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On a Property of Continuous Solutions of the Dilation Equation with Positive Coefficients

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Let N be integer and let $c_0, \ldots, c_N$ be positive reals summing up to 2. We prove that if $f: \mathbb{R} \to \mathbb{R}$ is a compactly supported and continuous solution of the dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n),$$

then either $f = 0$ or $f|_{(0,N)} > 0$ or $f|_{(0,N)} < 0$.

1. Introduction

It is easy to show (see [3]) that for every positive integer m the function $N_m: \mathbb{R} \to \mathbb{R}$, called the B-spline of the order m, defined as follows

$$N_1 = \chi_{[0,1)},$$

$$N_{m+1} = N_m \bigstar N_1 \quad \text{for every} \quad m \in \mathbb{N}$$

is compactly supported with $\operatorname{supp} N_m = [0,m]$ and for every $x \in \mathbb{R}$ satisfies

$$N_m(x) = \sum_{n=0}^{m} \frac{1}{2^{m-1}} \binom{m}{n} N_m(2x - n).$$

The present paper is motivated by the fact that for every $m \in \mathbb{N}$ the function N_m is positive on $(0,m)$. Our purpose is to show that this fact is a basic property

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of any continuous and compactly supported solution $f:\mathbb{R}\to\mathbb{R}$, positive at a point, of the dilation equation

$$f(x)=\sum_{n=0}^{N} c_n f(2x-n), \tag{1}$$

where the coefficients $c_0,\dots,c_N$ are positive reals and

$$\sum_{n=0}^{N} c_n = 2. \tag{2}$$

Solutions of (1) have been used in several fields such as wavelet theory (see [6], [4], [1], [5]), splines (see [19], [9]), subdivision schemes in approximation theory and curve design (see [12], [16], [2]), probability theory (see [11]). In the most significant applications of (1) the coefficients c_n's fulfill condition (2) (see [8]).

Equation (1) with nonnegative c_n's was studied in [15], [2], [20], [21], [11], [10], [18].

2. Two lemmas

We first notice that using similar argumentation as in [13] it can be proved that if $f:\mathbb{R}\to\mathbb{R}$ is a compactly supported solution of (1), then $\operatorname{supp} f\subset[0,N]$.

From [18] (cf. also [10]) follows that if c_n's are nonnegative reals satisfying (2), then any continuous and compactly supported solution of (1) is either nonnegative everywhere on $\mathbb{R}$ or nonpositive everywhere on $\mathbb{R}$. In the case of positive c_n's we know much more. Namely, according to [14], we get the following lemma.

Lemma 1. *Assume N is integer, $c_0,\dots,c_N$ are positive reals satisfying* (2). *If $f:\mathbb{R}\to\mathbb{R}$ is a continuous and compactly supported solution of* (1), *then f is either nonnegative everywhere on $\mathbb{R}$ or nonpositive everywhere on $\mathbb{R}$.*

Moreover, if *f is nonzero, then f is either positive almost everywhere on* $(0,N)$ *or negative almost everywhere on* $(0,N)$.

In our later considerations we will need only the first part of Lemma 1. The part moreover has been the second motivation to this paper.

Lemma 2. *Assume N is integer, $c_0,\dots,c_N$ are positive reals satisfying* (2). *If $f:\mathbb{R}\to\mathbb{R}$ is a continuous and compactly supported solution of* (1) *and if there exists an $x\in(0,N)$ such that $f(x)=0$, then there exist $y,z\in\mathbb{R}$ such that $y<x<z$, $|z-x|\le 1$, $|x-y|\le 1$ and $f(y)=f(z)=0$.*

Proof. According to Lemma 1 we can assume that f is nonnegative everywhere.

If the set of zeros of the function f is a dense subset of the real line, then the assertion of the lemma holds.

Fix now an interval $(a,b) \subset \mathbb{R}$ such that $f(x) > 0$ for every $x \in (a,b)$ and assume that (a,b) is an interval of the maximal length in the following sense:

$$\bigwedge_{((c,d)\subset \mathbb{R}} \left(f|_{(c,d)} > 0 \Rightarrow d - c \leq b - a\right). \tag{3}$$

Observe that to prove the lemma it is enough to show that $b - a \leq 1$.

Suppose that $b - a > 1$.

