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# Extendibility of Embeddings

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Lifting given embeddings is an important technique in core model theory. However, the structure constructed in the lifting process (called pseudoultrapower) is not necessarily well-founded. We are going to discuss criteria for getting well-foundedness. One of them will be a generalisation of the well-known *Frequent Extension of Embeddings Lemma.* In addition, we will show that this statement is in some sense optimal proven. This survey tries to give the idea behind this technique avoiding most details.

## 1. Introduction

We shall introduce a well-known basic technique of core model theory, exemplified by **L**, the simplest of all core models. We will look at related questions and its answers. All proofs in detail can be found in [Räsch00].

Let us start from an easy example where we can find and explain the problem – let us consider the large cardinal axiom $0^{\#}$. There are various common ways to characterise it. One of them is the existence of a non-trivial elementary embedding from **L** into itself. In fact, this is equivalent to have an embedding (from **L** into itself) which is at least $\Sigma_1$-preserving[1]. We can go one step further and look at a $\Sigma_0$-preserving map which is, in addition, cofinal which simply means that the ordinals in the domain are mapped cofinally into the ordinals of the range. Such map will be also $\Sigma_1$-preserving. Therefore, to check elementarity of a cofinal map

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[1] A map is $\Sigma_1$-preserving if t preserves all $\Sigma_1$-formulae with parameters.

we only have to look at atomic formulae. Nevertheless, sometimes to work with such large (class-sized) embedding is still a problem. Usually, it seems to be easier to work with set-sized maps. So, an interesting question might be whether there is a possibility to achieve $0^{\#}$ (characterised as above) starting from a set-sized map. The answer can be found in [Devlin84, p. 192] as follows:

**Example.** Suppose we have a non-trivial elementary embedding $\bar{\pi} : \mathbf{L}_{\bar{\alpha}} \to \mathbf{L}_{\bar{\beta}}$, where $\bar{\alpha}$ and $\bar{\beta}$ are limit ordinals and $\operatorname{crit}(\sigma) < |\bar{\alpha}|$. Then there is also a non-trivial elementary embedding $\pi : \mathbf{L} \to \mathbf{L}$, and hence $0^{\#}$ holds.

The proof uses well-known model theoretic techniques. Taking the usual ultrapower of $\bar{\pi}$ — but now with help of $\bar{\pi}$ rather than an explicit given ultrafilter[2] — we get such embedding $\pi$.

The hardest part thereby is to observe that the constructed ultrapower is indeed isomorphic to $\mathbf{L}$. The only thing we have to look at is the well-foundedness of this ultrapower. It is obviously a necessary property for such isomorphism but it is moreover also sufficient: If we were sure of the well-foundedness of the ultrapower we also would know that it must be isomorphic to $\mathbf{L}$ because of the $\mathbf{L}$-like behaviour[3]. In fact, this property is the only thing we have to look at.

In core model theory there are slightly different versions of this well-known construction used. The reason is to get as much information as possible. So, one such information we try to get is to have more control about the ultrapower map $\pi$. For, we are interested in a map such that $\pi \restriction \operatorname{dom}(\bar{\pi}) = \bar{\pi}$. Then we really have an *extension* of the given embedding. This is relatively easy possible: We have to put more things into the construction.

More exactly, we not only consider equivalent classes of constructible functions $f$ with domain $\operatorname{crit}(\bar{\pi})$, we now take constructible functions with an *arbitrary* domain lying in $\mathbf{L}_{\bar{\alpha}}$ and also a $\xi \in \bar{\pi}(\operatorname{dom}(f))$ and consider equivalent classes of the form $[\xi, f]$. Similar to the usual construction we now use the following definition of being equivalent:

$$[\xi_0, f_0]\, R[\xi_1, f_1] :\leftrightarrow \langle \xi_0, \xi_1 \rangle \in \bar{\pi}(\{\langle \eta_0, \eta_1 \rangle \mid f_0(\eta_0)\, R f_1(\eta_1)\}),$$

where $R$ is the $\in$- or $=$-relation. Such construction will be enough for getting an ultrapower map extending the given embedding. We call this kind of construction[4] the *canonical upward extension of* $\bar{\pi}$ *to* $\mathbf{L}_{\alpha}$ and since the construction is based on the usual ultrapower we also call this structure *pseudo-ultrapower.*

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[2] We can think of using the well-known ultrapower construction with $U := \{X \in \mathbf{L} \mid \kappa \in \bar{\pi}(X)\}$, where $\kappa$ is the so-called critical point of $\bar{\pi}$, which means that $\kappa$ is the smallest ordinal moved by $\bar{\pi}$.

