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A Topology Generated by Eventually Different Functions

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We investigate a topology on the set ω^ω, generated by eventually different functions, and the ideal $\mathcal{I}_\mathcal{E}$ of first category sets in this topology. We study the cardinal coefficients of $\mathcal{I}_\mathcal{E}$ and its relations to other ideals on ω^ω and to the eventually different real forcing.

0. Introduction

In this paper we use standard set theoretical notation. For example, ω denotes the first infinite cardinal number, which we shall identify with the set of natural numbers. We will study a topology on the set ω^ω of functions from natural numbers to natural numbers, generated by sets of the form

$$\{x \in \omega^\omega : s \subset x\} \cap \{x \in \omega^\omega : (\forall f \in F)(\forall i \in \omega)(x(i) \neq f(i))\},$$

where s is a finite sequence of natural numbers and F is a finite set of functions from ω^ω.

We investigate general properties of the topological space $\mathcal{X}_\mathcal{E}$, obtained in this way. We prove that $\mathcal{X}_\mathcal{E}$ is zerodimensional and completely regular, satisfies the Souslin condition, but is not separable. We also show that the Baire category theorem holds in $\mathcal{X}_\mathcal{E}$. This implies that the ideal $\mathcal{I}_\mathcal{E}$ of first category subsets of $\mathcal{X}_\mathcal{E}$ is nontrivial. We present examples of sets from this family and compare it with other ideals on the set ω^ω, such as the ideal $\mathcal{M}$ of first category subsets of $(\omega^\omega, \mathcal{N})$. Here $\mathcal{N}$ denotes the standard Baire topology generated by the sets $[s] = \{x \in \omega^\omega : s \subset x\}$. We prove that these two ideals are orthogonal.

Finally, we calculate the cardinal coefficients of $\mathcal{I}_\mathcal{E}$, and prove several consistency results concerning them. We also show the correspondence between the

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topological space $\mathscr{X}_{\mathscr{E}}$ and the notion of forcing adding eventually different functions.

Our results are parallel to [ŁR], where an analogous topology and ideal generated by dominating functions were investigated. We use several tools developed there.

Notation. Our notation is, in general, derived from [Je]. The quantifiers $\forall^\infty k$ and $\exists^\infty k$ stand for $(\exists n)(\forall k > n)$ and $(\forall n)(\exists k > n)$ respectively. We will distinguish between strict $(\subset)$ and non-strict inclusion $(\subseteq)$. Since all uncountable Polish topological spaces are Borel isomorphic, the term "real numbers" ("reals"), will refer to elements of any such space. The ideal of meager subsets of the real line (Cantor set, Baire space ...) is denoted by $\mathscr{M}$.

If X is a topological space, then $\mathrm{ro}(X)$ denotes the boolean algebra of regular open subsets of X. If $\mathbb{P}$ is a partially ordered set, then we will write $\mathrm{ro}(\mathbb{P})$ for the algebra of regular open subsets of $\mathbb{P}$ equipped with the order topology.

We use several special axioms of set theory, such as the classical Martin's Axiom, denoted by MA, and the Anti-Martin's Axiom, introduced by J. Cichoń and denoted by AMA. The detailed description of AMA can be found in [CW] or [Lu]. This axiom has several different formulations, and we will need the following one.

AMA. *For any Σ^1_2 notion of forcing $(\mathbb{P}, \leq)$ with c.c.c. there exists a sequence $\langle G_\alpha : \alpha \in \omega_1 \rangle$ of filters in $(\mathbb{P}, \leq)$, such that for each dense Σ^1_2 subset D of $\mathbb{P}$ we have* $(\exists \alpha < \omega_1)(\forall \beta > \alpha)(D \cap G_\beta \neq \emptyset)$.

Besides of MA and AMA in our consistency proofs we will use the method of forcing. We think about forcing as taking place over $\mathbf{V}$, the universe of all sets.

The cardinality continuum is denoted by $\mathfrak{c}$.

For functions $f, g \in \omega^\omega$ we consider the relation of eventual dominance. We write $f \leq^* g$, and say that g dominates f, if $(\forall^\infty k \in \omega)(f(k) \leq g(k))$. The dominating number $\mathfrak{d}$ is the minimal size of a family which dominates every function from ω^ω, and the unbounding number $\mathfrak{b}$ is the minimal size of a family of functions, which is not dominated by a single element of ω^ω.

We also use cardinal numbers $\mathfrak{p}$ and $\mathfrak{t}$, connected with combinatorial properties of natural numbers. For the definitions of these cardinals see [vD]. Here we will only need the Bell's theorem ([Be]), saying that $\mathfrak{p}$ is the minimal cardinal κ such that $\mathrm{MA}_\kappa(\sigma - \mathit{centered})$ fails.

For an ideal $\mathscr{I}$ on the set X we consider the following cardinal coefficients:

$$\mathrm{add}(\mathscr{I}) = \min \{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{A} \notin \mathscr{I}\},$$

$$\mathrm{cov}(\mathscr{I}) = \min \{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{A} = X\},$$

$$\mathrm{non}(\mathscr{I}) = \min \{|A| : A \subseteq X \ \& \ A \notin \mathscr{I}\},$$

$$\mathrm{cof}(\mathscr{I}) = \min \{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \ \& \ (\forall A \in \mathscr{I})(\exists B \in \mathscr{A})(A \subseteq B)\}.$$

A family $\mathscr{A}$ having the property from the last definition is called a basis for $\mathscr{I}$. A set $A \subset X$ is a Lusin set for the ideal $\mathscr{I}$, if A is uncountable, but $|K \cap A| \leq \omega$ for any $K \in \mathscr{I}$.

We will also use the Bartoszyński–Miller theorem which gives a combinatorial characterization of cardinal invariants of meager sets (see [Ba], [Mi]).

$$\operatorname{cov}(\mathscr{M}) = \min\{|F| : F \subseteq \omega^\omega \ \& \ (\forall g \in \omega^\omega)(\exists f \in F)(\forall^\infty n)(f(n) \neq g(n))\},$$
$$\operatorname{non}(\mathscr{M}) = \min\{|G| : G \subseteq \omega^\omega \ \& \ (\forall f \in \omega^\omega)(\exists g \in G)(\exists^\infty n)(f(n) \neq g(n))\}.$$

We shall deal with two topologies on ω^ω, namely the standard Baire topology $\mathscr{N}$ and the "eventually different" topology $\mathscr{E}$. The meaning of terms such as open, dense or closed should be clear from the context, but whenever necessary, we will also write $\mathscr{N}$-open or $\mathscr{E}$-open etc. to distinguish between the topologies.

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1. The topological space

In this section we define and investigate a topology on the set ω^ω, generated by eventually different functions.

For $s \in \omega^{<\omega}$ and a finite $F \subset \omega^\omega$ we set

$$U_{s,F} = \{x \in \omega^\omega : s \subseteq x \ \& \ (\forall f \in F)(\forall i \geq \operatorname{lh}(s))(f(i) \neq x(i))\}.$$

Lemma 1.1 *The family* $\{U_{s,F} : (s, F) \in \mathbb{E}\}$ *is a basis of a topology on the set* ω^ω.

Proof. It is enough to notice that if $U_{s,F} \cap U_{t,G} \neq \emptyset$ then we have $U_{s,F} \cap U_{t,G} = U_{s \cup t, F \cup G}$. □

The topology on the set ω^ω generated by the above basis will be denoted by $\mathscr{E}$ and the ideal of meager sets in the topological space $\mathscr{X}_\mathscr{E} = (\omega^\omega, \mathscr{E})$ will be called $\mathscr{I}_\mathscr{E}$. These objects are the main subject of our study.

Theorem 1.2.

(a) *The topology $\mathscr{E}$ is stronger than the Baire topology, and satisfied the Souslin condition.*

(b) *The space $\mathscr{X}_\mathscr{E}$ is zero-dimensional and therefore Tichonov.*

(c) *The Baire category theorem holds in $\mathscr{X}_\mathscr{E}$.*

Proof. (a) It is enough to notice that $U_{s,F} \cap U_{s,G} \neq \emptyset$ for any families $F, G \in [\omega^\omega]^{<\omega}$. So the basis of the topology $\mathscr{E}$ is a countable union of families having the finite intersection property.

