. It follows that $\mathrm{cc}(H'_{n-1}) \geqq \mathrm{cc}(H_n)$. Thus $\mathrm{cc}(H'_{n-1}) > \mathrm{cc}(H_n) - 1$.
– If v cones C_h, there exists $w \in V(H'_{n-1}) - V(C_h)$ such that v does not cone $\langle V(C_h) \cup \{w\}\rangle$, since H_n is hamiltonian. Since w does not cone C_h, we can construct a cycle C_{h+1}, whose vertex set is $V(C_h) \cup \{w\}$, which is non-coned in H_n. Then $\mathrm{cc}(H'_{n-1}) \geqq \mathrm{cc}(H_n) - 1$.

ii) Now consider the general case with $3 \leqq m \leqq n - 1$.
Two different possibilities must be considered:
1) There exists a chain of hamiltonian subtournaments $H'_{m+1}, H'_{m+2}, \ldots, H'_{n-1}$ such that $V(H'_m) \subset V(H'_{m+1}) \subset \ldots \subset V(H'_{n-1}) \subset V(H_n)$. Then, step by step, from i) we obtain $\mathrm{cc}(H'_m) \geqq \mathrm{cc}(H_n) - n + m$.
2) There exists a hamiltonian tournament H'_s $(m \leqq s < n)$, such that between H'_m and H'_s there is a chain of hamiltonian tournaments as in 1), whereas, for each tournament T'_{s+1} such that $V(H'_s) \subset V(T'_{s+1})$, T'_{s+1} is not hamiltonian, i.e. H'_s is coned by all the vertices of $V(H_n) - V(H'_s) = \{v_1, v_2, \ldots, v_{n-s}\}$. Then H_n is the composition $H_n = H'_{n-s+1}(\{v_1\}, \{v_2\}, \ldots, \{v_{n-s}\}, H'_s)$ and it follows that $\mathrm{cc}((H_n) = = \mathrm{cc}(H'_{n-s+1}) < n - s$ by Propositions 5 and 1. Hence, we obtain $\mathrm{cc}(H_n) - - n + m < -s + m < \mathrm{cc}(H'_s) - s + m \leqq \mathrm{cc}(H'_m)$, where the last inequality follows by 1). ■

Remark. We could have defined a characteristic cycle as a minimal cycle with maximal length. But in this case only a property similar to Proposition 5 would hold, whereas a property similar to Proposition 7 would fail. In fact in H_8 of Example b) the 5-cycle $x_2, x_3, \ldots, x_6, x_2$ is minimal with maximal length k, i.e. $8 - k = 3$, whereas:
– in $H_8 - x_4$ each minimal 3-cycle has maximal length k_1, i.e. $7 - k_1 = 4$;
– in $H_8 - x_1$ the 4-cycle x_2, x_3, x_4, x_5, x_2 is minimal with maximal length k_2, i.e. $7 - k_2 = 3$;
– in $H_8 - x_8$ the 5-cycle $x_2, x_3, \ldots, x_6, x_2$ is minimal with maximal length k_3, i.e. $7 - k_3 = 2$.

Corollary 8. If C is a characteristic cycle of H_n and $v \in P_C$ is a pole of C, it follows that $\mathrm{cd}(H_n) \geqq \mathrm{cd}(H_n - v) \geqq \mathrm{cd}(H_n) - 1$.

Proof. Since $v \in P_C$, C is also a non-coned cycle of $H_n - v$. Then $\mathrm{cc}(H_n - v) \leqq \leqq \mathrm{cc}(H_n)$. Therefore $n - 1 - \mathrm{cc}(H_n - v) \geqq n - 1 - \mathrm{cc}(H_n)$ i.e. $\mathrm{cd}(H_n - v) \geqq \geqq \mathrm{cd}(H_n) - 1$. ■

Remark. Afterwards, (see Lemma 15) we shall state when $\mathrm{cd}(H_n) = \mathrm{cd}(H_n - v)$.

Proposition 9. Let H'_m be a hamiltonian subtournament of H_n, then $\nu(H'_m) \leqq \nu(H_n)$.

