A. B. Kharazishvili

Some remarks on density points and the uniqueness property for invariant extensions of the Lebesgue measure

*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 35 (1994), No. 2, 33--39

Persistent URL: http://dml.cz/dmlcz/702011

# Some Remarks on Density Points and the Uniqueness Property for Invariant Extensions of the Lebesgue Measure

A. B. KHARAZISHVILI

Tbilisi*)

*Received 15. March 1994*

In this paper we construct an invariant extension $\mu$ of the classical Lebesgue measure such that $\mu$ has the uniqueness property and there exists a $\mu$-measurable set with only one density point.

There is a lot of papers devoted to the theory of invariant extensions of the classical Lebesgue measure and more general Haar measure (see, for instance, [1, 2, 3, 4, 5] and the references given in these works). There are several unsolved problems and questions in the mentioned theory which probably may be interesting for specialists of the modern mathematical analysis. In the present paper we consider the following question: does there exist a measure $\mu$ on the real line $\mathbf{R}$, invariant under the group of all isometrical transformations of $\mathbf{R}$, extending the usual Lebesgue measure $l$ on $\mathbf{R}$, having the uniqueness property on its domain of definition and such that some $\mu$-measurable subset of $\mathbf{R}$ has exactly one density point with respect to the standard Vitali's system in $\mathbf{R}$ consisting of all open intervals? Here we show that the answer to this question is positive. In particular, we obtain a positive answer to one of the questions formulated in the work [4].

First we recall some notions and definitions from the theory of invariant measures. Let $E$ be a basic set, $\Gamma$ be a group of transformations of $E$ and $\mu$ be a $\sigma$-finite $\Gamma$-invariant measure defined on some $\sigma$-algebra of subsets of $E$. Let $H$ be a subset of the group $\Gamma$. We say that the measure $\mu$ is metrically transitive with respect to the set $H$ if for every $\mu$-measurable set $X \subseteq E$ with $\mu(X) > 0$ there exists a countable family $(h_i)_{i \in I}$ of elements from $H$ such that

$$\mu(E \setminus \cup \{h_i(X) : i \in I\}) = 0 .$$

**Example 1.** Let $E$ be a $\sigma$-compact locally compact topological group with the $\sigma$-finite Haar measure $\mu$. Let $H$ be a subset of $E$ everywhere dense in $E$. Then the measure $\mu$ is metrically transitive with respect to the set $H$. This fact is well-known

*) Institute of Applied Mathematics, University of Tbilisi, University str. 2, 380 043 Tbilisi 43, Georgia.

in the theory of Haar measure and it can be easily proved if we use the uniqueness theorem for Haar measure and so called Steinhaus property of this measure. Recall here that the Steinhaus property of the measure $\mu$ means the following: for any $\mu$-measurable set $X$ the equality

$$\lim_{h \to e} \mu(h(X) \cap X) = \mu(X)$$

holds, where $e$ denotes the unit element of the group $E$.

Let again $E$ be a basic set, $\Gamma$ be a group of transformations of $E$ and $\mu$ be a $\sigma$-finite $\Gamma$-invariant measure defined on some $\sigma$-algebra of subsets of $E$. We say that the measure $\mu$ has the uniqueness property (on its domain of definition) if for any $\sigma$-finite $\Gamma$-invariant measure $\nu$ defined on $dom(\mu)$ there exists a coefficient $t \in \mathbf{R}$ (certainly, depending on $\nu$) such that $\nu = t \cdot \mu$.

For instance, every Haar measure has the uniqueness property in the mentioned sense. It is easy to see that if a $\sigma$-finite $\Gamma$-invariant measure $\mu$ has the uniqueness property then it is also metrically transitive with respect to the whole group $\Gamma$. The converse assertion is not true in general. However the following proposition holds.

**Lemma 1.** *Let $E$ be a basic set and $\Gamma$ be a group of transformations of $E$ containing an uncountable subgroup which acts freely in $E$. Let $\mu$ be a complete $\sigma$-finite $\Gamma$-invariant measure defined on some $\sigma$-algebra of subsets of $E$ and metrically transitive with respect to the whole group $\Gamma$. Then the measure $\mu$ has the uniqueness property.*

For the proof of Lemma 1 see [4]. Remark here that the essential role in the proof of this lemma plays the classical result of Ulam which asserts that there does not exist a non-zero $\sigma$-finite measure on the family of all subsets of the first uncountable cardinal $\omega_1$ vanishing on all one-element subsets of $\omega_1$.

For the further purposes we need also the following auxiliary proposition.

