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Automorphism Groups of Complements of Points

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A metrizable topological space (graph resp.) having no nontrivial automorphism with the automorphism groups of complements of points isomorphic to an arbitrary prescribed group is constructed.

1. Introduction

J. de Groot [3] in 1959 proved that every group is isomorphic to the group of all homeomorphisms of a topological space onto itself. His result was the starting point for further research in the area of representations of groups, semigroups, and categories in topological categories. It was shown in [8] that every monoid can be represented by the monoid of all nonconstant continuous mappings of a metrizable topological space into itself (i.e. for every monoid M there is a metrizable topological space such that the system such that the system of all of its nonconstant selfmaps forms a monoid which is isomorphic to M). Later it was proved by V. Koubek in [4] and V. Trnková in [7] respectively that every small and even every concrete category can be represented in the category of topological spaces. It turned out that in such representations the rigid spaces (i.e. spaces such that every continuous selfmap is either a constant or the identity) and automorphism rigid spaces (i.e. spaces such that the only autohomeomorphism is the identity) play very significant role. Useful example of a rigid space was constructed by H. Cook in [1]. In fact many of the results concerning representations in the category of topological spaces can be proved in the category of graphs first and then transferred into topological spaces using the "*arrow construction*" with the Cook continuum or another suitable

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rigid space. In such cases the theorem proved by P. Vopěnka, A. Pultr, and Z. Hedrlín in [9], namely that there exists a rigid binary relation on every set, is widely used. An interested reader can obtain complex information on the subject in [6].

The aim of the present paper is to show that the automorphism rigidity can be destroyed after deleting single arbitrarily chosen point (vertex resp.). However there are both a graph and a metrizable topological space which are automorphism rigid and they remain automorphism rigid after deleting any vertex (point resp.).

2. Preliminaries

Let Graph denote the category of all directed graphs, i.e. the category whose objects are all pairs (S, R), where S is a set and $R \subseteq S \times S$. A mapping $f: S \to S'$ is a morphism of Graph (denoted by $f: (S, R) \to (S', R')$) iff it satisfies the following: $(f(s_1), f(s_2)) \in R'$ whenever $(s_1, s_2) \in R$. We recall that the graph (S, R) is *rigid* iff the only existing homomorphism $f: (S, R) \to (S, R)$ is the identity. By Aut (A) we mean the group of all automorphisms of A, where A is either a topological space or a graph. $A \cong B$ means that A and B are isomorphic groups (graphs, top. spaces resp.). The following Theorems are necessary for the constructions contained in this paper. Their proofs are nontrivial and can be found for example in [6], therefore we do not present them here.

Theorem 2.1. *There exists a rigid binary relation on every set.*

Theorem 2.2. *For every group $\mathbb{G}$ there is a graph (X, R) with the group of all automorphism isomorphic to $\mathbb{G}$.*

There are full embeddings of Graph into its full subcategory of all symmetric graphs and the full subcategory of all graphs without cycles. So we may assume the graphs mentioned in Theorem 2.2 to be either symmetric or without cycles. Theorem 2.1 can be restated in the following way: *There are arbitrarily large rigid symmetric graphs (graphs without cycles resp.).* Let us remark that the graphs constructed in [9] are connected, which will be important for our later constructions.

Let us mention some properties of the Cook continuum, the topological space which is basic for many of the constructions not only in this paper. Recall that the *Cook continuum* $\mathscr{C}$ is a compact connected metric space of cardinality $2^{\aleph_0}$ satisfying the following two conditions:

1. $\mathscr{C}$ is a hereditarily indecomposable continuum.
2. For every subcontinuum A of $\mathscr{C}$ and every continuous mapping $f: A \to \mathscr{C}$, either f is a constant or the inclusion.

(A continuum is indecomposable iff it can not be obtained as a union of two of its proper subcontinua. Hereditarily indecomposable means that also every subcontinuum is indecomposable.)

The following two useful Lemmas concerning connected compact spaces can be found in [5].