Proof. At first we will prove the inequality for $m = n - 1$. Suppose u is a neutral vertex of $H_{n-1} = H_n - v$ and u is not a nautral vertex of H_n (i.e. $H_n - u$ is not hamiltonian). Then the simple quotient of $H_n - u$ is T_2 and a component is necessary $\{v\}$, otherwise also $H_n - \{u, v\}$ is not hamiltonian. Then it follows that either $u \rightarrow \rightarrow v \rightarrow H_n - \{u, v\}$ or $u \leftarrow v \leftarrow H_n - \{u, v\}$. Now if u' is a neutral vertex of $H_n - v$ different from u, since v can not cone $H_n - \{u', v\}$, u' is also a neutral vertex of H_n. Then there exists at most a neutral vertex of H_{n-1} which is not a neutral vertex of H_n. Since v is a neutral vertex of H_n and not of $H_n - v$, it follows that $\nu(H'_{n-1}) \leqq \nu(H_n)$.

In the general case, if there exists a chain of hamiltonian subtournaments $H'_{m+1}, \ldots \ldots, H'_{n-1}$ such that $V(H'_m) \subset V(H'_{m+1}) \subset \ldots \subset V(H'_{n-1}) \subset V(H_n)$, then $\nu(H'_m) \leqq \leqq \nu(H'_{m+1}) \leqq \ldots \leqq \nu(H_n)$. Otherwise, there exists H'_s as in 2) of Proposition 7. Then $\nu(H'_m) \leqq \nu(H'_s)$ and also $\nu(H'_s) \leqq \nu(H_n)$ by Proposition 6 since H'_s is a component of H_n. ∎

Proposition 10. For each $n \geqq 5$, the bineutral tournament A_n has its cyclic difference equal to 2 and it contains the only minimal cycle $x_2, x_3, \ldots, x_{n-1}, x_2$.

Proof. We proceed by induction on n.

For $n = 5$, x_2, x_3, x_4, x_2 is the only minimal cycle and $\text{cd}(A_5) = 2$.

Assume that $\text{cd}(A_{n-1}) = 2$ and A_{n-1} contains only the minimal cycle $x_2, x_3, \ldots \ldots, x_{n-2}, x_2$. Consider A_n obtained from A_{n-1} by adding the vertex x_n successor of x_{n-1} and predecessor of all the other vertices of A_{n-1}. In A_n all the $(n - 3)$-cycles are coned, in fact $x_2, x_3, \ldots, x_{n-2}, x_2$, non-coned in A_{n-1}, is coned by x_n and the only $(n - 3)$-cycle including x_n is $x_4, x_5, \ldots, x_n, x_4$, which is coned by x_1. Moreover, the $(n - 2)$-cycle $x_2, x_3, \ldots, x_{n-1}, x_2$ is the only one non-coned $(n - 2)$-cycle. Consequently $\text{cd}(A_n) = 2$. ∎

Theorem 11. (Classification Theorem). Let H_n $(n \geqq 5)$ be a hamiltonian tournament of order n, then $2 \leqq \text{cd}(H_n) \leqq n - 3$ $(3 \leqq \text{cc}(H_n) \leqq n - 2)$.

Conversely, for each $n \geqq 5$ and for each h such that $2 \leqq h \leqq n - 3$, there exist hamiltonian tournaments H_n with $\text{cd}(H_n) = h$.

Proof. The first part follows directly from Proposition 1.

Conversely, for each $n \geqq 5$ and $h = 2$, the bineutral tournament A_n satisfies the condition $\text{cd}(A_n) = 2$. Now let $V(A_{n-h+2}) = \{x_1, x_2, \ldots, x_{n-h+2}\}$, where A_{n-h+2} is the bineutral tournament of order $n - h + 2$ and $H_n = A_{n-h+2}(T_{h-1}, \{x_2\}, \ldots \ldots, \{x_{n-h+2}\})$ the composition obtained from A_{n-h+2} by replacing the singleton $\{x_1\}$ with any tournament T_{h-1} whatever. It follows that $\text{cc}(A_{n-h+2}) = n - h = \text{cc}(H_n)$ by Proposition 5. Therefore $\text{cd}(H_n) = h$. ∎

Remark 1. For $n = 4$ it is $\mathrm{cd}(H_4) = 1$ and $\mathrm{cc}(H_4) = 3$. For $n = 3$ it is $\mathrm{cd}(H_3) = 0$ and $\mathrm{cc}(H_3) = 3$.

Remark 2. A simple tournament H_n with $\mathrm{cd}(H_n) = h$ can be obtained from a bineutral tournament A_n by reversing the arc (x_i, x_{i+h-1}) for each $i \geqq 3$ and $h \geqq 3$ such that $i + h \leqq n - 1$. Thus, for each admissible h, there also exist simple tournaments H_n with $\mathrm{cd}(H_n) = h \geqq 4$. Moreover it is easy to construct simple tournaments H_n with $\mathrm{cf}(H_n) = 3$.