**Lemma 2.** *Let $\mathbf{T}$ be the unit circle in the Euclidean plane $\mathbf{R}^2$ considered as a commutative compact topological group (with respect to the natural group operation and to the induced topology from $\mathbf{R}^2$) and equipped with the invariant probability Lebesgue measure $\lambda$. Then there exists a homomorphism $\phi$ from the abstract group $\mathbf{R}$ into the abstract group $\mathbf{T}$ such that its graph*

$$\{(x, \phi(x)) : x \in \mathbf{R}\}$$

*is a $(l \times \lambda)$-massive subset (i.e. $(l \times \lambda)$-thick subset, according to the terminology of [6]) of the product-space $\mathbf{R} \times \mathbf{T}$. In particular, the homomorphism $\phi$ is everywhere discontinuous on its domain of definition (sometimes it is said that $\phi$ is a discontinuous character on $\mathbf{R}$).*

This lemma is also well-known. The required homomorphism $\phi$ can be constructed in a standard way, with the help of the method of transfinite recursion, if we consider the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers.

Now let $\phi$ be an arbitrary homomorphism from **R** into **T** the existence of which is established by Lemma 2. We recall how one can construct, starting with this homomorphism $\phi$, a certain extension of the Lebesgue measure $l$. For any set $Z$ from $dom(l \times \lambda)$ we put

$$Z' = \{x \in \mathbf{R} : (x, \phi(x)) \in Z\}.$$

Then we also put

$$S = \{Z' : Z \in dom(l \times \lambda)\}.$$

It is not difficult to check (see, for instance, [2] or [3]) that the family $S$ is a $\sigma$-algebra of subsets of the real line **R** invariant under the group of all isometrical transformations of **R**. Moreover, we have the inclusion $dom(l) \subsetneq S$ and if we put

$$\mu(Z') = (l \times \lambda)(Z), \qquad Z \in dom(l \times \lambda),$$

then it is not difficult to check that the last formula correctly defines a measure $\mu$ on the $\sigma$-algebra $S$, invariant under the group of all isometrical transformations of **R** and strictly extending the Lebesgue measure $l$.

We want to show now that the usual completion $\mu_1$ of the just constructed measure $\mu$ gives a positive solution of the question formulated at the beginning of this paper. First, let us establish that the measure $\mu_1$ gives a positive solution of the question formulated at the beginning of this paper. First, let us establish that the measure $\mu_1$ has the uniqueness property. Since the additive group of **R** is uncountable and acts freely in **R**, it is sufficient to prove that the measure $\mu$ is metrically transitive with respect to this group (see Lemma 1). Let a set $Y'$ belong to the domain of the definition of $\mu$ and let $\mu(Y') > 0$. Take a set $Y$ from $dom(l \times \lambda)$ such that

$$Y' = \{x \in \mathbf{R} : (x, \phi(x)) \in Y\}.$$

Of course, we have $(l \times \lambda)(Y') = \mu(Y') > 0$. Notice now that the product-measure $l \times \lambda$ is, in fact, the Haar measure on the $\sigma$-compact locally compact topological group $\mathbf{R} \times \mathbf{T}$. So $l \times \lambda$ is metrically transitive with respect to any everywhere dense subset of this group (see Example 1). If $(h_i)_{i \in I}$ is an arbitrary family of elements of **R** then we can write

$$\cup \{h_i + Y' : i \in I\} = \{X \in \mathbf{R} : (x, \phi(x)) \in \cup \{(h_i, \phi(h_i)) + Y : i \in I\}\}.$$

The graph of the homomorphism $\phi$ is a $(l \times \lambda)$-massive subset of the product-spase $(\mathbf{R} \times \mathbf{T}, l \times \lambda)$. Hence this graph is everywhere dense subset of the topological product $\mathbf{R} \times \mathbf{T}$. Let us take a countable family $(h_i)_{i \in I}$ of elements of $\mathbf{R} \times \mathbf{T}$ is everywhere dense in $\mathbf{R} \times \mathbf{T}$. Then, from the metrical transitivity of the measure $l \times \lambda$ and from the definition of the measure $\mu$ we will obtain

$$\mu(\mathbf{R} \setminus \cup \{h_i + Y' : i \in I\}) = 0.$$

Thus the measure $\mu$ is metrically transitive with respect to the additive group of **R**. In this way we have proved the following

**Lemma 3.** *The measure $\mu_1$ has the uniqueness property on its domain of definition.*

The next auxiliary proposition plays the key role in this paper.

**Lemma 4.** *There exist a $\mu$-measurable set $Z'$ having exactly one density point with respect to the standard Vitali's system in* **R**.

