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Władysław Kulpa

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Some Properties of n-Dimensional Generalized Cubes

W. KULPA

Katowice*)

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The fixed point property, the Bolzano property and some version of the domain invariance and the Jordan theorems are investigated for a subclass of the class of the limits of inverse sequences of n-dimensional cubes. It is proved a lemma of the Sperner type for combinatorial cubes.

1. A combinatorial Lemma

Let $(Z, +)$ be the group of integers, and Z^n – the Cartesian product of n-copies of the set Z^n;

$$Z^n := \{z : \{1, \dots, n\} \to Z \quad | \quad \text{z is a map}\}$$

The set Z^n will be equipped with a structure of a group, a partial order and a metric:

$$z = u + v \quad \textit{iff} \quad z(i) = u(i) + v(i) \quad \text{for each } i = 1, \dots, n$$
$$u \leq v \quad \textit{iff} \quad u(i) \leq v(i) \quad \text{for each } i = 1, \dots, n$$

where $u, v, z \in Z^n$.

Using the Cartesian notation let $0 := (0, \dots, 0)$ be the neutral element of the group Z^n, $e_i := (0, \dots, 0, 1, 0, \dots, 0)$, $e_i(i) = 1$, the i-th unit vector, and $e := (1, \dots, 1)$. Denote by P(n) the set of permutation of the set $(1, \dots, n)$;

$$\alpha \in P(n) \quad \textit{iff} \quad \alpha : \{1, \dots, n\} \to \{1, \dots, n\} \quad \text{is a 1-1 map}$$

Definition. *An ordered set $S = [z_0, \dots, z_n] \subset Z^n$ is said to be a (n-dimensional) simplex iff*

$$z_0 < z_1 < \dots < z_n = z_0 + e\,.$$

Let us note that the diameter of any simplex is equal 1. It is easy to observe that

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*) Instytut Matematyki, Uniwersytet Śląski, 40 007 Katowice, Poland

An ordered set $[z_0, \ldots, z_n]$ *is a simplex iff there exists a permutation* $\alpha \in P(n)$ *such that*

$$z_1 = z_0 + e_{\alpha(1)}, \qquad z_2 = z_1 + e_{\alpha(2)}, \ldots, \qquad z_n = z_{n-1} + e_{\alpha(n)}$$

or

An ordered set $[z_0, \ldots, z_n] \subset Z^n$ *is a simplex iff the two following conditions hold:*
(a) for each $i < n$ *there is an* $r \leq n$ *such that* $z_{i+1} - z_i = e_r$,
(b) for each $i \neq j$, $i + 1 \leq n$, $j + 1 \leq n$; $z_{i+1} - z_i \neq z_{j+1} - z_j$

Two simplexes $S, T \subset Z^n$ are said to be adjacent if they have n common points; $|S \cap T| = n$.

Observation. *Let* $S = [z_0, \ldots, z_n] \subset Z^n$ *be a simplex. Then for each point* $z_i \in S$ *there exists exactly one simplex* $T = S[i]$ *such that*

$$S \bigcap T = \{z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}.$$

Proof. We shall define i-neighbour S[i] of the simplex S
1. If $0 < i < n$, then $S[i] := [z_0, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n]$,
   where $x_i = z_{i-1} + (z_{i-+1} - z_i)$.
2. If i = 0, then $S[0] := [z_1, \ldots, z_n, x_0]$, where $x_0 = z_n + (z_1 - z_0)$,
3. If i = n, then $S[n] := [x_n, z_0, \ldots, z_{n-1}]$, where $x_n = z_0 - (z_{n+1} - z_n)$

We leave to the reader to prove that the simplexes S[i] are well defined and that they are the only possible i-neighbours of the simplex S. □
Any subset $[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n] \subset S$, i = 0, ..., n, is said to be ($(n - 1)$-dimensional) i-face of the simplex S. Let $k > 1$ be a natural number. A subset $C \subset Z^n$ of the form

$$C = \{0, \ldots, k\}^n$$

is said to be a combinatorial (n-dimensional) cube. Define the i-th opposite faces of C;

$$C_i^- := \{z \in C : z(i) = 0\}, \qquad C_i^+ := \{z \in C : z(i) = 1\}$$

and the boundary

$$\partial C := \bigcup \{C_i^- \cup C_i^+ : i = 1, \ldots, n\}$$

From the above lemma we get the following

Observation. *Any face of a simplex contained in the cube C is a face of exactly one or two simplexes from C, depending on whether or not it lies on the boundary* ∂C.