[3] We are using here the fact that $\mathbf{V} = \mathbf{L}$ can be expressed in a $\Sigma_1$-fashion such that this formula must be preserved by the ultrapower map $\pi$. But this means the ultrapower thinks it is $\mathbf{L}$. So, well-foundedness will be enough.

[4] We will mention another variant of this construction in the sixth section. There we still want to get more information of the ultrapower map and so we will extend the construction again.

To be more precisely, we will now switch our view of the constructible universe from the approximation via the $\mathbf{L}_\alpha$-hierarchy to the $\mathbf{J}_\alpha$-levels[5]. This will not change too much but with this changing we are able to speak about more general statements. We also have $\mathbf{L} = \bigcup_\alpha \mathbf{J}_\alpha$ but the stages now have better closure properties[6].

The problem we will consider now states as follows:

**Problem.** Consider a cofinal function $\bar{\pi} : \mathbf{J}_{\bar{\alpha}} \to \mathbf{J}_{\bar{\beta}}$, $\Sigma_0$-preserving, and an ordinal $\alpha > \bar{\alpha}$ is a cardinal[7] in $\mathbf{J}_\alpha$.

**Under what circumstances can we extend $\bar{\pi}$ to a cofinal and $\Sigma_0$-preserving embedding defined on $\mathbf{J}_\alpha$?**

Actually, we already know how to go on. We can always construct the ultrapower $\mathfrak{A}$ having $\bar{\pi}$ and $\alpha$ such that we get the following diagram and question:

$$\begin{tikzcd}
\mathbf{J}_\alpha \arrow[r, "\pi", "\text{cofinal, } \Sigma_0"'] & \mathfrak{A} \\
\mathbf{J}_{\bar{\alpha}} \arrow[u, "\subseteq"] \arrow[r, "\pi", "\text{cofinal, } \Sigma_0"'] & \mathbf{J}_{\bar{\beta}} \arrow[u, "\subseteq"']
\end{tikzcd}$$

**Question.** *When is $\mathfrak{A}$ well-founded—depending on $\bar{\pi}$ and $\alpha$?*

In case of a positive answer we find a $\beta$ such that $\mathfrak{A}$ is isomorphic to $\mathbf{J}_\beta$. Then we would have reached our goal[8]. Therefore, we will now try to answer *this* question in the remaining part of the paper.

In the second section we will look at an easy property concerning this construction. In the following third section we look at a first criterion for a positive answer to the question above which is folklore. It turns out that we are able to show this in case where $\alpha$ has (real) uncountable cofinality (and a bit more). But on the other hand, if $\bar{\alpha}$ has (real) countable cofinality, then we might run into problems. We will discuss this case in the fourth section. The ideas given there are all due to Ronald Jensen. In the fifth section we are going to look at the new version of the statement we will have considered in the section before. This has been done in a joint work with Ronald Jensen. We then show that both versions of the second criterion are in some sense

[5] *cf.* [Devlin84].

[6] You might think of a $\mathbf{J}_\alpha$ to be a suitable $\mathbf{L}_\beta$ where $\beta$ is a limit ordinal if you only look at the properties of such stage, in fact, for $\alpha = \omega$ both $\mathbf{L}_\alpha$ and $\mathbf{J}_\alpha$ are the same.

[7] This is a general assumption for technical reasons. In fact, we also want to prove a similar statement to the theorem of Łoś but now for the pseudo-ultrapower. To be able to do this we need a large enough new domain where we will find *all* subsets of $\bar{\alpha}$. With this property of $\bar{\alpha}$ being a cardinal in $\mathbf{J}_\alpha$ we can apply the method of acceptability to achieve this.

[8] In fact, considering the special case where, roughly, $\alpha$ is the class of all ordinals, that means $\mathbf{J}_\alpha$ is a symbol for the whole $\mathbf{L}$, our original example using $0^\#$ fits in this construction.

optimal[9]. In the sixth section we try to give an overview about the things really has been done using fine structure theory to get more general statements. In the last seventh section we give a sketch for a typical application of this kind of statements we will have proved then.

## 2. Why do we consider the Pseudo-Ultrapower?

There is one thing we should mention to motivate this kind of construction. We have risen a question given by a map $\bar{\pi}$ and a large domain given by an ordinal $\alpha$. The question now was whether there is *any* extension $\pi$. Our answer so far was a *special* kind of construction, in fact, we have constructed the pseudo-ultrapower and we only ask whether *this* special construction will work? Might there be another possibility to get such extension?