Lemma 2.3. *Let $\mathscr{C}$ be an indecomposable continuum. There are $2^{\aleph_0}$ pairwise disjoint nondegenerate subcontinua of $\mathscr{C}$.*

Lemma 2.4. *If A is a closed subset of a continuum X such that $0 \neq A \neq X$ then for every component C of the space A we have $C \cap \operatorname{Bd} A \neq 0$.*

Claim 2.5. *Let A be a subcontinuum of a Cook continuum Then $Aut(A) \cong \{1\}$ and $Aut(A\setminus\{x\}) \cong \{1\}$ for every $x \in A$.*

Proof. $\operatorname{Aut}(A) \cong \{1\}$ follows immediately from 2.

Let $x \in A$. In order to prove $\operatorname{Aut}(A\setminus\{x\}) \cong \{1\}$ let's observe that $A\setminus\{x\}$ is connected. Suppose on the contrary that there are C, D nonvoid disjoint clopen subsets of $A\setminus\{x\}$ such that $C \cup D = A\setminus\{x\}$. But then $\bar{C} = C \cup \{x\}$ and $\bar{D} = D \cup \{x\}$ are proper nondegenerate subcontinua of the hereditarily indecomposable continuum A such that $A = \bar{C} \cup \bar{D}$ – which is a contradiction.
To finish the proof it is sufficient to show that $\forall y \in A\setminus\{x\}$ there exists a nondegenerate subcontinuum of $A\setminus\{x\}$ containing y. This immediately follows from Lemma 2.4. □

3. Main theorems in Graph

Notation: Let $\mathscr{G} = (X, R)$ be a graph and $x \in X$. $\mathscr{G}\setminus\{x\}$ denotes the induced subgraph $(X\setminus\{x\}, R \cap (X\setminus\{x\}) \times (X\setminus\{x\}))$.

The aim of this section is to present a proof of the following:

Theorem 3.1. *For every group $\mathbb{G}$ there is a graph $\mathscr{G} = (X, R)$ having no nontrivial authomorphism such that for every $x \in X$ $Aut(\mathscr{G}\setminus\{x\}) \cong \mathscr{G}$.*

Lemma 3.2. *For every group $\mathbb{G}$ there is a graph $\mathscr{H}_{\mathbb{G}} = (Y, Q)$ with a distinguished vertex $x_{\mathbb{G}} \in Y$ such that*

1. $\operatorname{Aut}(\mathscr{H}_{\mathbb{G}}) \cong \{1\}$
2. $\operatorname{Aut}(\mathscr{H}_{\mathbb{G}}\setminus\{x_{\mathbb{G}}\}) \cong \mathbb{G}$
3. $\operatorname{Aut}(\mathscr{H}_{\mathbb{G}}\setminus\{y\}) \cong \{1\}$ for every $y \in Y$, $y \neq x_{\mathbb{G}}$.

Proof. By Theorem 2.2 there exists a graph $\mathscr{K}_{\mathbb{G}} = (Z, S)$ without cycles such that $\operatorname{Aut}(\mathscr{K}_{\mathbb{G}}) \cong \mathbb{G}$. Using Theorem 2.1 we obtain a rigid connected symmetric graph $\mathscr{R}_{\mathbb{G}} = (U, T)$ such that $card(U) \geqslant card(Z)$.

Put $Y = Z \cup \bigcup_{\substack{i=0,1 \\ z \in Z}} U \times \{z\} \times \{i\} \cup \{x_{\mathbb{G}}\}$.

To create the relation Q and Y we put:

1. $Q \restriction Z = S$
2. $\forall z \in Z, \forall i \in \{0, 1\}, \forall a, b \in U : ((a, z, i), (b, z, i)) \in Q$ iff $(a, b) \in T$
3. $\forall z \in Z, \forall a \in U : ((a, z, 0), (a, z, 1)) \in Q$
4. $\forall z \in Z, \forall a \in U : ((a, z, 1), z) \in Q$
5. For every $z \in Z$ we choose $x_z \in U$ such that $x_z \neq x_{z'}$ whenever $z \neq z'$ (it is possible since $card(U) \geqslant card(Z)$).
   $\forall z \in Z : (x_G, (x_z, z, 0)) \in Q$

There are no other arrows than the ones indicated by the rules 1.—5.