The Combinatorial Lemma. *Let* $\{F_i^-, F_i^+ : i = 1, \ldots, n\}$ *be a family of subsets of the combinatorial cube* $C = \{0, \ldots, k\}^n$ *such that*

(1) $\quad C = F_i^- \cup F_i^+, \quad C_i^- \subset F_i^-, \quad C_i^+ \subset F_i^+$ *for each* $i = 1, \ldots, n.$

Then there exists a simplex $S \subset C$ *with the following property*

(2) $$F_i^- \cap S \neq \emptyset \neq S \cap F_i^+ \quad \textit{for each } i = 1, \ldots, n$$

Moreover, the number of such simplexes is odd.

Proof. Since $C_i^- \subset F_i^-$ we infer that $C = F_i^- \cup \{F_i^+ \setminus C_i^-\}$. Thus without loss of generality we may assume that

(3) $$C_i^- \cap F_i^+ = \emptyset \quad \text{for each i} = 1, \ldots, \text{n}.$$

Define

(4) $\quad \varphi(x) := max\{j : x \in F_i^+ \quad \text{for each} \quad i = 0, \ldots, j\}, \quad \text{where } F_0^+ := C.$

The map $\varphi : C \to \{1, \ldots, n\}$ has the following properties:

(5) $\quad$ if $x \in C_i^-$, *then* $\varphi(x) < i$, and if $x \in C_i^+$, *then* $\varphi(x) \neq i - 1.$

From (5) it follows that for each simplex $S \subset C$;

(6) $$\varphi(S \cap C_i^\varepsilon) = \{0, \ldots, n-1\} \quad \text{implies that } i = n \text{ and } \varepsilon = -.$$

Let us note that from (4) and (1) we get

(7) $$\text{if } \varphi(x) = i - 1 \text{ and } \varphi(y) = i, \text{ then } x \in F_i^- \text{ and } y \in F_i^+.$$

Let us call n-dimensional simplex S to be proper if $\varphi(S) = \{0, \ldots, n\}$. From (7) it follows that the lemma will be proved if we show that the number ϱ of proper simplexes will be odd.

Our proof will be by induction on the dimensionality n of C. The lemma is obvious for n = 0, because $C = \{0\}, \varphi(0) = 0, \varrho = 1$.

Let us call an (n − 1)-dimensional face s to be proper if $\varphi(s) = \{0, \ldots, n-1\}$. According to (6) any proper face $s \subset \partial C$ lies in C_n^- and by our induction hypothesis the number α of such faces is odd. Let $\alpha(S)$ means the number of proper aces of a simplex $S \subset C$.

Now, if S is a proper simplex, clearly $\alpha(S) = 1$; while if S is not a proper simplex, we have $\alpha(S) = 2$ or $\alpha(S) = 0$ according as $\varphi(S) = \{0, \ldots, n-1\}$ or $\varphi(S) \neq \{0, \ldots, n-1\}$.
Hence

(8) $$\varrho = \sum \alpha(S), \text{ mod } 2$$

On the other hand, a proper face appears exactly once or twice in $\sum \alpha(S)$ according as it is in the boundary of C or not.
Accordingly

(9) $$\sum \alpha(S) = \alpha, \text{ mod } 2$$

Whence

(10) $$\alpha = \varrho, \bmod 2.$$

But α is odd. Thus ϱ is odd, too. □

2. Classical results

Let R^n be the Euclidean space

$$R^n := \{x : \{1, \dots, n\} \to R \mid \text{x is a map}\}$$

and let I^n be the n-dimensional cube

$$I^n := \{x \in R^n : \quad 0 \leq x(i) \leq 1, \quad i = 1, \dots, n\}$$

For each $i \leq n$ let us denote

$$I_i^- := \{x \in I^n : \quad x(i) = 0\}, \quad I_i^+ := \{x \in R^n : \quad x(i) = 1\}$$

the i-th opposite faces.

The Topological Lemma. Let $\{H_i^-, H_i^+ : i = 1, \dots, n\}$ *be a family of closed subsets of the cube* I^n *such that;* $I^n = H_i^- \cup H_i^+$, $I_i^- \subset H_i^-$, $I_i^+ \subset H_i^+$ *for each* $i = 1, \dots, n$.