Well, there is not. We can prove the following statement which says that our pseudo-ultrapower construction considered is somehow a *minimal* one such that if there is *any* other well-founded extension with our desired properties, then we are able to embed our pseudo-ultrapower into the given one in a $\Sigma_0$-preserving way. But this means that even our pseudo-ultrapower must also be well-founded. So, if there is any other well-founded extension, then even our special one will work. This means it will be sufficient to consider only this kind of construction.

**Lemma 2.1.** *Let $\bar{\pi} : \mathbf{J}_{\bar{\alpha}} \to \mathbf{J}_{\bar{\beta}}$ be any $\Sigma_0$-preserving and cofinal map. Let $\alpha$ be arbitrary such that $\bar{\alpha}$ is a cardinal in $\mathbf{J}_\alpha$. In addition, let $\pi' : \mathbf{J}_\alpha \to \mathfrak{B}$ a $\Sigma_0$-preserving and cofinal extension of $\bar{\pi}$ where $\mathfrak{B}$ is a well-founded $\mathbf{L}$-like structure.*[10] **Then** *also the canonical upward extension of $\bar{\pi} : \mathbf{J}_\alpha \to \mathfrak{A}$ is well-founded.*

In fact, the pseudo-ultrapower $\mathfrak{A}$ can be embedded into the given structure $\mathfrak{B}$ very easily in a $\Sigma_0$-fashion such that $\mathfrak{A}$ must be well-founded and we get the following diagram:

$$\begin{tikzcd}
& \mathfrak{B} \\
\mathbf{J}_\alpha \arrow[ur, "\pi'", "\Sigma_0"'] \arrow[r, "\pi", "\Sigma_0"'] & \mathfrak{A} \arrow[u, dotted, "\Sigma_0"'] \\
\mathbf{J}_{\bar{\alpha}} \arrow[u, "\subseteq"] \arrow[r, "\bar{\pi}", "\Sigma_0"'] & \mathbf{J}_{\bar{\beta}} \arrow[u, "\subseteq"']
\end{tikzcd}$$

[9] We will see what does this mean. Roughly, we have to make assumptions which seem to be made only for technical reasons but we will show that they are necessary.

[10] One can formulate this statement very general. However, for simplicity we may think of $\mathfrak{B}$ of being isomorphic to a $\mathbf{J}_\gamma$.

## 3. One easy first Criterion

Considering the following property we will see how useful it turns out to be for getting a positive answer to our question.

**Definition 3.1.** Say $\bar{\alpha}$ *looks nice in* $\alpha$ if
(a) $\alpha \leq \alpha$.
(b) If $\bar{\alpha} < \alpha$, then $\mathrm{cf}(\bar{\alpha}) > \omega$.
(c) If $\bar{\alpha} < \alpha$, then $\forall \xi < \bar{\alpha}\ \exists \tau \leq \bar{\alpha}(\xi < \tau \wedge \mathrm{cf}(\tau) > \omega \wedge \mathbf{J}_\alpha \models \tau$ is regular$)$.

Clearly, if $\bar{\alpha}$ looks nice in $\alpha$, $\bar{\alpha}$ is a cardinal in $\mathbf{J}_\alpha$. Obviously, the condition (b) looks like a *global* condition on $\bar{\alpha}$ (it do not depend on $\alpha$) and that turns out to be a problem. We will discuss this in the next section. The rather technical property (c) is not as hard as it might look. For instance, if $\bar{\alpha}$ itself is a regular cardinal in $\mathbf{J}_\alpha$, then (c) would hold. In this case, it would look more like a *local* condition (it only depends on $\alpha$). The only difference to the stated property (c) is the following: We need not really that $\bar{\alpha}$ looks like a regular cardinal in $\mathbf{J}_\alpha$ but there has to be cofinal many *copies* $\tau$ of $\bar{\alpha}$ having both properties, being regular in $\mathbf{J}_\alpha$ as well as having (real) uncountable cofinality. Now, having this property we get a positive answer to our asked question:

**Lemma 3.2.** *If $\bar{\alpha}$ looks nice in $\alpha$, then the canonical upward extension will be well-founded.*

The proof is not very hard and applies the technique of countable submodels. Using this we need the global assumption of $\bar{\alpha}$ on its cofinality: If $\bar{\alpha}$ has uncountable cofinality, then it cannot be captured by any sequence of the (small) countable model.

## 4. The second much harder Criterion

So far we have considered the case where $\bar{\alpha}$ has uncountable cofinality. But the countable case turned out to be more complicated. On the one hand we can find counterexamples such that we cannot hope to get the same statement as Lemma 3.2 for the countable case. On the other hand we need a similar statement to be able to cover all cases for $\tau$.