Remark 3. Obviously, the simple disconnected tournaments (see [2]) have cyclic characterists equal to 3.

Corollary 12. The collection $\mathscr{H}_{n,h} = \{H_n\colon \text{tournaments with } \mathrm{cd}(H_n) = h\}$, for each $n \geqq 5$ and for each h such that $2 \leqq h \leqq n - 3$, is a partition of the set of the hamiltonian tournaments of order $\geqq 5$. ■

Remark. If we add the classes $\mathscr{H}_{3,0} = \{H_3\colon \text{3-cycle}\}$ and $\mathscr{H}_{4,1} = \{H_4\colon \text{the hamiltonian tournament of order } 4\}$, we obtain a partition of all the hamiltonian tournaments.

6. Tournaments whose cyclic difference is equal to 2

Proposition 13. Each H_n $(n \geqq 6)$ with $\mathrm{cd}(H_n) = 2$ is simple.

Proof. Suppose H_n is non-simple and consider the composition $H_n = H_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$, where H_m is the simple quotient related to H_n. Now, let C be a non-coned $(m - 2)$-cycle of H_m (see the proof of Proposition 1). Since H_m can be identified with a subtournament of H_n, C can also be considered as a non-coned cycle of H_n, which yields the contradiction $\mathrm{cd}(H_n) > 2$. ■

Now it is possible to obtain the structural characterization of the tournaments H_n with $\mathrm{cd}(H_n) = 2$.

In fact, for each H_5 it follows that $\mathrm{cd}(H_5) = 2$ from Proposition 1.

For $n = 6$, it is easy to check that there are only two tournaments H_6 with $\mathrm{cd}(H_6) = 2$, namely the bineutral one A_6 and the tournament H'_6 obtained from A_6 by reversing the arc (x_2, x_5) (see Example c)).

Moreover, in the general case, we have:

Proposition 14. For $n \geqq 7$, the bineutral tournaments A_n are the only tournaments with cyclic differences equal to 2.

Proof. We proceed by induction on n.

For $n = 7$, with $\mathrm{cd}(H_7) = 2$ can only be obtained by adding a vertex v either to A_6 or to H'_6 (see Proposition 7).

If we consider A_6, since v must cone the characteristic cycle of A_6, we obtain eight different possibilities with regards to the adjacencies of v, but it is easy to check that only for $H_7 = A_7$, is the condition $\mathrm{cd}(H_7) = 2$ satisfied, whereas in the other cases either H_7 contains a hamiltonian subtournament with cyclic difference equal to 3 or it is not hamiltonian.

If we consider H'_6, since v must cone the four characteristic 4-cycles of H'_6, v must necessary cone H'_6. Then, starting from H'_6, a tournament H_7 with $\mathrm{cd}(H_7) = 2$ cannot be constructed.

Now assume that, for $n > 7$, A_n is the only tournament with $\mathrm{cd}(A_n) = 2$, and consider H_{n+1}. Since $\mathrm{cd}(H_{n+1})$ must be equal to 2 and H_{n+1} can not include a subtournament H_n with $\mathrm{cd}(H_n) > 2$, H_{n+1} can only be obtained by adding a vertex v to A_n. Similarly as before, we obtain eight different cases, but the condition $\mathrm{cd}(H_{n+1}) = 2$ is satisfied only when $H_{n+1} = A_{n+1}$. ■

Remark. From Corollary 3 and Proposition 14 it again follows that A_n is the only tournament with two neutral vertices for each $n \geqq 4$. (see [6]).

7. Cyclic difference of subtournaments

Lemma 15. Let C be a minimal k-cycle of H_n $(k > 3)$, $P_C = \{v_1, v_2, \ldots, v_p\}$ the set of poles of C and $W = \{w_1, w_2, \ldots, w_r\}$ the set of the neutral vertices of $\langle V(C) \rangle$. Then the collection of subsets of P_C:

$$P_C^i = \{v \in P_C / v \text{ cones } \langle V(C) - w_i \rangle\}, \quad \text{for each} \quad w_i \in W,$$

is a partition of $P_C - P_C^*$, where $P_C^* = \{v \in P_C / v$ does not cone $\langle V(C) - w_i \rangle$, $\forall i = 1, 2, \ldots, r\}$. Therefore $|P_C| = p \geqq r = |W|$.

Moreover, if C is a characteristic cycle and $\{v\} = P_C^i$ (i.e. P_C^i is a singleton) then $\mathrm{cd}(H_n - v) = \mathrm{cd}(H_n)$.