Then the intersection $\bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\}$ *is non-empty set.*

Proof. In order to prove the lemma it suffices to show that

$$\bigcap\{H_i^- \cap H_i^+ : \quad i = 1, \dots, n\} \neq \emptyset$$

Suppose to the contrary that it does not hold. Then, $\bigcup\{U_i^- \cup U_i^+ : i = 1, \dots, n\} = I^n$, where $U_i^\varepsilon := I^n \setminus H_i^\varepsilon$. Since the cube I^n is compact hence there is a real number $\delta > 0$ such that any subset of I^n of the diameter less than δ is contained in some set U_i^ε. For this number δ there is a natural number $k > 1$ such that the map $\varphi : C = \{0, \dots, k\}^n \to I^n$, where $\varphi(x) := \frac{x}{k}$, has the following property:
(a) for each simplex $S \subset C$ there exists a set U_i^ε such that $\varphi(S) \subset U_i^\varepsilon$.
(b) $\varphi(C_i^-) \subset I_i^-$ and $\varphi(C_i^+) \subset I_i^+$ for each i $= 1, \dots,$ n.
Now let us put; $F_i^- := \varphi^{-1}(H_i^-)$, $F_i^+ := \varphi^{-1}(H_i^+)$ for i $= 1, \dots,$ n
From the property (a) it follows that for each simplex $S \subset C$ there exist an $i \leq n$ such that

(1) $$S \cap F_i^- = \emptyset \quad \text{or} \quad S \cap F_i^+ = \emptyset.$$

On the other hand, for each i $= 1, \dots,$ n; $C = F_i^- \cup F_i^+$, $C_i^- \subset F_i^-$, $C_i^+ \subset F_i^+$
From the Combinatorial Lemma we infer that there is a simplex $S \subset C$ such that

(2) $$F_i^- \cap S \neq \emptyset \neq S \cap F_i^+ \quad \text{for each i} = 1, \dots, \text{n}.$$

Comparing (2) with (1) we get a contradiction. □

As corollaries we obtain

The Poincaré – Miranda Theorem. *Let* $f : I^n \to R^n$, $f = (f_1, \ldots, f_n)$, *be a continuous map such that for each* $i \leq n$, $f_i(I_i^-) \subset (-\infty, 0]$ *and* $f_i(I_i^+) \subset [0, +\infty)$. *Then there exists a point* $c \in I^n$ *such that* $f(c) = 0$.

Proof. For each i = 1, ..., n let us put; $H_i^- := f_i^{-1}(-\infty, 0]$, $H_i^+ := f_i^{-1}[0, \infty)$. The sets H's satisfy the assumptions of the Topological Lemma and therefore the intersection

$$C := \bigcap\{H_i^- \cap H_i^+ : \quad i = 1, \ldots, n\} \neq \emptyset$$

is not empty set. It is clear that $f(c) = 0$ for each $c \in C$. □

The Coincidence Theorem. *If maps* $g, h : I^n \to I^n$ *are continuous and for each* $i = 1, \ldots, n$; $h(I_i^-) \subset I_i^-$ *and* $h(I_i^+) \subset I_i^+$, *then they have a concidence property i.e., there exists a point c such that* $g(c) = h(c)$.

Proof. Let us put $f(c) := g(x) - h(x)$. The map f satisfies the assumptions of the Poincaré-Miranda Theorem and therefore there is a point $c \in I^n$ such that $f(c) = 0$. But this means that $g(c) = h(c)$. □
If h is the identity map the we get

The Bohl-Brouwer Fixed Point Theorem. *Any continuous map* $g : I^n \to I^n$ *has a fixed point.*
And applying the Coincidence Theorem to constant maps; $g(x) = a$, $a \in I^n$, we get

Corollary. *Any continuous map* $h : I^n \to I^n$ *satisfying for each* $i = 1, \ldots, n$; $h(I_i^-) \subset I_i^-$ *and* $h(I_i^+) \subset I_i^+$ *is "onto".*

The Borsuk Non-Retraction Theorem. *Let* $f : X \to R^n$ *be a continuous map from a compact set* $X \subset R^n$. *If* $f(x) = x$ *for each* $x \in Bd\, X$, *then* $X \subset f(X)$.