(b) Let us take any $s \in \omega^{<\omega}$ and $F \in [\omega^\omega]^{<\omega}$ and let $n = \mathrm{lh}(s)$. We want to show that $U_{s,F}$ is closed in $\mathscr{E}$. We have

$$\omega^\omega \setminus U_{s,F} = \bigcup\{U_{t,\emptyset} : (t \in \omega^m) \;\&\; ((t \nsupseteq s) \vee (s \subset t \;\& \\ (\forall f \in F)(\forall i \in [n, m-1))(t(i) \neq f(i)) \;\& \\ (\exists f \in F)(t(m-1) = f(m-1))))\}$$

This shows that $U_{s,F}$ is closed even in the standard topology.

(c) Let U_{t_0,H_0} be a basic open set and let $\{G_n : n \in \omega\}$ be a sequence of dense open subsets of $(\omega^\omega, \mathscr{E})$. We shall show that $U_{t,H_0} \cap \bigcap_{n\in\omega} G_n \neq \emptyset$. We may assume that $G_n \supseteq G_{n+1}$ for all n. By induction on $n \in \omega$, using the density of G_n, we can define $t_n \in \omega^{<\omega}$, $H_n \in \omega^\omega$ such that $U_{t_{n+1},H_{n+1}} \subseteq G_{n+1} \cap U_{t_n,H_n}$ and $\mathrm{lh}(t_n) \geq n$. Then the function $x = \bigcup_{n\in\omega} t_n$ belongs to $U_{t,H_0} \cap \bigcap_{n\in\omega} G_n$, which is therefore nonempty. □

Part (b) of the above theorem, which says that the basic open sets of $\mathscr{E}$ are $\mathscr{N}$-Borel, may suggest that the topologies $\mathscr{E}$ and $\mathscr{N}$ have the same Borel sets. However, this assumption is false, since we can construct a $\mathscr{E}$-open set, which is not Borel in the classical topology. To do this, we take a family $\mathscr{A}$ of $\mathfrak{c}$ many almost disjoint subsets of ω and define A to be the set of all characteristic functions of elements of $\mathscr{A}$. There are $2^{\mathfrak{c}}$ subsets of A, so we may find a subset $B \subseteq A$, which is not $\mathbf{\Pi}^1_1$. Let c_0, c_1 be the functions constantly equal to zero and one on ω and let

$$G = \bigcup\{U_{\emptyset,\{f,c_0\}} : f \in B\}.$$

Then we can express the set B as

$$B = \{x \in \omega^\omega : x \in G \;\&\; (\forall y \in G)[(\forall n)(y(n) \leq x(n)) \Rightarrow y = x]\}.$$

But B is just a translation of A $(B = A + c_1)$, so it is also not $\mathbf{\Pi}^1_1$, and therefore G cannot be Borel in $\mathscr{N}$.

Now we will consider congruence modulo $\mathscr{I}_{\mathscr{E}}$ instead of equality of sets. Next lemma, although quite simple, has many important consequences.

Lemma 1.3. *For every $\mathscr{E}$-open set U there exists an $\mathscr{E}$-open and $\mathscr{N}$-Borel set $U^* \subseteq U$, such that $U \setminus U^* \in \mathscr{I}_{\mathscr{E}}$.*

Proof. Let $\mathscr{U}$ be a maximal family of disjoint basic open subsets of U. Then $\mathscr{U}$ is countable because of the Souslin condition, hence we may enumerate it as $\mathscr{U} = \langle U_{s_n,F_n} : n \in \omega\rangle$ and define $U^* = \bigcup_{n\in\omega} U_{s_n,F_n}$. Because all basic open sets of $\mathscr{E}$ are $\mathscr{N}$-closed, U^* is an F_σ set in $\mathscr{N}$. The fact that $(U \setminus U^*) \in \mathscr{I}_{\mathscr{E}}$ follows from the maximality of $\mathscr{U}$. □

Corollary 1.4. *Every $\mathscr{E}$-Borel set is congruent to an $\mathscr{N}$-Borel set modulo $\mathscr{I}_{\mathscr{E}}$.*

Proof. By induction on the Borel complexity.

Corollary 1.5. *The ideal $\mathscr{I}_{\mathscr{E}}$ has a basis consisting of $\mathscr{N}$-Borel sets.*

Proof. For a set $K \in \mathscr{I}_{\mathscr{E}}$ there are dense open in $\mathscr{E}$ sets D_n, $n \in \omega$, such that $K \cap \bigcap_{n\in\omega} D_n = \emptyset$. Using lemma 1.3 we contruct sets D_n^*. Then we have

$$K \subseteq (\bigcap_{n\in\omega} D_n^*)^c = \bigcup_{n\in\omega} (D_n^*)^c$$

and the last set is $G_{\delta\sigma}$ in the topology $\mathscr{N}$. □

At the end of this section we will calculate the density of the topology $\mathscr{E}$. It is at least ω_1, so $\mathscr{X}_{\mathscr{E}}$ is not separable. Together with the Souslin condition it imlies that $\mathscr{X}_{\mathscr{E}}$ is also not metrizable (see [En], theorem 4.1.15).

Theorem 1.6. *The size of a minimal dense subset of $\mathscr{X}_{\mathscr{E}}$ is equal to the covering of category.*

Proof. Let X be a dense subset of $\mathscr{X}_{\mathscr{E}}$. Then for any $f \in \omega^\omega$ we have $U_{\emptyset,\{f\}} \cap X \neq \emptyset$, and from the Bartoszyński–Miller theorem we get $\mathrm{cov}(\mathscr{M}) \leq |X|$.

To prove the other inequality, we consider a sequence $\langle F_n \subseteq \omega^\omega : n \in \omega\rangle$, of subsets of ω^ω such that

$$(\forall H \in [\omega^\omega]^{\leq n})(\exists f \in F_n)(\forall h \in H)(\forall^\infty k \in \omega)(f(k) \neq h(k))$$

and each F_n has minimal possible cardinality. For $s \in \omega^{<\omega}$ and $g \in \omega^\omega$ let $s^\frown g$ be the element of ω^ω, obtained by replacing the initial part of g with s. Then the set

$$X = \bigcup_{n\in\omega} \{s^\frown f : s \in \omega^{<\omega} \;\&\; f \in F_n?$$

is dense in $\mathscr{X}_{\mathscr{E}}$ and $|X| = \sup\{|F_n| : n \in \omega\}$. Because the Bartoszyński–Miller theorem says that $\mathrm{cov}(\mathscr{M}) = |F_1|$, it is enough to show that $|F_n| = |F_1|$ for any $n \in \omega$. This was implicitly proved in [Ba], but we show it also here for completenes.

Of course $|F_1| \leq |F_n|$, so we prove only the other inequality. For fixed $n \in \omega$, we take an arbitrary family $G \subseteq \omega^\omega$ such that $|G| < |F_n|$. We will find $x \in \omega^\omega$ such that

$$(\forall g \in G)(\exists^\infty i \in \omega)(x(i) = g(i))$$

and this will give $|G| < |F_1|$.

For $g \in G$ and $k \in \omega$ let $g'(k) = g \restriction \{n(k-1)+1, \ldots, nk\}$ and let $G' = \{g' : g \in G\}$. So g' maps ω into the set $Y = \bigcup_{k\in\omega} \omega^{\{n(k-1)+1,\ldots,nk\}}$, which is countable, and thus has the same combinatorial properties as ω. As $|G'| < |F_n|$, there exists a function $\varphi : \omega \to [Y]^{\leq n}$ such that $(\forall g \in G)(\exists^\infty i \in \omega)(g'(i) \in \varphi(i))$ (notice that here we identify $([Y]^{\leq n})^\omega$ with $[Y^\omega]^{\leq n}$). For any $k \in \omega$ let

$$\varphi(k) = \{h_1^k, h_2^k, \ldots, h_n^k\} \quad \text{where} \quad h_1^k, h_2^k, \ldots, h_n^k \in \omega^{\{n(k-1)+1,\ldots,nk\}}$$

We define the function x putting $x(kn+1) = h_i^k(kn+i)$ for $k \in \omega$ and $i \leq n$. It is easy to check that x has the desired property. □

In this section we would like to present examples of sets from the ideal $\mathscr{I}_{\mathscr{E}}$ and to compare it to several other ideals on the set ω^ω. The first example shows the most typical member of the ideal $\mathscr{I}_{\mathscr{E}}$.