This is the point where we have to become more *modest* in our goals: If we cannot extend *one* arbitrary embedding to get a well-founded ultrapower, let us try to consider *many* of them extend them all and hope that at least one of them would be a well-founded one.

But what does the phrase '*many* embeddings' mean?—One answer is due to Jensen in 1974, when he proved the well-known covering lemma we will consider in the last section. The main idea is to take terms of *large sets* we already known.

In fact, take the term of stationarity speaking about ordinals such that all we have to do is to *code* a set of (suitable) embeddings via a (suitable) coding function into a set of ordinals.

Of course, that do not seem to be an easier problem. However, the idea of this coding goes as follows: Assume $\tau$ has countable cofinality[11]. To get more information about the embeddings we will consider elementary substructures $X_\alpha$ of $\mathbf{J}_\tau$ coded by an ordinal $\alpha$ as we will see below. Such substructure then generates an elementary embedding $\sigma_\alpha$ where its domain $\mathbf{J}_{\tau_\alpha}$ is the Mostowski collapse of $X_\alpha$. Having these embeddings we will consider arbitrary upward extensions of them. More exactly, let $\gamma < \tau$ be uncountable, regular, and $f : \gamma \to \mathbf{J}_\tau$ be a surjection[12]. Set $X_\alpha := f" \alpha$ and

$$C := \{\alpha < \gamma \mid X_\alpha \prec \mathbf{J}_\tau \wedge X_\alpha \cap \gamma = \alpha \wedge \sup(X_\alpha \cap \mathrm{On}) = \tau \wedge \gamma \in X_\alpha\}.$$

Clearly, $C$ is a club subset of $\gamma$. Let $\sigma_\alpha : \mathbf{J}_{\tau_\alpha} \to X_\alpha$ be the (inverse) Mostowski collapse of $X_\alpha$ for $\alpha \in C$.

A short comment to the more technical properties in the definition of $C$. The first one is necessary to get $\sigma_\alpha$ via a condensation argument[13]. The second ensures $\mathrm{crit}(\sigma_\alpha) = \alpha$ and therefore the fourth gives $\sigma_\alpha(\alpha) = \gamma$. The third property asserts that the related embedding to $X_\alpha$ will be cofinal.

There is a reason we will discuss in Lemma 5.4 which forces us to restrict ourselves to take a smaller but still large subset of $C$. Therefore, set $D := \{\alpha \in C \mid \mathrm{cf}(\alpha) > \omega\}$, the subset of all codes with uncountable cofinality. Then $D$ is obviously stationary. Considering this large set we get the following lemma, the so-called *Frequent Extension Lemma*:

**Lemma 4.1.** *Let $S \subseteq D$ be stationary in $\gamma$. For $\alpha \in S$ let $\mu_\alpha > \tau_\alpha$ be arbitrary chosen such that $\tau_\alpha$ is a cardinal in $\mathbf{J}_{\mu_\alpha}$. In addition, let $\tilde{\sigma}_\alpha : \mathbf{J}_{\mu_\alpha} \to \mathfrak{A}_\alpha$ the canonical upward extension of $\sigma_\alpha$.* ***Then*** *there is a club set $C \subseteq \gamma$ such that the pseudo-ultrapower $\mathfrak{A}_\alpha$ is well-founded for every $\alpha \in S \cap C$, i.e., the set $\{\alpha \in S \mid \mathfrak{A}_\alpha$ is* ***not*** *well-founded$\}$ is* ***not*** *stationary.*

The original proof can be found in [DevJen75]. This statement can be proved for all $\tau$ as well, *i.e.*, even in the uncountable cofinality case.

[11] We switch here in our notation from $\bar{\alpha}$ to $\tau$ because we now have another situation where we will start from: This $\tau$ will not be the domain of the given embedding. In fact, we will copy $\tau$ using elementary substructures as we shall see to get many embeddings at once.

[12] This will be our *coding function* but to fix such a surjection might be another problem, especially if $\tau$ is a (singular) cardinal.

[13] The condensation property of $\mathbf{L}$ is very important. It says having an elementary substructure of $\mathbf{L}_\alpha$ (or even $\mathbf{J}_\beta$) where $\alpha$ is a limit ordinal we know that is must be isomorphic to a $\mathbf{L}_\alpha$ (or $\mathbf{J}_\beta$, respectively).