Proof. Let J^n be an n-dimensional cube such that $X \subset J^n$ and extend the map f to a continuous map $h : J^n \to J^n$ such that h(x) = x for each $x \in J^n \backslash X$. It is clear that for any i; $h(J_i^-) \subset J_i^-$ and $h(J_i^+) \subset J_i^+$. From the above corollary we infer that $J^n \subset h(J^n)$, and in consequence $X \subset f(X)$. □

3. New results

In this part we shall introduce a class of spaces for which the results of the previous paragraph hold.

Defionition. *A space X belongs to the class* K_n, $X \in K_n$, *provided that X is the limit of an inverse sequence of n-dimensional cubes,*
$X = \lim inv \{p_{k,l} : I^n \to I^n;\ k \geq l,\ k, l \in N\}$,
where the bonding maps $p_{k,l}$ *are continuous and satisfy the following condition*

(B) $p_{k,l}(I_i^\varepsilon \subset I_i^\varepsilon$, *for each* $i = 1, \ldots, n$ *and* $\varepsilon = -, +$.
Denote by $p_k : X \to I^n$, $k \in N$, *the projection maps. And finally, let us say that* $X \in K$ *provided that* $X \in K_n$ *for some* $n \in N$

Observation. *If* $X \in K_n$ *and* $Y \in K_m$ *then* $X \times Y \in K_{n+m}$.
Indeed, let X = lim inv $\{p_{k,l}, : I^n \to I^n\}$ and Y = lim inv $\{q_{k,l} : I^m \to I^m\}$. Then $X \times Y =$ lim inv $\{r_{k,l} : I^{n+m} \to I^{n+m}\}$, where $r_{k,l}(x, y) := (p_{k,l}(x), q_{k,l}(y))$. Assuming that maps p's and q's satisfy the condition (B) one can verify that

$$r_{k,l}(x_1, \ldots, \eta, \ldots, x_n, y_1, \ldots, y_n) = (s_1, \ldots, \eta, \ldots, s_n, t_1, \ldots, t_n)$$

and

$$r_{k,l}(x_1, \ldots, x_n, y_1, \ldots, \eta, \ldots, y_n) = (s_1, \ldots, s_n, t_1, \ldots, \eta, \ldots, t_n),$$

where $\eta \in \{0, 1\}$, but this means that the maps $r_{k,l}$ satisfy the condition (B).
The class K_1 contains spaces of so complicated structure as pseudoarc being the field of investigation of many authors. In 1951 Hamilton [4] has shown that the pseudoarc has the fixed point property. From the results which are presented in this paper it follows that the Cartesian product of arbitrary many pseudoarcs has the fixed point property.
For a given space $X \in K_n$ let us fix an inverse system $\{p_{k,l} : I^n \to I^n\}$ having the property (B). Define for each i = 1, ..., n;
$A_i :=$ lim inv $\{p_{k,l}|I_i^- : I_i^- \to I_i^-\}$ and $B_i :=$ lim inv $\{p_{k,l}|I_i^+ : I_i^+ \to I_i^+\}$,
where I_i^- and I_i^+ mean, as usual, the i-th opposite faces of the cube I^n.

The Bolzano Theorem. *Let* $f : X \to R^n$, $f = (f_1, \ldots, f_n)$, *where* $X \in K_n$, *be a continuous map such that for each* $i = 1, \ldots, n$
(1) $f_i(A_i) \subset (-\infty, 0]$ *and* $f_i(B_i) \subset [0, \infty)$.
Then there exist a point $c \in X$ *such that* $f(x) = 0$.

Proof. Define for each i = 1, ..., n, and $m \in N$

$$H_{i,m}^- := p_m(f_i^{-1}(-\infty, 0]), \quad H_{i,m}^+ := p_m(f_i^{-1}[0, \infty)).$$

From the assumption (1) and the definition of the sets A_i, B_i it follows that;

$$I^n = H_{i,m}^- \cup H_{i,m}^+ \quad A_i \subset H_{i,m}^- \quad B_i \subset H_{i,m}^+.$$

According to the Topological Lemma the set

$$C_m := \bigcap \{H_{i,m}^- \cap H_{i,m}^+ : i = 1, \ldots, n\}$$

is non-empty. Moreover, the sets C_m are compact and $C_{m+1} \subset C_m$ for each $m \in N$. Hence the intersection

$$C := \bigcap \{C_m : m \in N\}$$

is a non-empty set. It is clear that f(c) = 0 for each $c \in C$. Thus the Bolzano theorem is proved. □