Example 2.1. *For any function $g \in \omega^\omega$ the set*

$$E_g = \{x \in \omega^\omega : (\exists^\infty k \in \omega)(x(k) = g(k))\}$$

belongs to the ideal $\mathscr{I}_{\mathscr{E}}$. Moreover it is nowhere dense in $\mathscr{E}$.

Proof. Let us consider the complement of E_g:

$$D_g = \omega^\omega \setminus E_g = \{x \in \omega^\omega : (\forall^\infty k \in \omega)(x(k) \neq g(k))\} = \bigcup_{s \in \omega^{<\omega}} U_{s,\{g\}}.$$

From the last equality it is easy to see, that D_g is dense open in $\mathscr{E}$. □

Recall that two ideals $\mathscr{I}$ and $\mathscr{J}$ on the set X are called *orthogonal* $(\mathscr{I} \perp \mathscr{J})$ if there are sets $A \in \mathscr{I}$ and $B \in \mathscr{J}$ such that $X = A \cup B$. Then (if $\mathscr{I}$ and $\mathscr{J}$ are proper) we have $B \notin \mathscr{I}$ and $A \notin \mathscr{J}$.

The set $\omega^\omega \setminus E_g$ is meager in the standard topology $\mathscr{N}$, so from example 2.1 it follows that $\mathscr{I}_{\mathscr{E}} \perp \mathscr{M}$.

Let $\mathscr{I}_{\mathscr{E}}^\sigma$ be the ideal on ω^ω generated by those elements of $\mathscr{I}_{\mathscr{E}}$ which are also F_σ sets in the standard topology $\mathscr{N}$. It is clear that we have $\mathscr{I}_{\mathscr{E}}^\sigma \subseteq (\mathscr{I}_{\mathscr{E}} \cap \mathscr{M})$. We will show that this ideal is much smaller. Before giving examples of sets from $\mathscr{I}_{\mathscr{E}}^\sigma$, we prove a technical lemma describing the structure of this ideal.

Lemma 2.2. *A set $A \subseteq \omega^\omega$ belongs to the ideal $\mathscr{I}_{\mathscr{E}}^\sigma$ if and only if there exists a sequence $\langle s_n : n \in \omega \rangle$ of elements of $\omega^{<\omega}$ such that*

(i) $A \cap \bigcap_{m \in \omega} \bigcup_{n>m} [s_n] = \emptyset$

(ii) *for any $t \in \omega^{<\omega}$ and $F \in [\omega^\omega]^{<\omega}$ there are infinitely many $n \in \omega$ such that* $[t \subseteq s_n \ \& \ (\forall i \in \mathrm{dom}(s_n \setminus t))(\forall f \in F)(s_n(i) \neq f(i))]$

Proof. Suppose that $A \in \mathscr{I}_{\mathscr{E}}^\sigma$. Then there is a sequence $\langle D_k : k \in \omega \rangle$ of $\mathscr{N}$-open $\mathscr{E}$-dense sets, such that $A \cap \bigcap_{k \in \omega} D_k = \emptyset$. Each of these sets can be represented as $D_k = \bigcup_{j \in \omega} [t_{j,k}]$, where $\mathrm{lh}(t_{j,k}) \geq k$ and $[t_{j,k}] \cap [t_{i,k}] = \emptyset$ for $i \neq j$. Let $\{s_n : n \in \omega\}$ be an enumeration of the family $\{t_{j,k} : j, k \in \omega\}$. We claim that this is the desired sequence. Since $\bigcap_{m \in \omega} \bigcup_{n>m} [s_n] = \bigcap_{k \in \omega} D_k$, the condition (i) holds. To prove (ii) fix $k \in \omega$, $t \in \omega^{<\omega}$ and $F \in [\omega^\omega]^{<\omega}$. Since D_k is $\mathscr{E}$-dense we find $x \in U_{t,F} \cap D_k$. Since D_k is $\mathscr{N}$-open, we can find j, n such that $x \in [t_{j,k}]$ and $t_{j,k} = s_n$. Then $s_n(i) \neq f(i)$ for all $i \in \mathrm{dom}(s_n \setminus t)$ and $f \in F$. Since $\mathrm{lh}(s_n) \geq k$ and for every k we can find such n, there are infinitely many n's with this property.

To prove the other implication it is enough to notice that the sets $D_m = \bigcup_{n>m} [s_n]$ are $\mathscr{N}$-open and $\mathscr{E}$-dense. □

Example 2.3. *Let $g \in \omega^\omega$ and let a, b be two disjoint subsets of ω, a infinite. Then the set*

$$E_{g,a,b} = \{x \in \omega^\omega : (\exists^\infty k \in a)(x(k) = g(k)) \,\&\, (\forall k \in b)(x(k) \neq g(k))\}$$

belongs to $\mathscr{I}_\mathscr{E} \setminus \mathscr{I}_\mathscr{E}^\sigma$. Moreover, if b is infinite, then $E_{g,a,b} \in \mathscr{M}$.

Proof. $E_{g,a,b}$ belongs to $\mathscr{I}_\mathscr{E}$, since $E_{g,a,b} \subseteq E_g$. Now assume for contradiction that $E_{g,a,b} \in \mathscr{I}_\mathscr{E}^\sigma$. Using lemma 2.2 we find a sequence $\langle s_n : n \in \omega \rangle$ of elements of $\omega^{<\omega}$, satisfying (i) and (ii). Then we construct sequences $\langle t_m : m \in \omega \rangle$ and $\langle t'_m : m \in \omega \rangle$ such that

(a) $t_0 = \emptyset$ and for all $m \in \omega$ we have $t'_m \subseteq t_m \subseteq t'_{m+1}$

(b) there is $i \in \mathrm{dom}(t_m \setminus t'_m) \cap a$ such that $t_m(i) = g(i)$

(c) for all $i \in \mathrm{dom}(t_m \setminus t'_m) \cap b$, $t_m(i) \neq g(i)$

(d) $t'_{m+1} = s_n$, where n is minimal with the property that $t_m \subseteq s_n$ and for all $i \in \mathrm{dom}(s_n \setminus t_m)\,(s_n(i) \neq g(i))$ (using (ii) from lemma 2.2)

Then let $x = \bigcup_{m \in \omega} t_m$. Conditions (b) and (c) imply that $x \in E_{g,a,b}$, and from (d) it follows that $x \in \bigcap_{m \in \omega} \bigcup_{n > m} [s_n]$, but this is a contradiction with (i) from lemma 2.2. □

The ideals $\mathscr{I}_\mathscr{E}$ and $\mathscr{I}_\mathscr{E}^\sigma$, although similarily defined, are quite far from each other. In the last section we will show that their cardinal coefficients can (consistently with ZFC) be different.

Now we want to look at the difference $(\mathscr{I}_\mathscr{E} \cap \mathscr{M}) \setminus \mathscr{I}_\mathscr{E}^\sigma$ and find $\mathfrak{c}$ many disjoint sets there. To do this we take an arbitrary function $g \in \omega^\omega$ and a family $\mathscr{A}$ of almost disjoint subsets of ω with $|\mathscr{A}| = \mathfrak{c}$. Then we consider sets $E_{g,a,\omega \setminus a}$ for $a \in \mathscr{A}$. They are disjoint because of the almost disjointness of $\mathscr{A}$, and the example 2.3 says exactly that all these sets are in $(\mathscr{I}_\mathscr{E} \cap \mathscr{M}) \setminus \mathscr{I}_\mathscr{E}^\sigma$.