Let's verify that the required conditions are fulfiled.

a) Let's prove that $\mathrm{Aut}(\mathscr{H}_G) \cong \{1\}$: Let $g : \mathscr{H}_G \to \mathscr{H}_G$ be an automorphism. $g(x_G) = x_G$ since $\deg^+_{\mathscr{H}_C}(x_G) = \mathrm{card}(\{y : (y, x_G) \in Q\}) = 0$ and x_G is the only such vertex. So $\forall z \in Z\, \exists z' \in Z$ such that $g((x_z, z, 0)) = (x'_z, z', 0)$. $(U \times \{z\} \times \{0\}, Q \restriction U \times \{z\} \times \{0\})$ are rigid connected symmetric graphs for all $z \in Z$. There is just one morphism between any two of them hence $z = z'$ and $g \restriction \bigcup_{z \in U} U \times \{z\} \times \{0\} = \mathrm{id}$. Also $g \restriction \bigcup_{z \in Z} U \times \{z\} \times \{1\} = \mathrm{id}$ which implies $\forall z \in Z\ g(z) = z$ so $g = \mathrm{id}_{\mathscr{H}_C}$.

b) Let g be an automorphism of $\mathscr{H}_G \setminus \{x_G\}$. Then obviously $g(\bigcup_{z \in Z} U \times \{z\} \times \{i\}) \subseteqq \subseteqq \bigcup_{z \in Z} U \times \{z\} \times \{i\}$ for $i \in \{0, 1\}$ and $g(Z) \subseteqq Z$.

So every automorphism of $\mathscr{H}_G \setminus \{x_G\}$ is uniquely determined by an automorphism of $\mathscr{K}_\mathscr{G}$ and every automorphism of $\mathscr{K}_\mathscr{G}$ determines the unique automorphism of $\mathscr{H}_G \setminus \{x_G\}$. Therefore we may conclude that $\mathrm{Aut}(\mathscr{H}_G \setminus \{x_G\}) \cong \mathbb{G}$

c) The proof that for $\forall y \in Y, y \neq x_G$ $\mathrm{Aut}(\mathscr{H}_G \setminus \{y\}) \cong \{1\}$ is easy using the same arguments as in a). □

This proposition can be of some interest on itself but we state it in order to prove Theorem 3.1.

Proof of Theorem 3.1. Let an arbitrary group $\mathbb{G}$ be given. Consider the graph $\mathscr{H}_G = (Y, Q)$ from Lemma 3.2. Put $I = \bigcup_{n=0}^{\infty} {}^n(Y \setminus \{x_G\})$ (where ${}^nX$ denotes the set of all functions from n to X). Take for every $f \in I$ one copy of the graph $\mathscr{H}_G$ (say $\mathscr{H}^f_G$).

Put $\tilde{\mathscr{G}} = \bigotimes_{f \in I} \mathscr{H}^f_G$, $\mathscr{G} = \tilde{\mathscr{G}} / \sim$,

where $\sim$ is defined as follows : $\forall f, g \in I$ such that $g = f \smile \{y\}$: $y^f \sim x^g_G$ ($\smile$ stands for the concatenation).

Let us remark that the factorisation and the coproduct are made in the category Graph. By an easy induction it can be verified that the resulting graph $\mathscr{G} = (X, R)$ is rigid with respect to automorphisms.

$Aut(\mathscr{G}\backslash\{x_G^0\}) \cong \mathbb{G}$ holds since automorphisms of $\mathscr{G}\backslash\{x_G^0\}$ are in one-to-one correspondence with the automorphisms of $\mathscr{K}_G^0$ and $Aut(\mathscr{K}_G^0) \cong \mathbb{G}$.
Let $x \in X$, $x \neq x_G^0$ and $g : \mathscr{G}\backslash\{x\} \to \mathscr{G}\backslash\{x\}$ be an automorphism. Then $g(x_G^0) = x_G^0$ since $\deg_{\mathbb{G}}^+(x_G^0) = 0$ and x_G^0 is the only such vertex. After deleting the vertex x the graph $\mathscr{G}$ falls apart into two parts. The component containing x_G^0 is mapped identically onto itself and the rest is isomorphic to $\mathscr{G}\backslash\{x_G^0\}$ and is mapped onto itself as well. So $Aut(\mathscr{G}\backslash\{x\}) \cong \mathbb{G}$. □

4. Topological version of the main theorem

The main goal of this section is to prove the following.