The Coincidence Theorem. *Let $h : X \to X$, where $X \in K_n$, be a continuous map such that for each $i = 1, \ldots, n$*

(2) $h(A_i) \subset A_i$ and $h(B_i) \subset B_i$

Then for any continuous map $g : X \to X$ there exists a point $a \in X$ such that

$$g(a) = h(a).$$

Proof. Now, let a map $h : X \to X$ satisfies the assumptions of the Coincidence Theorem and let $g : X \to X$ be an arbitrary continuous map. For each $m \in N$ let us put

$$f_m(x) := (p_m \circ g)(x) - (p_m \circ h)(x), \quad x \in X$$

According to the Bolzano Theorem the set

$$A_m := \{x \in X : f_m(x) = 0\}$$

is non-empty. Moreover, it is a compact set and $A_{m+1} \subset A_m$ for each $m \in N$. Thus the intersection

$$A := \bigcap \{A_m : m \in N\}$$

is non-empty set. It is clear that $g(a) = h(a)$ for each $a \in A$. □

Now, let $h : X \to X$ be the identity map. Then from the Coincidence Theorem we get

The Fixed-point Theorem. *If $X \in K$, then each continuous map $g : X \to X$ has a fixed point.*

R. H. Bing [1, p. 103] gives an example of compact set X in R^3 which is an intersection of a sequence of 3-cells but for which there is a fixed point free homeomorphism of X onto itself. Thus the assumption (B) is essential.

4. The Bolzano property

A family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of non-empty disjoint closed subsets of a topological space X is said to be an *n-dimensional boundary system* whenever for each continuous map $f : X \to R^n$, $f = (f_1, \ldots, f_n)$, satisfying for each $i \leq n$ the Bolzano condition;

$$f_i(A_i) \subset (-\infty, 0], \quad f_i(B_i) \subset [0, \infty)$$

there exists a point $c \in X$ such that f(c) = 0. If a space X has an n-dimensional boundary system then we say that X has an n-dimensional Bolzano property, $X \in B_n$. The following relation holds

$$I^n \in K_n \subset B_n.$$

One can prove that a subset $X \subset R^n$ has the n-dimensional Bolzano property if and only if it has non-empty interior.

Let us assume that for any space $X \in B_n$ we have established an n-dimensional boundary system and let us define; $\partial X := \bigcup\{A_i \cup B_i : i = 1, \ldots, n\}$.

As in section 3 we get

The Coincidence Theorem. *If $X \in B_n$ and $h : X \to R^n$ is a continuous map such that $h(A_i) \subset I_i^-$ and $h(B_i) \subset I_i^+$, for each $i = 1, \ldots, n$, then $I^n \subset h(X)$ and for each continuous map $g: X \to I^n$ there exists a point $c \in C$ such that $g(c) = h(c)$.*

In view of the Coincidence Theorem let us observe that an existence of a normal space $X \in B_n$ implies the Brouwer fixed point theorem.

Indeed, let $(A_1, B_1), \ldots, (A_n, B_n)$ be a boundary system of a normal space X, and let $h_i : X \to [0, 1]$, i = 1, ..., n, be continuous functions such that $h_i(A_i) = \{0\}$ and $h_i(B_i) = \{1\}$. Then for the map $h := (h_1, \ldots, h_n) : X \to I^n$ we have; $h(A_i) \subset I_i^-$ and $h(B_i) \subset I_i^+$ for each i = 1, ..., n. Now, let $g : I^n \to I^n$ be an arbitrary continuous map. According to the Coincidence Theorem there exists a point $a \in X$ such that $h(a) = (g \circ h)(a)$. Thus the point c = h(a) is a fixed point of the map g.

The dimension theory says nothing how to construct an n-dimensional boundary system for a space $X \in B_n$. We show that the Combinatorial Lemma gives a possibility to find such a system. Let $X_1, \ldots, X_n$ be compact connected Hausdorff spaces. For each $i \leq n$ choose two distinct points $a_i, b_i \in X_i$. In the Cartesian product $X := X_1 \times \ldots \times X_n$ define $A_i := \{x \in X : x_i = a_i\}$ and $B_i := \{x \in X : x_i = b_i\}$. We shall show that the pairs $(A_i, B_i), \ldots, (A_i, B_i)$ form an n-dimensional boundary system.