Example 2.4. *Let $f \in \omega^\omega$ be unbounded and let $\varphi : \omega \to P(\omega)$ be such that $(\forall n \in \omega)(|\varphi(n)| = f(n))$. Then*

$$A_\varphi = \{x \in \omega^\omega : (\forall^\infty k \in \omega)(x(k) \notin \varphi(k))\} \in \mathscr{I}_\mathscr{E}^\sigma .$$

Proof. Let $D_m = \{x \in \omega^\omega : (\exists k > m)(x(k) \in \varphi(k))\}$. Then $A_\varphi \cap \bigcap_{m \in \omega} D_m = \emptyset$ and each D_m is $\mathscr{N}$-open. To prove that it is also $\mathscr{E}$-dense, fix $t \in \omega^{<\omega}$ and $F \in [\omega^\omega]^{<\omega}$. Since $|\varphi(k)|$ is unbounded, there is $k > \max\{\mathrm{lh}(t), m\}$, such that $|\varphi(k)| > |F|$, so there exists $x \in U_{t,F} \cap D_m$, and D_m is indeed $\mathscr{E}$-dense. □

Example 2.5. *For an unbounded function $f \in \omega^\omega$ we have*

$$B_f = \{x \in \omega^\omega : (\forall^\infty k \in \omega)(x(k) \geq f(k))\} \in \mathscr{I}_\mathscr{E}^\sigma .$$

Proof. It is enough to take $\varphi(k) = f(k)$ in example 2.4. □

Example 2.6. *If A is an infinite subset of ω then*

$$C_A = \left\{f \in \omega^\omega : (\forall n \in \omega)\left(\sum_{k=0}^{n} f(k) \notin A\right)\right\} \in \mathscr{I}_\mathscr{E}^\sigma .$$

Proof. Let $D_n = \{f \in \omega^\omega : \sum_{k=0}^{n} f(k) \in A\}$. Then $\omega^\omega \setminus C_A = \bigcup_{n\in\omega} D_n$ and similarily as in example 2.4 one can show that D_n is an $\mathcal{N}$-open, $\mathcal{E}$-dense subset of ω^ω. □

Now we want to discuss the correspondence between $\mathcal{I}_\mathcal{E}$, $\mathcal{I}_\mathcal{E}^\sigma$ and two analogous ideals arising from the dominating topology. The detailed description of these ideals can be found in [ŁR]. Here we just recall the definitions.

For $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ let

$$U_{s,f} = \{x \in \omega^\omega : s \subset x \ \& \ (\forall k \geq \mathrm{lh}(s))\,(x(k) \geq f(k))\}.$$

The family $\{U_{s,f} : s \in \omega^{<\omega}, f \in \omega^\omega\}$ is a basis for another topology on ω^ω, called the *dominating topology* and denoted by $\mathcal{D}$. The space $(\omega^\omega, \mathcal{D})$ satisfies the Baire category theorem, so we may define $\mathcal{I}_\mathcal{D}$ to be the ideal of meager subsets of this space, and $\mathcal{I}_\mathcal{D}^\sigma$ to be the ideal generated by those elements of $\mathcal{I}_\mathcal{D}$, which are also standard F_σ sets.

Lemma 2.7. $\mathcal{I}_\mathcal{D}^\sigma \subseteq \mathcal{I}_\mathcal{E}^\sigma$.

Proof. Let $K \in \mathcal{I}_\mathcal{D}^\sigma$. We find $\mathcal{N}$-open, $\mathcal{D}$-dense sets D_n such that $K \cap \bigcap_{n\in\omega} D_n = \emptyset$. We want to show that the same sequence D_n witnesses that $K \in \mathcal{I}_\mathcal{E}^\sigma$.

For $F \in [\omega^\omega]^{<\omega}$ we define $g_F(k) = \max\{f(k) : f \in F\} + 1$. It is easy to notice that for any $s \in \omega^{<\omega}$ we have $U_{s,F} \supseteq U_{s,g_F}$, so any set from the basis of $\mathcal{E}$ contains a set from the basis of $\mathcal{D}$. Hence $\mathcal{D}$-density implies $\mathcal{E}$-density. Thus we have shown that all the sets D_n for $n \in \omega$ are indeed $\mathcal{E}$-dense. □

We have shown the inclusions $\mathcal{I}_\mathcal{D}^\sigma \subseteq \mathcal{I}_\mathcal{E}^\sigma \subseteq \mathcal{I}_\mathcal{E}$ and $\mathcal{I}_\mathcal{D}^\sigma \subseteq \mathcal{I}_\mathcal{D}$. In fact no other inclusion between the four ideals holds. Recall first that in example 2.3 we proved that $\mathcal{I}_\mathcal{E} \setminus \mathcal{I}_\mathcal{E}^\sigma \neq \emptyset$. The corresponding fact that $\mathcal{I}_\mathcal{D} \setminus \mathcal{I}_\mathcal{D}^\sigma \neq \emptyset$ was shown in [ŁR].

We fix an unbounded function $f \in \omega^\omega$. Then the set B_f, considered in example 2.5 belongs to $\mathcal{I}_\mathcal{E}^\sigma$. On the other hand let $C_f = \omega^\omega \setminus B_f$. In [ŁR] it was proved that $C_f \in \mathcal{I}_\mathcal{D}$ (compare this with the example 2.1). Thus the pair (B_f, C_f) witnesses that $\mathcal{I}_\mathcal{E}^\sigma \perp \mathcal{I}_\mathcal{D}$, which implies also $\mathcal{I}_\mathcal{E} \perp \mathcal{I}_\mathcal{D}$.

Finally, the same pair can be used to show that the inclusion $\mathcal{I}_\mathcal{D}^\sigma \subseteq \mathcal{I}_\mathcal{E}^\sigma$ is strict. Namely $B_f \in (\mathcal{I}_\mathcal{E}^\sigma \setminus \mathcal{I}_\mathcal{D}^\sigma)$, since if $B_f \in \mathcal{I}_\mathcal{D}^\sigma$, then $\omega^\omega = B_f \cup C_f \in \mathcal{I}_\mathcal{D}$.

3. Connections between ideal and forcing

In this section we present connections between the topological space and the theory of forcing. Let $\mathbb{E}$ denote the eventually different real forcing. In this section we are going to show that there is a correspondence between the forcing $\mathbb{E}$ and the ideal $\mathcal{I}_\mathcal{E}$, which is similar to the one between Cohen forcing and the classical ideal of first category sets on the real line.

Let us recall definition of the forcing $\mathbb{E}$ ([Mi]).

$$\mathbb{E} = \{(s,F) : s \in \omega^{<\omega} \ \& \ F \subseteq [\omega^\omega]^{<\omega}\}$$

$$(s, F) \leq (t, G) \text{ iff } t \subseteq s \ \& \ G \subseteq F \ \& \ (\forall i \in \mathrm{dom}(s \setminus t))\,(\forall g \in G)\,(s(i) \neq g(i)) .$$

If $G \subseteq \mathbb{E}$ is generic over V, then we define the canonical $\mathbb{E}$-generic real $e_G \in \omega^\omega$ putting

$$e_G = \bigcup\{s \in \omega^{<\omega} : (\exists F \in [\omega^\omega]^{<\omega})\,((s, F) \in G)\} .$$

We say that $x \in \omega^\omega$ is $\mathbb{E}$-generic if $x = e_G$ for some G.

The definition of basic open set $U_{s,F}$ can be now expressed in terms of forcing. Namely, if we denote the canonical name for the $\mathbb{E}$-generic real by $\dot{e}$ then

$$(s, F) \Vdash \text{“}\dot{e} \in U_{s,F}\text{”} ,$$

and this is all that we know about $\dot{e}$ so far.

Theorem 3.1. *The boolean algebra of regular open subsets of* $\mathbb{E}$ *is isomorphic to the quotient* $\mathcal{N}$-Borel/$\mathcal{I}_{\mathcal{E}}$, *of the algebra of standard Borel sets modulo the ideal* $\mathcal{I}_{\mathcal{E}}$.

Proof. We will prove it in via a chain of simple isomorphism theorems. Namely

$$\mathrm{ro}(\mathbb{E}) = \mathrm{ro}(\mathcal{X}_{\mathcal{E}}) = \mathcal{E}\text{-Baire}/\mathcal{I}_{\mathcal{E}} = \mathcal{E}\text{-Borel}/\mathcal{I}_{\mathcal{E}} = \mathcal{N}\text{-Borel}/\mathcal{I}_{\mathcal{E}} .$$

For the first equality let us consider the mapping $\psi : \mathbb{E} \to \mathcal{E}$, defined by $\psi((s, F)) = U_{s,F}$. The function ψ is of course an isomorphism of the forcing $\mathbb{E}$ with the basis of $\mathcal{E}$ ordered by inclusion. The set $\mathbb{E}$ is dense in $\mathrm{ro}(\mathbb{E})$ and basic open sets are dense in $\mathrm{ro}(\mathcal{X}_{\mathcal{E}})$, and two boolean algebras with isomorphic dense subsets are isomorphic too.

The second and third equality holds in any space satisfying the Baire category theorem which was proved for $\mathcal{X}_{\mathcal{E}}$ in theorem 1.2 (c).