Theorem 4.1. *For every group $\mathbb{G}$ there is a metrizable topological space X having no nontrivial autohomeomorphism such that $\forall x \in X$ $Aut(X\backslash\{x\}) \cong \mathbb{G}$.*

In 2. Preliminaries we have shown that the autohomeomorphism groups of complements of points in a subcontinuum of a Cook continuum are trivial. This fact solves the problem for $\mathbb{G}$ trivial and will be of use later on. As in the previous section we shall prove the following proposition first:

Lemma 4.2. *For every group $\mathbb{G}$ there is a metrizable topological space X_G with a distinguished point x_G such that*

1. $Aut(X_G) \cong \{1\}$
2. $Aut(X_G\backslash\{x_G\}) \cong \mathbb{G}$
3. $\forall y \in X_G,\ y \neq x_G : Aut(X_G\backslash\{y\}) \cong \{1\}$.

Let $\{A_i : i = 1, 2, \ldots\} \cup \{B\}$ be a pairwise disjoint collection of nondegenerate subcontinua of a Cook continuum $\mathscr{C}$. Multiplying their metrics inherited from $\mathscr{C}$ by a suitable constant we may suppose that

$\operatorname{diam}(B) = 1$ and $\operatorname{diam}(A_i) = 2^{-i}$ for $i = 1, 2, \ldots$

For every i we choose two points $a_i^0, a_i^1 \in A_i$ such that $p(a_i^0, a_i^1) = \operatorname{diam}(A_i)$ and b^0, $b^1 \in B$ such that $p(b^0, b^1) = \operatorname{diam}(B) = 1$.
A metric space U is obtained from the coproduct $\bigotimes_{i=1,2,\ldots} A_i$ by the identification a_i^1 with a_{i+1}^0 for all $i = 1, 2, \ldots$ Let us consider all spaces A_i to be contained in U as subspaces and $A_i \cap A_{i+1} = \{a_i^1\} = \{a_{i+1}^0\}$. The coproduct and the identification is made in Metr — the category of all metric spaces of the diameter not exceeding 1 and nonexpanding mappings (The factorisation differs in general from that made in Top).
Denote by $\bar{U}$ the completion of U obtained from U by adding one point, say u.

Claim 4.3. *$Aut(U) \cong \{1\}$.*

Proof. It is sufficient to prove that

$\forall i \in \{1, 2, \ldots\}$ $\forall f : A_i \to U$ continuous: f is either a constant or the inclusion.
Put $P = \{a_k^0 : k = 1, 2, \ldots\} \cup A_i$.

First we shall show that f is constant whenever $f[A_i]$ intersects $U\backslash P$. Let us suppose that $f[A_i] \cap V \neq 0$ where $V = A_j \backslash \{a_j^0, a_j^1\}$ for some $j \neq i$. Obviously if $f[A_i] \subseteqq V$ then f is constant because A_i and A_j are disjoint nondegenerate subcontinua of the Cook continuum $\mathscr{C}$. If $A_i \backslash f^{-1}[V] \neq 0$, take any component K of $f^{-1}[V]$. By Lemma 2.4 its closure $\bar{K}$ intersects the boundary of $f^{-1}[V]$. So K is a nondegenerate subcontinuum of A_i which is mapped onto a nondegenerate subcontinuum of A_j which is imposible. We conclude that f is either constant or the inclusion. □.

Claim 4.4. $Aut(\bar{U}) \cong \{1\}$.

Proof. The proof is analogous to that of the preceeding claim. □

Let $\mathbb{G}$ be a fixed group. Let us take the graph $\mathscr{H}_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\} = (T, S)$, where $\mathscr{H}_{\mathbb{G}}$ is the graph with the distinguished point $x_{\mathbb{G}}$ constructed in the previous section Let B be the subcontinuum of the Cook continuum $\mathscr{C}$ with diameter equal to 1 we have already chosen. $b^0, b^1 \in B$ are the points the distance of which is equal to the diameter of B.