Proof. Since $k > 3$, W is not empty. Moreover, since C is minimal, at least one vertex $v_i \in P_C$ must cone $\langle V(C) - w_i \rangle$, i.e. $P_C^i \neq \emptyset$, $\forall i = 1, 2, \ldots, r$. Finally, $P_C^i \cap P_C^j = \emptyset$, $\forall i, j = 1, 2, \ldots, r$, $i \neq j$, since, otherwise, $v \in P_C^i \cap P_C^j$ would cone C. Then $\{P_C^i\}_{i=1,2,\ldots,r}$ is a partition of $P_C - P_C^*$ and $p \geqq r$.

Now, if C is a characteristic cycle and $\{v\} = P_C^i$ is a singleton, $\langle V(C) - w_i \rangle$ is non-coned in $H_n - v$. Then $\mathrm{cc}(H_n - v) \leqq k - 1 = \mathrm{cc}(H_n) - 1$, i.e. $\mathrm{cd}(H_n - v) = \mathrm{cd}(H_n)$. ■

Proposition 16. For each H_n $(n \geqq 7)$ with $\mathrm{cd}(H_n) \geqq 3$ and for each k such that $6 \leqq k \leqq n$, there exists a subtournament H_k of H_n with $\mathrm{cd}(H_k) \geqq 3$. In particular there exists a subtournament H_6 with $\mathrm{cd}(H_6) = 3$.

Proof. If $\mathrm{cd}(H_n) = 3$, let C be a characteristic $(n - 3)$-cycle of H_n and $P_C = \{v_1, v_2, v_3\}$ the set of the poles of C. Since in $\langle V(C) \rangle$ there are at least two neutral vertices w_1, w_2, the partition of $P_C - P_C^* \subseteqq \{v_1, v_2, v_3\}$ contains at least two sets and one at least is a singleton. Thus in H_n there exists a subtournament H_{n-1} with $\mathrm{cd}(H_{n-1}) = 3$ by Lemma 15.

If $\mathrm{cd}(H_n) = h > 3$, let C be a characteristic $(n - h)$-cycle of H_n and $v \in P_C$ a pole of C. Since $h \geqq 4$, it follows that $\mathrm{cd}(H_n - v) \geqq h - 1 \geqq 3$ from Corollary 8.

By recurrence, the assertion follows for each k such that $6 \leqq k \leqq n$, in particular the equality follows for $k = 6$ since it is in general $\mathrm{cd}(H_n) \leqq 3$. ■

Remark. In general, it is not true that:

– in each H_n with $\mathrm{cd}(H_n) = h \geqq 3$ there exists a subtournament H_{n-1} with $\mathrm{cd}(H_{n-1}) = h - 1$ (see Example d));

– in each H_n with $\mathrm{cd}(H_n) = h' \geqq 4$, there exists a subtournament H_{n-1} with $\mathrm{cd}(H_{n-1}) = h'$. For example, consider $H_7 = A_5\{T_2, \{x_2\}, \{x_3\}, \{x_4\}, T_2\}$, obtained from A_5 by replacing the two neutral vertices x_1, x_5 of A_5 with T_2.

8. Some properties of the bineutral tournaments

Theorem 17. A tournament $H_n\,(n \geqq 5)$ is isomorphic to the bineutral tournament A_n if and only if there exists k $(5 \leqq k \leqq n)$ such that each subtournament H_k is isomorphic to A_k.

Proof. If $H_n \simeq A_n$ then, for each k, the condition holds.

Conversely, for $n = 6$ and $k = 5$, it follows that:

– in $H_6' \not\simeq A_6$ with $\mathrm{cd}(H_6') = 2$ (see Example c)) there is, e.g., $\langle x_1, \ldots, x_5, x_1 \rangle \not\simeq A_5$;

– in H_6 with $\mathrm{cd}(H_6) > 2$, if $V(H_6) = \{x_1, \ldots, x_6\}$ and $C\colon x_1, x_2, x_3, x_1$ is a characteristic 3-cycle of H_6, the three subtournaments $\langle x_1, \ldots, x_5, x_1 \rangle$, $\langle x_1, \ldots, x_4, x_6, x_1 \rangle$ and $\langle x_1, x_2, x_3, x_5, x_6, x_1 \rangle$ cannot be at the same time isomorphic to A_5.

Thus only A_6 contains all the subtournaments H_5 isomorphic to A_5.