The last follows from the corollary 1.4. □

Now we want to present a characterization of those elements of ω^ω, which can be generically added by the forcing $\mathbb{E}$ in terms of the ideal $\mathcal{I}_{\mathcal{E}}$. This idea is classical and stems from the Solovay's characterization of random and Cohen reals. Similar construction, as presented below, may be carried out for many c.c.c. forcing notions adding reals. The most important thing here is a suitable coding of the forcing by some real numbers.

Let us fix some effective enumeration $\{t_m : m \in \omega\}$ of $\omega^{<\omega}$. We say that a real $c \in {}^{\omega\times\omega}\omega$ codes the following $\mathcal{E}$-open set

$$U_c = \bigcup\{U_{t_m, \{f_1^m, \ldots, f_n^m\}} : c(m, 0) > 0\}$$

$$\text{where } n = c(m,0) - 1 \text{ and } f_k^m(i) = c(m, (k-1)\,n + i + 1) \text{ for } m, k \in \omega .$$

Recall also the standard coding of Borel sets (see [Je]). The set BC of all Borel codes is a $\mathbf{\Pi}_1^1$ subset of ω^ω, and for every real $c \in BC$, the set B_c is assigned so that the relations $B_c \subseteq B_d$, $B_c = \emptyset$, $B_d = B_c \cap B_d$, $B_d = B_c \setminus B_b$, $B_d = \bigcup_{n\in\omega} B_{c_n}$, etc., are all $\mathbf{\Pi}^1_1$. One can easily describe a Borel function $F: {}^{\omega\times\omega}\omega \to BC$ such that $U_c = B_{F(c)}$ for all $c \in {}^{\omega\times\omega}\omega$.

Lemma 3.2. *The predicate "$c \in {}^{\omega\times\omega}\omega$ codes an open dense subset of $(\omega^\omega, \mathscr{E})$ which is the union of a disjoint family of basic sets" is $\mathbf{\Pi}^1_1$.*

Proof. The predicate "U_c is $\mathscr{E}$-dense" may be described as follows.

$$\begin{aligned}(\forall n)(\forall f \in \omega^\omega)(\exists m)(\exists k)\Big(t_k = t_m \cup t_n \ \&\ c(m,0) > 0\ \& \\ \&\ (\forall i \in \mathrm{dom}(t_k \setminus t_m))(\forall 0 < j < c(m,0)) \\ (t_k(i) \neq c(m,(j-1)(c(m,0)-1)) + i + 1)\ \& \\ \&\ (\forall i \in \mathrm{dom}(t_k \setminus t_n))(\forall 0 < j < f(0)) \\ (t_k(i) \neq f((j-1)(f(0)-1)) + i + 1)\Big).\end{aligned}$$

Note that the function f in the above formula encodes a set from $[\omega^\omega]^{<\omega}$, similarily as a single "row" of $c \in {}^{\omega\times\omega}\omega$ does. Since the part in big brackets is arithmetical, the whole formula is indeed $\mathbf{\Pi}^1_1$.

The fact whether the intersection of two basic sets depends only on the finite sequences determining them. Thus the predicate "U_c is the union of a disjoint family of basic sets" is arithmetical, and the whole predicate stated in the lemma is a $\mathbf{\Pi}^1_1$ predicate. □

Lemma 3.3. *The predicate $B_a \in \mathscr{I}_{\mathscr{E}}$ is $\mathbf{\Sigma}^1_2$.*

Proof. Let us observe that $B_a \in \mathscr{I}_{\mathscr{E}}$ if and only if

$$(\exists c \in {}^{\omega}({}^{\omega\times\omega}\omega))\Big(B_a \cap \bigcap_{n\in\omega} U_{c(n)} = \emptyset\ \&\ (\forall n)(U_{c(n)} \text{ is open dense in } \mathscr{E})\Big)$$

or equivalently

$$(\exists c \in {}^{\omega}({}^{\omega\times\omega}\omega))\Big(B_a \cap \bigcap_{n\in\omega} B_{F(c(n))} = \emptyset\ \&\ (\forall n)(U_{c(n)} \text{ is open dense in } \mathscr{E})\Big).$$

From this characterization and the previous lemma we obtain the required result. □

Theorem 3.4. *Let $x \in \omega^\omega$. The following conditions are equivalent.*

(i) *x is an $\mathbb{E}$-generic real over $\mathbf{V}$;*

(ii) *for each $c \in {}^{\omega\times\omega}\omega \cap \mathbf{V}$, if U_c is open dense in $(\omega^\omega, \mathscr{E})$ and if U_c is a disjoint union of basic sets then $x \in U_c$;*

(iii) *for each $a \in BC \cap \mathbf{V}$, if $B_a \in \mathscr{I}_{\mathscr{E}}$, then $x \notin B_a$.*

Proof. The equivalence (i) $\leftrightarrow$ (ii) follows from the correspondence between maximal antichains in $\mathbb{E}$ from $\mathbf{V}$, and $c \in {}^{\omega\times\omega}\omega \cap \mathbf{V}$ for which U_c satisfies all the conditions in (ii).

The implication (iii) → (ii) is obvious, and we prove only (i) → (iii).

Let x be an $\mathbb{E}$-generic real, let $a \in BC \cap \mathbf{V}$ be such that $B_a \in \mathscr{I}_{\mathscr{E}}$. There is a sequence $\{c_n : n \in \omega\} \subseteq {}^{\omega \times \omega}\omega$, such that $B_a \cap \bigcap_{n \in \omega} B_{F(c_n)} = \emptyset$ and U_{c_n} are open dense in $\mathscr{I}_{\mathscr{E}}$. But Σ^1_2 properties are absolute, so we can find the sequence $\{c_n : n \in \omega\}$ with the same property in $\mathbf{V}$ and hence $x \in U_{c_n}$ for all n which means that $x \notin B_a$. □

4. Cardinal coefficients

Let us consider the function $\varphi : \omega^\omega \to 2^\omega$ defined by $\varphi(x)(k) = x(k) \bmod 2$, for $x \in \omega^\omega$. If the Cantor set 2^ω is equipped with the standard topology, and the set ω^ω with any of the topologies $\mathscr{N}$ or $\mathscr{E}$, then one can easily see that the mapping φ is continuous and open. Hence we can apply for both topologies on ω^ω the following obvious lemma.

Lemma 4.1. *If a mapping $\varphi : X \to Y$ is continuous and open then $\varphi^{-1}(U)$ is an open dense subset of X whenever U is open dense subset of Y.*

Theorem 4.2. $\mathrm{cov}(\mathscr{I}_{\mathscr{E}}^\sigma) = \mathrm{cov}(\mathscr{M})$, $\mathrm{non}(\mathscr{I}_{\mathscr{E}}^\sigma) = \mathrm{non}(\mathscr{M})$.

Proof. From the previous lemma we deduce that for any open dense set $U \subseteq 2^\omega$, $\varphi^{-1}(U)$ is $\mathscr{N}$-open and $\mathscr{E}$-dense subset of ω^ω.

Let $\langle B_\alpha : \alpha < \mathrm{cov}(\mathscr{M})\rangle$ be a covering of the Cantor set with closed nowhere dense sets. Then $\langle \omega^\omega \setminus \varphi^{-1}(2^\omega \setminus B_\alpha) : \alpha < \mathrm{cov}(\mathscr{M})\rangle$ is a covering of ω^ω with the sets from $\mathscr{I}_{\mathscr{E}}^\sigma$ and we have $\mathrm{cov}(\mathscr{I}_{\mathscr{E}}^\sigma) \le \mathrm{cov}(\mathscr{M})$.

Take a set $X \subseteq \omega^\omega$ with $|X| < \mathrm{non}(\mathscr{M})$. Then $|\varphi(X)| < \mathrm{non}(\mathscr{M})$, so we can find a sequence $\langle D_n : n \in \omega\rangle$ of dense open subsets of the Cantor set such that $\varphi(X) \cap \bigcap_{n \in \omega} D_n = \emptyset$. Then also $X \cap \bigcap_{n \in \omega} \varphi^{-1}(D_n) = \emptyset$, so $X \in \mathscr{I}_{\mathscr{E}}^\sigma$, which shows that $\mathrm{non}(\mathscr{I}_{\mathscr{E}}^\sigma) \ge \mathrm{non}(\mathscr{M})$.