Put $\bar{H}_{\mathbb{G}} = \bigotimes_{(x,y) \in S} B_{(x,y)}$, where $B_{(x,y)} \cong B$ for all $(x, y) \in S$.

Let $b^0_{(x,y)}, b^1_{(x,y)} \in B_{(x,y)}$ be the points corresponding to $b^0, b^1 \in B$.

Put $H_{\mathbb{G}} = \bar{H}_{\mathbb{G}}/\sim$,

where the equivalence $\sim$ is defined as follows:

$$b^0_{(x,y)} \sim b^0_{(u,v)} \text{ iff } x = u$$
$$b^1_{(x,y)} \sim b^1_{(u,v)} \text{ iff } y = v$$
$$b^0_{(x,y)} \sim b^1_{(u,v)} \text{ iff } x = v$$

The coproduct and the factorisation are made in Metr again.

Claim 4.5. *$H_{\mathbb{G}}$ is a metrizable topological space such that $Aut(H_{\mathbb{G}}) \cong \mathbb{G}$.*

Proof. $H_{\mathbb{G}}$ is really a metrizable space because both the coproduct and the factorisation were made in Metr.
In is sufficient to show that $H_{\mathbb{G}}$ has the same autohomeomorphism group as the graph $\mathscr{H}_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}$. In order to prove this we shall show that every $f : B \to H_{\mathbb{G}}$ continuous and one-to-one mapping is the only existing homeomorphism between B and some $B_{(x,y)}$.
Suppose $f[B] \subseteqq B_{(u,v)}$ for some $(u, v) \in S$. Then $f[B] = b(u,v)$ and f is the homeomorphism since B is a subcontinuum of the Cook continuum. If it's not the case there would be some $(x, y), (u, v) \in S$, $(x, y) \neq (u, v)$ such that $B_{(x,y)} \cap B_{(u,v)}$ is nonempty and contained in $f[B]$ and for the component K of $B_{(x,y)} \cap B_{(u,v)}$ in $f[B] \cap (B_{(x,v)} \cup B_{(u,v)})$ the following holds true:

$$K \cup (B_{(x,y)} \backslash \{b^0_{(x,y)}, b^1_{(x,y)}\}) \neq 0 \text{ and } K \cap (B_{(u,v)} \backslash \{b^0_{(u,v)}, b^1_{(u,v)}\}) \neq 0$$

But then $f^{-1}[K \cap B_{(x,y)}]$, $f^{-1}[K \cap B_{(u,v)}]$ and $f^{-1}[K]$ are nondegenerate subcontinua of B and $f^{-1}[K] = f^{-1}[K \cap B_{(x,y)}] \cup f^{-1}[K \cap B_{(u,v)}]$ which leads to contradiction since B is herediterily indecomposable.
Therefore f fulfils the required conditions and consequently $\mathrm{Aut}(H_{\mathbb{G}}) \cong \mathbb{G}$. □

The last claim was an instance of the "*arrow construction*". Instead of every arrow we glued into the graph $\mathscr{H}_{\mathbb{G}} \setminus \{x_{\mathbb{G}}\}$ one copy of the continuum B. The glueing points were exactly the vertices of the graph. So we may consider T the set of all vertices of the graph $\mathscr{H}_{\mathbb{G}} \setminus \{x_{\mathbb{G}}\}$ to be contained in the space $H_{\mathbb{G}}$.

Construction of $X_{\mathbb{G}}$: Let $\mathbb{G}$ be a group. Take the space $H_{\mathbb{G}}$ and the spaces U, $\bar{U}$ constructed at the beginning of this section. Consider J to be the set of all the points corresponding to the original vertices $(a, z, 0) \in T$. For every $z \in Z$ (which is a subset of T) we have chosen a vertex $(x_z, z, 0) \in T$ (recall the construction of the graph $\mathscr{H}_{\mathbb{G}}$). Assign by K the subset of J containing all the points corresponding to these vertices.