Fix $n \in \{0,\ldots, N\}$ and $x \in \left(\frac{a+n}{2},\frac{b+n}{2}\right)$. Then $2x - n \in (a,b)$ and hence

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n) \geq c_n f(2x - n) > 0.$$

We thus get

$$f|_{\bigcup_{n=0}^{N}\left(\frac{a+n}{2},\frac{b+n}{2}\right)} > 0. \tag{4}$$

Since $b - a > 1$, it follows that $\bigcup_{n=0}^{N}\left(\frac{a+n}{2},\frac{b+n}{2}\right) = \left(\frac{a}{2},\frac{b+N}{2}\right)$. Now, according to (4) and (3) we see that $\frac{b+N}{2} - \frac{a}{2} \leq b - a$ and hence that $(0,N) = (a,b)$. This implies that $f(x) > 0$ for every $x \in (0,N)$; contradiction. ◊

3. Main result

Theorem. *Assume N is an integer, $c_0,\ldots, c_N$ are positive reals satisfying* (2). *If $f:\mathbb{R} \to \mathbb{R}$ is a continuous and compactly supported solution of* (1), *then $f(x) = 0$ for every $x \in \mathbb{R}\setminus(0,N)$ and either $f(x) = 0$ for every $x \in (0,N)$ or $f(x) > 0$ for every $x \in (0,N)$ or $f(x) < 0$ for every $x \in (0,N)$.*

Proof. According to Lemma 1 we can assume that f is nonnegative everywhere. Assume, moreover, that there exists an $x \in (0,N)$ such that $f(x) = 0$. The proof will be completed if we show that the set of zeros of the function f is a dense subset of the real line.

Using Lemma 2 we fix a real number y such that $|\frac{N}{2} - y| \leq \frac{1}{2}$ and $f(y) = 0$. Then

$$0 = f(y) = \sum_{n=0}^{N} c_n f(2y - n)$$

and hence $f(2y - n) = 0$ for every $n \in \{0,\ldots, N\}$. Since $\operatorname{supp} f \subset [0,N]$ we get $f(2y - n) = 0$ for every $n \in \mathbb{Z}\setminus\{0,\ldots, N\}$. Therefore $f(2y + k) = 0$ for every $k \in \mathbb{Z}$, which gives

$$f(y + k) = \sum_{n=0}^{N} c_n f(2y + 2k - n) = 0$$

for every $k \in \mathbb{Z}$.

Finally, by a simple induction on m we conclude that

$$f\left(\frac{y+k}{2^m}\right) = 0$$

for any $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$. ◊

Remark. In the Theorem we cannot omit any of the assumptions. Indeed:

1. Among all compactly supported solutions of the equation

$$f(x) = f(2x) + f(2x-1)$$

obtained in [17] there are solutions having both positive and negative values;

2. Since for every positive integer m the B-spline N_{m+2} is of the class C^m and since for every $x \in \mathbb{R}$ we have

$$N'_{m+2}(x) = N_{m+1}(x) - N_{m+1}(x-1)$$

(see [3]), then the function N'_{m+2} has both positive and negative values and, moreover, is a continuous and compactly supported solution of the equation

$$N'_{m+2}(x) = \sum_{n=0}^{m+2} \frac{1}{2^m} \binom{m+2}{n} N'_{m+2}(2x-n);$$

3. In [6] it is showed that the equation

$$f(x) = \frac{1+\sqrt{3}}{4} f(2x) + \frac{3+\sqrt{3}}{4} f(2x-1) + \frac{3-\sqrt{3}}{4} f(2x-2) + \\ + \frac{1-\sqrt{3}}{4} f(2x-3)$$

has a continuous and compactly supported solution having both positive and negative values;

4. From [13] it follows that the equation

$$f(x) = \frac{1}{2} f(2x) + f(2x-1) + \frac{1}{2} f(2x-2)$$

has continuous solutions having both positive and negative values.

Corollary. *Let N be an integer and let $c_0, \ldots, c_N$ be positive reals satisfying* (2). *If* (1) *has a non-trivial L^1-solution, then there exists a representative f of that L^1-solution such that $f(x) = 0$ for every $x \in \mathbb{R} \setminus (0,N)$ and either $f(x) > 0$ for every $x \in (0,N)$ or $f(x) < 0$ for every $x \in (0,N)$.*

Proof. From [7] we get that the L^1-solution of (1) is compactly supported with support contained in $[0,N]$ and from [16] (cf. also [18]) we conclude that there exists a continuous representative f of that L^1-solution such that $f(x) = 0$ for every $x \in \mathbb{R} \setminus (0,N)$. Now it is enough to use the Theorem.

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