The reverse inequalities $\mathrm{cov}(\mathscr{I}_{\mathscr{E}}^\sigma) \ge \mathrm{cov}(\mathscr{M})$, $\mathrm{nov}(\mathscr{I}_{\mathscr{E}}^\sigma) \le \mathrm{non}(\mathscr{M})$ follow immediately from the inclusion $\mathscr{I}_{\mathscr{E}}^\sigma \subseteq \mathscr{M}$. □

Corollary 4.3. $\mathrm{cov}(\mathscr{I}_{\mathscr{E}}) \le \mathrm{cov}(\mathscr{M})$, $\mathrm{non}(\mathscr{I}_{\mathscr{E}}) \ge \mathrm{non}(\mathscr{M})$.

Proof. This is immediate from the inclusion $\mathscr{I}_{\mathscr{E}}^\sigma \subseteq \mathscr{I}_{\mathscr{E}}$. □

Lemma 4.4. $\mathrm{non}(\mathscr{I}_{\mathscr{E}}) \ge \mathrm{cov}(\mathscr{M})$, $\mathrm{cov}(\mathscr{I}_{\mathscr{E}}) \le \mathrm{non}(\mathscr{M})$.

Proof. We can rewrite the Bratoszyński–Miller theorem using the sets E_g in the following way

$$\mathrm{cov}(\mathscr{M}) = \min\{|F| : F \subseteq \omega^\omega \;\&\; \neg(\exists g \in \omega^\omega)(F \subseteq E_g)\}$$

$$\mathrm{non}(\mathscr{M}) = \min\{|G| : G \subseteq \omega^\omega \;\&\; \bigcup_{g \in G} E_g = \omega^\omega\}.$$

To prove the first inequality take $F \subseteq \omega^\omega$, $F \notin \mathscr{I}_{\mathscr{E}}$, of the minimal size. Then $F \nsubseteq E_g \in \mathscr{I}_{\mathscr{E}}$ for any $g \in \omega^\omega$, and therefore $\mathrm{cov}(\mathscr{M}) \le |F| = \mathrm{non}(\mathscr{I}_{\mathscr{E}})$.

The second inequality is also clear. If a family G satifies the requirements from the definition of non($\mathcal{M}$), it immediately yields an $\mathcal{I}_{\mathcal{E}}$-covering of ω^ω of the same size. □

Now we have an upper bound for cov($\mathcal{I}_{\mathcal{E}}$) and a lower bound for non($\mathcal{I}_{\mathcal{E}}$). Applying Martin's Axiom to the forcing $\mathbb{E}$ we easily see that cov$(\mathcal{I}_{\mathcal{E}}) = \mathfrak{c}$ under MA. Since $\mathbb{E}$ is σ-centered, this fact can be translated to the inequality $\mathfrak{p} \leq$ cov($\mathcal{I}_{\mathcal{E}}$).

We do not know any reasonable upper bound for non($\mathcal{I}_{\mathcal{E}}$), but it is consistent that non$(\mathcal{I}_{\mathcal{E}}) < \mathfrak{c}$. To see this we shall apply the Anti-Martin's Axiom. But we need some additional definitions.

Let $G \subseteq \mathbb{E}$ be a filter. We say that G is ω^ω-generic if for any $n \in \omega$.

$$G \cap \{(s, F) \in \mathbb{E} : \mathrm{lh}(s) \geq n\} \neq \emptyset .$$

Then we may define a function $e_G \in \omega^\omega$, arising from the filter G, namely

$$e_G = \bigcup \{s \in \omega^{<\omega} : (\exists F \in [\omega^\omega]^{<\omega})((s, F) \in G)\} .$$

Lemma 4.5. *Let $D \subseteq \omega^\omega$ be $\mathcal{E}$ open dense. There exists a $\boldsymbol{\Sigma}^1_2$ dense set $D^* \subseteq \mathbb{E}$ such that $G \cap D^* \neq \emptyset \Rightarrow e_G \in D$ for any ω^ω-generic filter $G \subseteq \mathbb{E}$.*

Proof. We may suppose without loss that D is a sum of a disjoint sequence of basic open sets U_{s_n, F_n} (use lemma 1.3 otherwise). Define $D^* = \{(t, H) \in \mathbb{E} : U_{t,H} \subseteq D\}$. So we have

$$(t, H) \in D^* \Leftrightarrow (\forall x \in \omega^\omega)(x \in U_{t,H} \Rightarrow (\exists n \in \omega)(x \in U_{s_n, F_n})) .$$

Since the predicate $x \in U_{t,H}$ is $\mathcal{N}$-Borel, the set D^* is even a $\boldsymbol{\Pi}^1_1$ set. Other required properties of D^* are clear. □

Theorem 4.6. *Assuming* AMA *we have* non$(\mathcal{I}_{\mathcal{E}}) = \omega_1$.

Proof. We want to apply AMA to the forcing $\mathbb{E}$. We leave it to the reader to verify that this is a $\boldsymbol{\Sigma}^1_2$ notion of forcing. As we know, $\mathbb{E}$ satisfies the c.c.c. condition, so we may use AMA and obtain an AMA-sequence $\langle G_\alpha : \alpha < \omega_1 \rangle$. We can suppose that all G_α's are ω^ω-generic, since otherwise we just skip some initial part of the sequence.

We will show that $X = \{e_{G_\alpha} : \alpha < \omega_1\}$ is a Lusin set for the ideal $\mathcal{I}_{\mathcal{E}}$. Then in particular $X \notin \mathcal{I}_{\mathcal{E}}$.

Take a set $K \in \mathcal{I}_{\mathcal{E}}$ and find a sequence $\langle D_n : n \in \omega \rangle$ of dense open subsets of $(\omega^\omega, \mathcal{E})$, such that $K \cap \bigcap_{n \in \omega} U_n = \emptyset$. Use the lemma 4.5 to find $\boldsymbol{\Sigma}^1_2$ dense subsets D^*_n of $\mathbb{E}$.

Using AMA for each $n \in \omega$ we can find α_n such that $(D^*_n \cap G_\beta \neq \emptyset)$ for any $\beta > \alpha_n$. Let $\alpha = \bigcup_{n \in \omega} \alpha_n$. Then lemma 4.5 implies that

$$(\forall \beta > \alpha)(\forall n \in \omega)(e_{G_\beta} \in D_n)$$

so $(K \cap X) \subseteq \{e_{G_\beta} : \beta \leq \alpha\}$ and the last set is countable. □

It is known (see [CW] or [Lu] that AMA + $(\mathfrak{c} = \omega_2)$ is consistent with ZFC. We have therefore proved that $\text{non}(\mathcal{I}_{\mathcal{E}})$ may consistently be equal to ω_1 and $\mathfrak{c}$ while the continuum is ω_2. The same applies to $\text{cov}(\mathcal{I}_{\mathcal{E}})$. However we still do not know if these two cardinals can be expressed in terms of other, better known cardinal invariants.

If we consider only those from the Cichoń's diagram (see [Fr]), then our upper bound for $\text{cov}(\mathcal{I}_{\mathcal{E}})$ and lower bound for $\text{non}(\mathcal{I}_{\mathcal{E}})$ seem the best possible, since both inequalities $\mathfrak{b} < \text{cov}(\mathcal{I}_{\mathcal{E}})$ and $\text{non}(\mathcal{I}_{\mathcal{E}}) < \mathfrak{d}$ are consistent. The first holds in the model $\mathbf{V}[G_{\omega_2}]$, where $\mathbf{V} \models \text{CH}$ and G_{ω_2} is a generic over the ω_2 finite support iteration of $\mathbb{E}$. The second in true in $\mathbf{V}[G_{\omega_1}]$, where $\mathbf{V} \models \text{MA}$ and G_{ω_1} is a generic over the ω_1 finite support iteration of $\mathbb{E}$ (see [Mi]). So we can formulate.

Question. *Are the following inequalities consistent with ZFC?*
(a) $\text{cov}(\mathcal{I}_{\mathcal{E}}) < \min\{\text{non}(\mathcal{M}), \text{cov}(\mathcal{M})\}$
(b) $\text{non}(\mathcal{I}_{\mathcal{E}}) > \max\{\text{non}(\mathcal{M}), \text{cov}(\mathcal{M})\}$

Theorem 4.7. *(J. Brendle)* $\text{add}(\mathcal{I}_{\mathcal{E}}) = \omega_1$ *and* $\text{cof}(\mathcal{I}_{\mathcal{E}}) = \mathfrak{c}$.