In U we have one distinguished point, namely $a_1^0 \stackrel{\text{def}}{=} u^0$ and in $\bar{U}$ two distinguished points: $a_1^0 \stackrel{\text{def}}{=} \bar{u}^0$ and $u \stackrel{\text{def}}{=} \bar{u}^1$.

Put $\tilde{X}_{\mathbb{G}} = H_{\mathbb{G}} \otimes \bigotimes_{x \in J \setminus K} U_x \otimes \bigotimes_{x \in K} \bar{U}_x \otimes \{x_{\mathbb{G}}\}$,

where $U_x \cong U$ for all $x \in J \setminus K$ and $\bar{U}_x \cong \bar{U}$ for all $x \in K$ and $x_{\mathbb{G}}$ is a new point. Finally $X_{\mathbb{G}} = \tilde{X}_{\mathbb{G}}/\sim$,
where the equivalence relation $\sim$ is defined as follows:

$$\forall x \in J \setminus K \;\; x \sim u_x^0 \text{ where } u_x^0 \in U_x$$
$$\forall x \in K \;\; x \sim \bar{u}_x^0 \text{ where } \bar{u}_x^0 \in \bar{U}_x$$
$$\forall x \in K \;\; \bar{u}_x^1 \sim x_{\mathbb{G}}$$

Let us remark that we work in Metr again.

Let us now state some useful claims concerning the space $X_{\mathbb{G}}$. Their proofs are easy and similar to that of Claim 4.5 so they are left to the reader.

Claim 4.6. *Every $f : \bar{U} \to X_{\mathbb{G}}$ continuous one-to-one mapping is the canonical embedding of some $\bar{U}_x$, where $x \in K$.*

Claim 4.7. *Every $f : U \to X_{\mathbb{G}}$ continuous one-to-one mapping is the canonical embedding of some U_x, where $x \in J \setminus K$, or $f[U] \subset \bar{U}_x$ for some $x \in K$ and f is the unique such one-to-one mapping.*

Claim 4.8. *Every $f : B \to X_{\mathbb{G}}$ continuous one-to-one mapping is the only existing homeomorphism between B and some $B_{(x,y)}$.*

Proof of Lemma 4.2. Let an arbitrary group $\mathbb{G}$ be given.
a) Let $g : X_{\mathbb{G}} \to X_{\mathbb{G}}$ be an autohomeomorphism. $g(x_{\mathbb{G}}) = x_{\mathbb{G}}$ holds true due to

Claim 4.6. So $g(x) = x'$ for $x, x' \in K$. As a consequence of Claim 4.8 we obtain that $g(H_{\mathbb{G}}) \subset H_{\mathbb{G}}$ and nothing else is mapped into $H_{\mathbb{G}}$ · Thus $g \restriction H_{\mathbb{G}}$ is an autohomeomorphism of $H_{\mathbb{G}}$. From the proof of Claim 4.5 we know that every autohomeomorphism of $H_{\mathbb{G}}$ respects the original vertices and that no autohomeomorphism can send x to x' for $x \neq x'$. So $g(x) = x$ for every $x \in T$ and consequently $g \restriction H_{\mathbb{G}} = \mathrm{id}_{H_{\mathbb{G}}}$. We can conclude that due to Claim 4.7 $g = \mathrm{id}_{X_{\mathbb{G}}}$.
b) Let g be an autohomeomorphism of $X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}$. It is easy to verify that $g \restriction H_{\mathbb{G}}$ is an autohomeomorphism of $H_{\mathbb{G}}$ · Every autohomeomorphism of $H_{\mathbb{G}}$ uniquely determines an autohomeomorphism of $X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}$. So the autohomeomorphisms of $X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}$ are essentially the same as the autohomeomorphisms of $H_{\mathbb{G}}$. Using Claim 4.5 we can conclude that $\mathrm{Aut}(X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}) \cong \mathbb{G}$.
c) Let $y \in X_{\mathbb{G}}$, $y \neq x_{\mathbb{G}}$ and $g : X_{\mathbb{G}} \backslash \{y\} \to X_{\mathbb{G}} \backslash \{y\}$ be an autohomeomorphism. In order to prove that $g = \mathrm{id}_{X_{\mathbb{G}} \backslash \{y\}}$ let us make some observation contained in the following:

Claim 4.9. *Let $y \in X_{\mathbb{G}}, y \neq x_{\mathbb{G}}$ and let $f : B \to X_{\mathbb{G}} \backslash \{y\}$ be a continuous one-to-one mapping. Then f is the unique embedding of B onto some $B_{(u,v)}$, where $(u, v) \in S$.*

Proof. The proof is easy and therefore left to the reader. □

Claim 4.10. *Let $y \in X_{\mathbb{G}}$, $y \neq x_{\mathbb{G}}$ and let A_k be one of the subcontinua of the Cook continuum from which the space U was constructed. Then every continuous one-to-one mapping $f : A_k \to X_{\mathbb{G}} \backslash \{y\}$ is the canonical embedding of A_k into some U_x or $\bar{U}_x$.*

Proof. This proof is left to the reader as well. □

Let us return to the proof of Lemma 4.2. Using Claim 4.10 we obtain that $g(x_{\mathbb{G}}) = x_{\mathbb{G}}$. Due to the Claims we presented it is sufficient to restrict our attention to the original vertices.
Let $y \in X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\}$ and $y \notin T$. Following the proof of Lemma 3.2 and using the facts we already know we easily achieve that $g \restriction T = \mathrm{id}_T$. Then having proved Claim 4.10, Claim 4.9 and Claim 2.5 we can conclude that $g = \mathrm{id}_{X_{\mathbb{G}}}$.
The situation is analogous for $y \in T$. In this case we shall consider $T \backslash \{y\}$. Claim 4.9 assures that $g \restriction (T \backslash \{y\})$ respects the graph structure on $T \backslash \{y\}$. So the only thing that has to be verified is that $\forall x \in K\ g(x) = x$. To prove this remember how the graph $\mathscr{H}_{\mathbb{G}}$ was constructed. □

Proof of Theorem 4.1. Let an arbitrary group $\mathbb{G}$ be given. Consider the space $X_{\mathbb{G}}$ from Lemma 4.2. Put $I = \bigcup_{n=0}^{\infty} {}^{n}(X_{\mathbb{G}} \backslash \{x_{\mathbb{G}}\})$. Take for every $f \in I$ one copy of the space $X_{\mathbb{G}}$ (say $X^f_{\mathbb{G}}$).

Put $\tilde{X} = \bigotimes_{f \in I} X^f_{\mathbb{G}}$,

$$X = \tilde{X}/\sim ,$$

where $\sim$ is derfined as follows: $\forall f,\, g \in I$ such that $g = f \smile \{y\}$: $y^f \sim x^g_{\mathrm{G}}$.
The factorisation and the coproduct are made in the category Metr.

By an induction it can be verified that the space X is rigid with respect to automorphisms. We use the act that B (A_k resp.) can be mapped into X just onto a copy of itself. So every autohomeomorphism sends x^0_{G} onto x^0_{G} and hence the whole of X^0_{G} onto itself and so on.
Aut $(X\backslash\{x^0_{\mathrm{G}}\}) \cong \mathbb{G}$ holds since autohomeomorphisms of $X\backslash\{x^0_{\mathrm{G}}\}$ are in one-to-one correspondence with the autohomeomorphisms of $X^0_{\mathrm{G}}\backslash\{x^0_{\mathrm{G}}\}$ and $\mathrm{Aut}(X^0_{\mathrm{G}}\backslash\{x^0_{\mathrm{G}}\}) \cong \mathbb{G}$.
Let $x \in X$, $x \neq x^0_{\mathrm{G}}$ and $g : X\backslash\{x\} \to X\backslash\{x\}$ be an autohomeomorphism. Then $g(x^0_{\mathrm{G}}) = x^0_{\mathrm{G}}$ using standard arguments. After deleting x the space X falls apart into two parts. The component containing x^0_{G} is mapped identically onto itself and the rest is isomorphic to $X\backslash\{x^0_{\mathrm{G}}\}$ and is mapped onto itself as well.
So $\mathrm{Aut}(X\backslash\{x\}) \cong \mathbb{G}$. □

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