**Proof.** In the further considerations it is convenient to assume that the product-space $\mathbf{R} \times \mathbf{T}$ is canonically embedded in the three-dimensional Euclidean space $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$. So in this three-dimensional space the set $\mathbf{R} \times \mathbf{T}$ is an unbounded cylindrical surface with axis of symmetry coinciding with the axis of abscissae. More precisely, the equation of cylindrical surface is

$$x_2^2 + x_3^2 = 1 \qquad ((x_1, x_2, x_3) \in \mathbf{R}^3).$$

Let us consider the next three points on our surface:

$$(0, 1, 0), \ (-1, -1, 0), \ (1, -1, 0).$$

Let us draw two geodezic lines on our surface with the end-points (0, 1, 0) and (−1, −1, 0). Draw also two geodezic lines on our surface with the end-points (0, 1, 0) and (1, −1, 0). These four geodezic lines divide our surface into three connected parts. Only one of these parts is bounded. Let $Z$ denote the closure of the bounded part. It is clear that

$$Z \in dom(l \times \lambda), \qquad (0, 1, 0) \in Z.$$

Let $\phi$ be a homomorphism from the group **R** into the group **T** about existence of which is said in Lemma 2. Let us consider the set

$$Z' = \{x \in \mathbf{R} : (x, \phi(x)) \in Z\}.$$

Obviously, $0 \in Z'$. We assert that 0 is the unique density point of $Z'$ with respect to the standard Vitali's system on the real line. To show this let us fix a point $z \in \mathbf{R}$. Let $V(z)$ be any open interval in **R** containing the point $z$. We can write

$$V(z) \cap Z' = \{x \in R : (x, \phi(x)) \in (V(z) \times \mathbf{T}) \cap Z\}.$$

Hence, we have the relation

$$\frac{\mu(C(z) \cap Z')}{\mu(V(z))} = \frac{(l \times \lambda)((V(z) \times \mathbf{T}) \cap Z)}{(l \times \lambda)(V(z) \times \mathbf{T})}$$

By this relation,

$$\lim_{l(V(Z)) \to 0} \frac{\mu(V(z) \cap Z')}{\mu(V(z))}$$

is equal to $1 - |z|$ if $|z| \leqq 1$ and is equal to 0 if $|z| > 1$. Indeed, to establish the last fact it is sufficient to cut our cylindrical surface through the straight line containing the point (0, 1, 0) and parallel to the axis of abscissae and then to unfold this surface on the plane. Obviously, the set $Z$ will become a rhomb and the set $V(z) \times \mathbf{T}$ will become a rectangle, so the needed calculations can be done without any difficulties. Hence we see that the function $f : \boldsymbol{R} \to [0, 1]$ defined by the formula

$$f(z) = \lim_{l(V(z)) \to 0} \frac{\mu(V(z) \cap Z')}{\mu(V(z))}$$

is piecewise linear and continuous. Moreover, only for $z = 0$ we have $f(z) = 1$. Thus 0 is the unique density point of the set $U'$.

Summarizing all the results above we obtain the following theorem.

**Theorem.** *There exists a measure $\mu_1$ on the real line* **R** *such that*

1. *$\mu_1$ extends the classical Lebesgue measure 1 on* **R***;*
2. *$\mu_1$ is invariant under the group of all isometrical transformations of* **R***;*
3. *$\mu_1$ has the uniqueness property;*
4. *there exist a $\mu_1$-measurable subset of* **R** *having exactly one density point with respect to the standard Vitali's system in* **R**.

**Remark.** It is easy to see that for any natural number $n$ there exists a $\mu_1$-measurable subset of **R** having exatly $n$ density points with respect to the standard Vitali's system. There is also a $\mu_1$-measurable subset of **R** having an infinite countable set of density points with respect to the same Vitali's system. Of course, there exist some $\mu_1$-measurable subsets of **R** with strictly positive $\mu_1$-measure having no density point with respect to the mentioned Vitali's system. Notice also that the arguments similar to the proof of Lemma 4 show that for every continuous function $g : \mathbf{R} \to [0, 1]$ one can find a $\mu_1$-measurable set $Z'$ with the density function

$$f(x) = \lim_{l(V(x)) \to 0} \frac{\mu(V(x) \cap Z')}{\mu(V(x))}$$

coinciding with $g$. Finally, notice that the constructed measure $\mu_1$ does not have the Steinhaus property.