Proof. We work in the topology $\mathcal{E}$. Let us take a family $\{f_\alpha : \alpha < \mathfrak{c}\}$ of eventually different functions (i.e. $\alpha \neq \beta \Rightarrow (\forall^\infty k)(f_\alpha(k) \neq f_\beta(k))$). We define

$$E_\alpha = \{x \in \omega^\omega : (\exists^\infty k \in \omega)(x(k) = f_\alpha(k))\}.$$

From example 2.1 we know that E_α is closed nowhere dense, for any $\alpha < \mathfrak{c}$. We claim that for any $K \in \mathcal{I}_{\mathcal{E}}$ the set $\{\alpha : E_\alpha \subset K\}$ is countable. This, together with corollary 1.5, which implies that $\text{cof}(\mathcal{I}_{\mathcal{E}}) \leq \mathfrak{c}$, will finish the proof.

Suppose hence that $K \in \mathcal{I}_{\mathcal{E}}$ and let us fix a sequence $\langle G_n : n \in \omega \rangle$ of dense open sets, such that $K \cap \bigcap_{n\in\omega} G_n = \emptyset$. We may assume that each G_n has the form

$$G_n = \bigcup_{m\in\omega} U_{s^n_m, F^n_m}.$$

For a finite set $A \subset \omega \times \omega$ we define $F_A = \bigcup\{F^n_m : (n, m) \in A\}$ and we say that F_A *covers* a function $x \in \omega^\omega$ if $(\forall^\infty j)(\exists f \in F_A)(x(j) = f(j))$. Notice that because of the almost disjointness of the family $\{f_\alpha : \alpha < 2^\omega\}$ each F_A may cover only finitely many its elements. So there are only countably many functions f_α, which are covered by F_A for some set $A \in (\omega \times \omega)^{<\omega}$.

From now on we fix α, so that the function f_α is not covered. We have to show that $E_\alpha \cap \bigcap_{n\in\omega} G_n \neq \emptyset$. We construct a sequence $\langle t_i : i \in \omega \rangle$, which approiximates a function $x = \bigcup_{i\in\omega} t_i$, such that $x \in E_\alpha \cap \bigcap_{n\in\omega} G_n$. Along the construction we want to preserve the following conditions:
(a) $t_i \subset t_{i+1}$
(b) $\text{dom}(t_i) \geq i$
(c) $(\exists k \in \text{dom}(t_{i+1} \setminus t_i))(t_{i+1}(k) = f_\alpha(k))$
(d) $(\forall j < i)(\exists m(j))(s^j_{m(j)} \subset t_i \ \& \ (\forall k \in \text{dom}(t_i \setminus s^j_{m(j)})(\forall g \in F^j_{m(j)})(t_i(k) \neq g(k))$

We start with $t_0 = F_0 = \emptyset$. Assume we have already constructed t_i correctly and we want to construct t_{i+1}. Set $A = \bigcup_{j=0}^{i} \{j\} \times m(j)$ so that

$$F_A = F^0_{m(0)} \cup F^1_{m(1)} \cup \ldots \cup F^i_{m(i)} .$$

We know that f_α is not covered by F_A, so

$$(\exists^\infty k \in \omega)\,(\forall g \in F_A)\,(g(k) \neq f_\alpha(k)) .$$

In particular

$$(\exists k > \mathrm{lh}(t_i))\,(\forall g \in F_A)\,(g(k) \neq f_\alpha(k)) .$$

We define t'_i putting

$\mathrm{dom}(t'_i) = k + 1$

$t'_i(k) = f_\alpha(k)$

$t'_i(l) =$ any j such that $j \notin \{g(l) : g \in F_A\}$ for all $l \in \mathrm{dom}(t'_n \setminus t_i)$.

Now the conditions (a), (b) and (c) are already satisfied. To satisfy also (d) we just have to notice that $U_{t'_i, F_A} \cap G_{i+1} \neq \emptyset$. This means that

$$(\exists m(i+1))\,(U_{t'_i, F_A} \cap U_{s^{i+1}_{m(i+1)}, F^{i+1}_{m(i+1)}} \neq \emptyset)$$

and we may take $t_{i+1} = t'_i \cup s^{i+1}_{m(i+1)}$. Now it is quite clear that $x \in E_\alpha \cap \bigcap_{n \in \omega} G_n$, so the proof is complete. □

From example it follows that the sets E_α used in the above proof are elements of $\mathscr{I}_{\mathscr{E}} \setminus \mathscr{I}^\sigma_{\mathscr{E}}$. So the last theorem says nothing about the additivity and cofinality of the ideal $\mathscr{I}^\sigma_{\mathscr{E}}$. We are going to investigate these two cardinals now.

In order to do this, we define a notion of forcing, called the amoeba for $\mathbb{E}$ and denoted by $\mathbb{A}\mathbb{E}$. The purpose is to add generically a set $K \in \mathscr{I}^\sigma_{\mathscr{E}}$, which covers every old member of the ideal $\mathscr{I}^\sigma_{\mathscr{E}}$.

$$\mathbb{A}\mathbb{E} = \{(\sigma, D) : \sigma \in (\omega^{<\omega})^{<\omega} \;\&\; D \subseteq \omega^\omega \;\&\; D \text{ is open and } \mathscr{E}\text{-dense}\} .$$

The ordering on $\mathbb{A}\mathbb{E}$ is rather natural:

$$(\sigma, D) \leq (\rho, C) \Leftrightarrow \rho \subseteq \sigma \;\&\; D \supseteq C \;\&\; (\forall i \in \mathrm{dom}(\sigma \setminus \rho))\,([\sigma(i)] \subseteq C) .$$

If $G \subseteq \mathbb{A}\mathbb{E}$ is a generic filter, then using a density argument we may define a generic function $g : \omega \to \omega^{<\omega}$ and a generic set $A_G \subseteq \omega^\omega$ putting

$$g = \bigcup \{s : (\exists D \subseteq \omega^\omega)\,((s, D) \in G)\}$$

$$A_G = \bigcap_{m \in \omega} \bigcup_{n > m} [g(n)]$$

$$K_G = \omega^\omega \setminus A_G .$$

Again a standard density argument shows that if $D \subseteq \omega^\omega$ is dense and open in $\mathscr{E}$ and encoded in $\mathbf{V}$, then we have $\mathbf{V}[G] \models A_G \subseteq D$. So the notion of forcing $\mathbb{A}\mathbb{E}$ satisfies most of our requirements. The only problem is to show that $K_G \in \mathscr{I}^\sigma_{\mathscr{E}}$.

Lemma 4.8. *Suppose that $t \in \omega^{<\omega}$, $k \in \omega$ and suppose that A is an open $\mathscr{E}$-dense subset of ω^ω. Then we can find $l \in \omega$ and a sequence $\langle t_i : i \in l\rangle$ of elements of $\omega^{<\omega}$, such that*

(i) $t \subseteq t_i$ *and* $[t_i] \subseteq A$ *for any* $i \in l$;

(ii) $(\forall F \in [\omega^\omega]^{\leq k}) (\exists i \in l) (\forall j \in \mathrm{dom}(t_i \setminus t)) (\forall f \in F) (t_i(j) \neq f(j))$.

Proof. For $s \in \omega^{<\omega}$ we put

$$O_s = \{F \in [\omega^\omega]^{\leq k} : (\forall j \in \mathrm{dom}(s \setminus t)) (\forall f \in F) (s(j) \neq f(j))\}.$$

Let us consider the cofinite topology on the set ω of natural numbers. This gives rise to the product topologies on the set ω^ω and then on the set $[\omega^\omega]^{\leq k}$. Since the cofinte topology is compact, the Tichonov theorem implies that both product topologies are compact as well.

For any $s \in \omega^{<\omega}$ the set O_s is a typical basic open set for the product topology.