Indeed, let f $= (f_1, \ldots, f_n) \to R^n$ be a continuous map such that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. Let us put $H_i^- := f_i^{-1}(-\infty, 0]$ and $H_i^+ := f^{-1}[0, \infty)$. Then it is clear that $f(c) = 0$ if and only if $c \in \bigcap\{H_i^- \cap H_i^+ : i \leq n\}$. Suppose to contrary that the intersection $\bigcap\{H_i^- \cap H_i^+ : i \leq n\}$ is empty. Then the family $Q := \{U_i^\varepsilon : i = 1, \ldots, n, \varepsilon = +, -\}$ is an open covering of the space X, where $U_i^\varepsilon := X \setminus H_i^\varepsilon$. Let a covering $P = P_1 \times \ldots \times P_n$ be an open refinement of the covering Q, where each P_i is an open covering of the space X_i. Now, for each $i \leq n$ let us choose a chain $U_{i,1}, \ldots, U_{i,k_i}$ of elements from the covering P_i such that $a_i \in U_{i,1}$, $b_i \in U_{i,k_i}$ and $U_{i,j-1} \cap U_{i,j} \neq \emptyset$ for each $j \leq k_i$. And then choose points $c_{i,1}, \ldots, c_{i,k_i}$ from X_i such that $a_i = c_{i,1}$, $b_i = c_{i,k_i}$ and $c_{i,j-1}, c_{i,j} \in U_{i,j}$ for each $j \leq k_i$. Let k $= \max\{k_i : i \leq n\}$ and define;

$$\varphi_i(j) = \begin{cases} c_{i,j} & \text{if } j \leq k_i \\ c_{i,k_i} & \text{if } j > k_i \end{cases}$$

The map $\varphi := (\varphi_1, \ldots, \varphi_n) : \{0, \ldots, k\}^n \to X$ has the following property; if $F_i^- := \varphi^{-1}(H_i^-)$, $F_i^+ := \varphi^{-1}(H_i^+)$ then $C = F_i^- \cup F_i^+$ and $C_i^- \subset F_i^-$, $C_i^+ \subset F_i^+$ for each i = 1, ..., n. From the Combinatorial Lemma it follows that there is a simplex $S \subset X$ such that $S \cap F_i^\varepsilon \neq \emptyset$ for each $i \leq n$ and $\varepsilon = -, +$. On the other hand, $\varphi(S) \subset U_i^\varepsilon$ for some $i \leq n$ and $\varepsilon \in \{-,+\}$. This contradiction concludes our remark.

Let us state without proof the following

Theorem on Two Maps. *Let be given two continuous maps $h : X \to R^n$ and $g : h(X) \to R^n$, where $X \in B_n$ and $h(X)$ is a compact subspace of R^n, such that $g(h(A_i)) \subset I_i^-$, and $g(h(B_i)) \subset I_i^+$ for each $i = 1, \ldots, n$. Then $R^n \backslash h(\partial I^n)$ is not a connected set and for each point $a \in Int\, I^n$, $g^{-1}(a) \cap Int\, h(X) \neq \emptyset$.*

As corollaries we obtain

The Domain Invariance Thorem. *If $f : I^n \to R^n$ is a one-to-one continuous map the $f(Int\, I^n)$ is an open subset of R^n.*

The Non-Squeezing Theorem. *Let $h : X \to R^n$ be a continuous map from a compact space $X \in B_n$ such that $h(A_i) \cap h(B_i) = \emptyset$ for each $i = 1, \ldots, n$. Then $R^n \backslash h(\partial X)$ is not a connected set and the set $Int\, h(X)$ is not empty.*

In [6] it was formulated a theorem equivalent to the Brouwer fixed point theorem; the indexed open covering theorem. We are going to strengthen this theorem by proving

Theorem on indexed open families. *If $U_1, \ldots, U_n$ are families of open pairwise disjoint sets of a normal space $X \in B_n$ and $X = \bigcup\{U \in U_i : i = 1, \ldots, n\}$, then there exists an index $i \leq n$ and a set $U \in U_i$ such that $A_i \cap U \neq \emptyset \neq U \cap B_i$.*

More precisely, we shall prove the following

Theorem. *Let $\{(A_i, B_i) : i = 1, \ldots, n\}$ be a family of non-empty disjoint closed subsets of a normal space X. Then the following statements are equivalent:*

(i). If $f = (f_1, \ldots, f_n) : X \to R^n$ is a continuous map such that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$ for each $i \leq n$, then there exists a point $c \in X$ such that $f(c) = 0$.