**Example 2.** At the present time various methods of constructing of invariant extensions of Lebesgue (respectively, Haar) measure are known. Such extensions may (or not) have some interesting properties. The most famous work in this area of measure theory is the paper [1] which gives a construction of a non-separable invariant extension of the Lebesgue measure. But we must notice here that the invariant extensions of Lebesgue measure obtained in [1] do not have the uniqueness property on their domain of definition. It is undoubtedly, that for different problems of mathematical analysis the uniqueness property of invariant measure is very important. In connection with this fact let us notice that it can be shown, essentially using Lemma 1, that the nonseparable invariant extensions of the Lebesgue measure obtained by the method of Kodaira and Kakutani [2] have the uniqueness property. To see this let us consider, for example, the commutative compact topological group $T^{\mathbf{c}}$, where $\mathbf{c}$ is the cardinality continuum. Let $\lambda_{\mathbf{c}}$ be the probability Haar measure on this group. Of course, $\lambda_{\mathbf{c}}$ is a non-separable measure. As in Lemma 2 there exist a homomorphism

$$\phi : \mathbf{R} \to \mathbf{T}^{\mathbf{c}}$$

such that its graph is a $(l \times \lambda_{\mathbf{c}})$-massive subset of the product-space $\mathbf{R} \times \mathbf{T}^{\mathbf{c}}$. The existence of the mentioned homomorphism can be established by the method of transfinite recursion if we consider the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and use the fact that the cardinality of the Baire's $\sigma$-algebra $B_0(\mathbf{R} \times \mathbf{T}^{\mathbf{c}})$ is equal to $\mathbf{c}$. Recall here that $\mathrm{B}_0(\mathbf{R} \times \mathbf{T}^{\mathbf{c}})$ is the smallest $\sigma$-algebra in the space $\mathbf{R} \times \mathbf{T}^{\mathbf{c}}$ containing all compact $G_\delta$-subsets of $\mathbf{R} \times \mathbf{T}^{\mathbf{c}}$ and the Haar measure $l \times \lambda_{\mathbf{c}}$ is inner regular with respect to this $\sigma$-algebra (see, for instance, [6]). Basing on these facts it is not difficult to construct the required homomorphism $\phi$ by transfinite recursion. Therefore, we can define a measure $\nu$ on the real line $\mathbf{R}$ by the following formula:

$$\nu(\{x \in \mathbf{R} : (x, \phi(x)) \in Z\}) = (l \times \lambda_{\mathbf{c}})(Z),$$

where $Z \in dom(l \times \lambda_{\mathbf{c}})$. It is easy to check that $\nu$ is a non-separable extension of the measure $l$ invariant under the group of all isometrical transformations of the real line $\mathbf{R}$. Using the metrical transitivity of the Haar measure $l \times \lambda_{\mathbf{c}}$ we can prove that the measure $\nu$ defined above is metrically transitive with respect to the additive group of $\mathbf{R}$ (the proof is completely analogical to the arguments preceding Lemma 3). Hence, applying Lemma 1, we obtain that the measure $\nu_1$ (the completion of $\nu$) has the uniqueness property on its domain of definition. Moreover, we also see that in $dom(\nu_1)$ there exists a set $X$ having exactly one density point with respect to the standard Vitali's system in $\mathbf{R}$. Indeed, it is sufficient to define $X$ by the following equality:

$$X = \{x \in \mathbf{R} : (x, \phi(x)) \in Z \times \mathbf{T}^{\mathbf{c} \setminus \{0\}}\},$$

where $Z$ is the subset of $\mathbf{R} \times \mathbf{T}$ described in the proof of Lemma 4.

Now, let $n$ be a non-zero natural number, $E_n$ be the $n$-dimensional Euclidean space, $G_n$ be the group of all isometrical transformations of the space $E_n$ and let $l_n$ be the classical $n$-dimensional Lebesgue measure on $E_n$. Combining the method of Kakatuni and Oxtoby with the method of Kakutani and Kodaira it can be constructed a non-separable $G_n$-invariant extension $\mu$ of $l_n$ such that there exists a $\mu$-measurable set having exactly one density point with respect to the standard Vitali's system in the space $E_n$ (consisting of all open cubes in $E_n$).

## References


[1] KAKUTANI, S., OXTOBY, J., *Construction of a non-separable invariant extension of the Lebesgue measure space*, Ann. Math., 52 (1950), 580–590.

[2] KODAIRA, K., KAKUTANI, S., *A non-separable translation invariant extension of the Lebesgue measure space,* Ann. Math., 52 (1950), 574–579.

[3] HULANICKI, A., RYLL-NARDZEWSKI, C., *Invariant extensions of the Haar measure*, Colloq. Mathem., 42 (1979), 223–227.

[4] KHARAZISHVILI, A. B., *Invariant extensions of the Lebesgue measure*, Tbil. Gos. Univ., Tbilisi, 1983 (in Russian).

[5] KHARAZISHVILI, A. B., *Vitali's theorem and its generalizations*, Tbil. Gos. Univ., Tbilisi, 1991 (in Russian).

[6] HALMOS, P. R., *Measure theory*, D. Van Nostrand, New York, 1950.