We fix a $\mathscr{E}$-dense set $A \subseteq \omega^\omega$ and $t \in \omega^{<\omega}$. For any $F \in [\omega^\omega]^{\leq k}$ we have $U_{t,F} \cap A \neq \emptyset$, so we can find $s \in \omega^{<\omega}$ satisfying (i), such that $F \in O_s$. It follows that the family

$$\{O_s : s \in Seq \ \& \ [s] \subseteq A \ \& \ t \subseteq s\}.$$

is a covering of $[\omega^\omega]^{\leq k}$ with basic open sets of the product-cofinite topology. Using compactness we may find a finite subcovering and this completes the proof. □

Lemma 4.9. *Let $\dot{g}$ be the canonical name for an $\mathbb{AE}$-generic function. Then for any $m \in \omega$*

$$\Vdash_{\mathbb{AE}} \bigcup_{n>m} [\dot{g}(n)] \ \textit{is dense in the topology } \mathscr{E}.$$

Proof. We work in **V**. Suppose to the contrary, that there are $t \in \omega^{<\omega}$, $k, m \in \omega$, a name $\dot{H}$ for an element of $(\omega^\omega)^k$ and a condition $(\sigma, A) \in \mathbb{AE}$ satisfying

$$(\sigma, A) \Vdash \bigcup_{n>m} [\dot{g}(n)] \cap U_{t,\dot{H}} = \emptyset.$$

Without loss we may assume that $\mathrm{lh}(\sigma) > m$. Applying the previous lemma for t, k, A we get a sequence $\langle t_i : t \in l\rangle$. Then putting $\sigma' = \sigma^\wedge\langle t_0, \ldots, t_{l-1}\rangle$ we have $(\sigma', A) \leq (\sigma, A)$. Let $N = \max\{\mathrm{lh}(t_i) : i \in l\}$. We find $p \leq (\sigma', A)$ which decides N values of every function from $\dot{H}$. Namely, for every $r \in k$ let $f_r \in \omega^\omega$ be such that

$$p \Vdash \dot{h}_r \restriction N = f_r \restriction N.$$

But then the condition (ii) of the lemma gives us that

$$p \Vdash (\exists i \in l) (\forall j \in \mathrm{dom}(t_i \setminus t)) (\forall r \in k) (t_i(j) \neq \dot{h}_r(j)).$$

So $p \Vdash U_{t,\dot{H}} \cap [\dot{g}(\mathrm{lh}(\sigma) + i)] = U_{t,\dot{H}} \cap [t_i] \neq \emptyset$, and since $\mathrm{lh}(\sigma) + i > m$, this is a contradiction. □

Theorem 4.10. add $(\mathcal{I}_{\mathcal{E}}^{\sigma}) \leq \mathfrak{b}$ *and* $\mathrm{cof}(\mathcal{I}_{\mathcal{E}}^{\sigma}) \geq \mathfrak{d}$.

Proof. We use the conection between partial orderings $(\mathcal{M}(2^{\omega}), \subseteq)$ and $(\omega^{\omega}, \leq^*)$, established by Miller. He constructed mappings

$$\alpha : \mathcal{M} \mapsto \omega^{\omega} \quad \text{and} \quad \beta : \omega^{\omega} \mapsto \mathcal{M}\,,$$

such that for any $g \in \omega^{\omega}$ and $K \in \mathcal{M}$

$$\text{if} \quad \beta(g) \subseteq K \quad \text{then} \quad g \leq^* \alpha(K)\,.$$

The construction and proof of the described property can be found in [Fr] or [Mi]. Here we will only need the fact that $\beta(g) = \{x \in 2^{\omega} : (\forall n \in \omega)\,(x(g(n)) = 0)\}$.

We want to construct a similar two mappings $\tilde{\alpha}$ and $\tilde{\beta}$ between the partially ordered sets $(\mathcal{M}(2^{\omega}), \subseteq)$ and $(\omega^{\omega}, \leq^*)$. For $f \in \omega^{\omega}$ let $\gamma(f)$ be the characteristic function of the set $A_f = \{\sum_{k=0}^{n} f(k) : n \in \omega\}$. Then γ is a homeomorphism between the set $(\omega^{\omega}, \mathcal{N})$ and the set $2^{\omega} \backslash \{x \in 2^{\omega} : |x^{-1}(1)| < \omega\}$ with a topology inherited from the Cantor set. So γ maps meager sets onto meager sets and in particular if $K \in \mathcal{I}_{\mathcal{E}}^{\sigma}$ then $\gamma[K] \in \mathcal{M}$.

We can therefore define $\tilde{\alpha}(K) = \alpha(\gamma[K])$ and $\tilde{\beta}(g) = \gamma^{-1}[\beta(g)]$. Since γ obviously preserves inclusion of sets, the fact that if $\tilde{\beta}(g) \subseteq K$ then $g \leq^* \tilde{\alpha}(K)$ follows from the properties of α and β. So we just need to check that $\tilde{\beta}(g) \in \mathcal{I}_{\mathcal{E}}^{\sigma}$ for any $g \in \omega^{\omega}$. But we have

$$\tilde{\beta}(g) = \{f \in \omega^{\omega} : \gamma(f) \in \beta(g)\} = \{f \in \omega^{\omega} : (\forall n \in \omega)\,(\gamma(f)\,(g(n)) = 0)\} =$$
$$= \left\{ f \in \omega^{\omega} : (\forall n \in \omega) \left(\sum_{k=0}^{n} f(k) \notin \mathrm{rng}(g) \right) \right\}$$

and the fact that $\tilde{\beta}(g) \in \mathcal{I}_{\mathcal{E}}^{\sigma}$ follows from example 2.6.

Using the above mapping we can see that if $F \subseteq \omega^{\omega}$ is an unbounded family, then $\{\tilde{\beta}(f) : f \in F\}$ is a nonadditive subfamily of $\mathcal{M}$, and similarily that if $\mathcal{B}$ is a basis of the ideal $\mathcal{M}$, then $\{\tilde{\alpha}(B) : B \in \mathcal{B}\}$ is a dominating family. This finishes the proof. □

Next proposition follows from the equalities (due to Chicoń, Miller and Truss) $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$. Putting them together with theorem 4.11 and corollary 4.3 we obtain the following corollary.

Corollary 4.11. $\mathrm{add}(\mathcal{I}_{\mathcal{E}}^{\sigma}) \leq \mathrm{add}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{I}_{\mathcal{E}}^{\sigma}) \geq \mathrm{cof}(\mathcal{M})$.

We do not know if the above inequalities can be strict. This leeds us to formulating the following.

Question. *Is any of the inequalities* $\mathrm{add}(\mathcal{I}_{\mathcal{E}}^{\sigma}) < \mathrm{add}(\mathcal{M})$ *or* $\mathrm{cof}(\mathcal{I}_{\mathcal{E}}^{\sigma}) > \mathrm{cof}(\mathcal{M})$ *consistent with ZFC?*

Theorem 4.12.
(a) *Assuming* MA *we have* $\mathrm{add}(\mathcal{I}_{\mathcal{E}}^{\sigma}) = \mathfrak{c}$;
(b) *Assuming* AMA *we have* $\mathrm{cof}(\mathcal{I}_{\mathcal{E}}^{\sigma}) = \omega_1$.

Proof. The arguments are standard. To prove (a) we apply MA to the amoeba for eventually different forcing, and to prove (b) we apply AMA to $\mathbb{AE}$. □

An equivalent formulation of (a) is the inequality $\mathfrak{p} \leq \mathrm{add}(\mathscr{I}_{\mathscr{E}}^{\sigma})$. But $\mathfrak{p}$ is not a good lower bound for the additivity of the ideal $\mathscr{I}_{\mathscr{E}}^{\sigma}$. Namely, it is consistent that $\mathfrak{p} = \mathfrak{t} < \mathrm{add}(\mathscr{I}_{\mathscr{E}}^{\sigma})$. To see this, we start from a model of $(\mathrm{CH}\ \&\ 2^{\omega_1} > \omega_2)$, and force with finite support iteration of the forcing $\mathbb{AE}$ of length ω_2. Then the extension satisfies $\mathrm{add}(\mathscr{I}_{\mathscr{E}}^{\sigma}) = \omega_2 = \mathfrak{c}$ and $2^{<\omega_2} = 2^{\omega_1} > \mathfrak{c}$. But it is well known (see e.g. [vD]), that $\mathfrak{p} \leq \mathfrak{t}$ and $2^{<\mathfrak{t}} = \mathfrak{c}$, so we must have $\mathfrak{t} = \mathfrak{p} = \omega_1$ in the extension.

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