(ii). If pairs (H_i^-, H_i^+), $i = 1, \ldots, n$, of closed sets are such that $X = H_i^- \cup H_i^+$ and $A_i \subset H_i^-$, $B_i \subset H_i^+$, then the intersection $\bigcap\{H_i^- \cap H_i^+ : i \leq n\}$ is non-empty.

(iii). If $U_1, \ldots, U_n$ are families of open pairwise disjoint sets such that $X = \bigcup\{U \in U_i : i = 1, \ldots, n\}$, then there exists an index $i \leq n$ and a set $U \in U_i$ such that $A_i \cap U \neq \emptyset \neq U \cap B_i$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} = \emptyset$. Since X is a normal space there exist continuous functions $g_i, h_i : X \to [0,1]$ such that

$$H_i^- \subset g^{-1}(0) =: C_i, \quad H_i^+ \subset h^{-1}(0) =: D_i \quad \text{and} \quad \bigcap\{C_i \cap D_i : i \leq n\} = \emptyset.$$

Define for each $i \leq n$ and $x \in X$,

$$f_i(x) := g_i(x) - h_i(x)$$

It is clear that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. From (i) it follows that there exists a point $c \in X$ such that $f_i(c) = 0$. This means that for each $i \leq n$,

$g_i(c) = h_i(c)$. But, since $X = C_i \cup D_i$ and $c \in X$, we infer that $g_i(c) = 0 = h_i(c)$ for each i = 1, ..., n. This implies that $c \in \bigcap\{C_i \cap D_i : i \leq n\}$. And this leads to a contradiction with our supposition.

$(ii) \Rightarrow (iii)$. Suppose that for each $i \leq n$, and $U \in U_i$, $A_i \cap U = \emptyset$ or $B_i \cap U = \emptyset$. Define

$$G_i^- := \bigcup\{U \in U_i : U \cap A_i^- \neq \emptyset\}, \quad G_i^+ := \bigcup\{U \in U_i : U \cap A_i = \emptyset\}$$

and then let us put

$$H_i^- := X \setminus G_i^+, \quad \text{and} \quad H_i^+ := X \setminus G_i^-$$

We have; $A_i \subset H_i^-$ and $B_i \subset H_i^+$. Since $G_i^- \cap G_i^+ = \emptyset$ we get, $X = H_i^- \cup H_i^+$. Now, from (ii) it follows that $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} \neq \emptyset$. But $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} = \bigcap\{X \setminus (G_i^- \cup G_i^+) : i \leq n\} = X \setminus \bigcup\{G_i^- \cup G_i^+ : i \leq n\} = X \setminus \bigcup\}U \in U_i : i \leq n\} = \emptyset$, a contradiction.

$(iii) \Rightarrow (i)$. Let $f = (f_1, ..., f_n) : X \to R^n$ be a continuous map such that for each $i \leq n$, $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$ and suppose that $0 \notin f(X)$. Define for each $i \leq n$ $U_i := \{V_i, W_i\}$, where $V_i := \{x \in X : f_i(x) < 0\}$ and $W_i := \{x \in X : f_i(x) > 0\}$ From the supposition $0 \notin f(X)$ it follows that $X = \bigcup\{U \in U_i : i \leq n\}$.

But according to (iii) we infer that there exists an index $i \leq n$ and a set $U \in U_i$ and points $a, b \in U$ such that $a \in A_i$ and $b \in B_i$. We have $f_i(a) \leq 0$ and $f_i(b) \geq 0$. But it is impossible, because in the case when $U = V_i$ we have $f_i(a), f_i(b) < 0$ and in the case when $U = W_i$; $f_i(a), f_i(b) > 0$. □

There is a strict connection between dimension and the Bolzano property. One can prove that for a space X such that $X \times [0, 1]$ is a normal space, $\dim X \leq n$ if and only if $X \notin B_{n+1}$.

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