Now suppose that $H_n \not\simeq A_n$ for $n \geqq 7$ and there exists k $(5 \leqq k \leqq n)$ such that $H_k \simeq A_k$ for each subtournament H_k.

Then $\mathrm{cd}(H_n) \geqq 3$ from Proposition 14, which immediately yields a contradiction since, by Proposition 16, if $k \geqq 6$, there exists $H_k \not\simeq A_k$ and, if $k = 5$, there exists a H_6 with $\mathrm{cd}(H_6) = 3$, which contains a $H_5 \not\simeq A_5$, as seen earlier. ■

Proposition 18. If H_n $(n \geqq 5)$ is not isomorphic to the bineutral tournaments A_n, then it contains at least $n - k + 2$ H_k hamiltonian subtournaments with $4 \leqq k \leqq n - 1$.

Proof. For $n = 5$, since H_5 is not isomorphic to A_n, it contains three neutral vertices (see [6]) i.e. three hamiltonian subtournaments of order 4.

For $n = 6$, since $H_6 \not\approx A_6$, by Theorem 17 we can consider a subtournament $H_5 \not\approx A_5$. Then in H_6 there are three H_4 contained in H_5 and one more, including the vertex v such that $H_5 = H_6 - v$, since each vertex is contained in a k-cycle $(3 \leqq k \leqq n)$ (see [7]). Moreover, if $\mathrm{cd}(H_6) = 3$, in H_6 there are at least three neutral vertices i.e. at least three H_5; otherwise $H_6 = H'_6$ of Example c) and it contains four H_5.

Then the property holds for $n = 5, 6$.

Now we proceed by induction on n. Suppose each H_{n-1} $(n - 1 \geqq 7)$, not isomorphic to A_{n-1}, contains at least $n - k + 1$ H_k with $4 \leqq k \leqq n - 2$ and consider H_n such that $H_n \not\approx A_n$. Then $\mathrm{cd}(H_n) \geqq 3$. Consider a subtournament H_{n-1} with $\mathrm{cd}(H_{n-1}) \geqq 3$ by Proposition 16. It follows that $H_{n-1} \not\approx A_{n-1}$ and in H_{n-1} there are at least $n - k + 1$ H_k $(4 \leqq k \leqq n - 2)$. Then H_n contains $n - k + 2$ H_k, such that $n - k + 1$ are contained in H_{n-1} and one includes the vertex v such that $H_{n-1} = H_n - v$, since each vertex is contained in a k-cycle.

Finally, consider the three hamiltonian subtournaments $H_{n-1} - w_1$, $H_{n-1} - w_2$, $H_{n-1} - w_3$ of H_{n-1}, where w_1, w_2, w_3 are the neutral vertices of H_{n-1}. The vertex v cones at most one of the $H_{n-1} - w_i$, $i = 1, 2, 3$, e.g. $H_{n-1} - w_1$, since H_n is hamiltonian. Then $H_n - w_2$, $H_n - w_3$, H_{n-1} are hamiltonian subtournaments of order $n - 1$.

Hence the proposition holds for each $k = 4, 5, \ldots, n - 1$. ■

Remark. As a consequence, we again obtain the following extremal property, proved by Las Vergnas in [6]: "If $H_n \not\approx A_n$, then it contains at least $n - k + 2$ k-cycles for $4 \leqq k \leqq n - 1$".

References

[1] Beineke L. W. and Reid K. B., Tournaments, Selected Topics in Graph Theory. Edited by Beineke L. W. and Wilson R. J.. Academic Press, New York (1979).

[2] Burzio M. and Demaria D.C., On simply disconnected tournaments to appear in Ars Combin. (Waterloo, Ont.).

[3] Burzio M. and Demaria D. C., Characterization of Tournaments by Coned 3-cycles, Acta Univ. Carolin. - Math. Phys., Prague, Vol 28. No 2 (1987), 25—30.

[4] Douglas R. J., Tournaments that admit exactly one Hamiltonian circuit, Proc. London Math. Soc. 21 (1970), 716—730.

[5] Harary F., Norman R. Z. and Cartwright D., Structural models, An Introduction to the Theory of Directed Graphs. Wiley, New York (1965).

[6] Las Vergnas M., Sur le nombre de circuit dans un tournoi fortement connexe, Cahiers du CERO, Bruxelles 17 (1975), 261—265.

[7] Moon J. W., On subtournaments of a tournament, Canad. Math. Bull., vol. 9, no. 3 (1966), 297—301.

[8] Müller V., Nešetřil J. and Pelant J., Either tournaments or algebras?, Discrete Math. 11 (1975), 37—66.