## 5. The new Version of the second Criterion

In the last two sections we have seen two criteria which cover most cases where we try to answer the question above. But to use the second one (Lemma 4.1) we have to fix the coding function. But such coding map would collapse the given $\tau$. So, it seems for cardinals we need a new statement. However, in typical examples of an application of such arguments we only need an answer in an indirect proof towards a contradiction. So, if we do not have such kind of coding function we could *force* to have one, which means we switch to a forcing extension where we can find such coding function and derive there a contradiction. Then we know there must already a contradiction in our ground model. Therefore, we are done with this kind of proof.

But nevertheless, it would be nice to have a statement which avoids such coding function. Indeed, it is possible as we will see below. So far we have used terms of largeness speaking about ordinals which code suitable substructures of the given $\mathbf{J}_\tau$. But to speak about large sets of subsets we already know terms we could take: Consider the well-known structure theory[14] of '$\mathscr{P}_\kappa(\lambda)$' and therefore small subsets of $\mathbf{J}_\tau$, *i.e.*, $[\mathbf{J}_\tau]^{<\gamma} := \{X \subseteq \mathbf{J}_\tau \mid |X| < \gamma\}$ where $\max\{\omega_1, \mathrm{cf}(\tau)\} \leq \gamma \leq \tau$ regular.

**Definition 5.1.** A subset $\mathscr{C}$ of $[\mathbf{J}_\tau]^{<\gamma}$ is called *club* if it is *closed* under (arbitrary) chains of length less than $\gamma$, *i.e.*, the union of such chain is also a member of $\mathscr{C}$ if all elements of the chain are, and it is *unbounded, i.e.*, for all $u \in [\mathbf{J}_\tau]^{<\gamma}$ there is a superset of $u$ in $\mathscr{C}$.

This term of a club set in $[\mathbf{J}_\tau]^{<\gamma}$ gives us in the usual way a term of stationarity: We call a subset $\mathscr{S} \subseteq [\mathbf{J}_\tau]^{<\gamma}$ stationary if it meets all club sets. But unfortunately, this term does not work as we will state in Lemma 5.4. Even in the first part of this statement we do not know how to handle with the restriction of the whole set $C$ to $D$, the subset of $C$ with ordinals of uncountable cofinality. The solution seems to be very unusual at first but we will see that this will be the right way: We will change our term of stationarity in the following way:

**Definition 5.2.** A subset $\mathscr{C}$ of $[\mathbf{J}_\tau]^{<\gamma}$ is called *club** if it is closed under chains of uncountable length[15] and is unbounded in the usual way.

We call a subset $\mathscr{S} \subseteq [\mathbf{J}_\tau]^{<\gamma}$ now *stationary** if it meets all club* sets. We obviously have got (in general) a stronger term of stationarity because such set has to meet more closed and unbounded sets in the new context. We can again show the usual properties of such terms like the theorem of Fodor or the pigeon hole principle.

In fact, we have not created a new kind of term. Considering this terms in the world of ordinals we see that for subsets of $D$—defined at the end of the fourth

[14] *cf.* [Jech71].

[15] We only consider chains of a l ngth $\delta$ where $\delta$ has uncountable cofinality.

section – both terms stationarity and stationarity* describe the same collection of large sets: To see this it is sufficient to consider a stationary $S \subseteq D$ set and to show now that $S$ meets an arbitrary chosen club* set $C$. For, consider the full closure of $C$, say $C'$, then $C'$ meets $S$ but an ordinal in the intersection of both has uncountable cofinality and every member of $C' \setminus C$ has obviously countable cofinality such that this ordinal must be inside $C$. This means that even we have change the term of largeness re-translate to the scope of ordinals we get the same term for subsets of $D$. So, for subsets of $D$ we have not changed the *term* but the *view* of it. Now, this kind of term gives us the following statement:

**Lemma 5.3.** *Let $\mathscr{S} \subseteq \mathscr{C}$ be stationary* in $[\mathbf{J}_\tau]^{<\gamma}$. For $u \in \mathscr{S}$ let $\mu_u > \tau_u$ be arbitrary chosen such that $\tau_u$ is a cardinal in $\mathbf{J}_{\mu_u}$. In addition, let $\tilde{\sigma}_u : \mathbf{J}_{\mu_u} \to \mathfrak{A}_u$ the canonical upward extension of $\sigma_u$.* ***Then*** *there is a club* set $\mathscr{C} \subseteq [\mathbf{J}_\tau]^{<\gamma}$ such that the pseudo-ultrapower $\mathfrak{A}_u$ is well-founded for every $u \in \mathscr{S} \cap \mathscr{C}$, i.e., the set $\{u \in \mathscr{S} \mid \mathfrak{A}_u$ is* ***not*** *well-founded$\}$ is* ***not*** *stationary*.*

Here $\mathscr{C}$ means the canonical translation of $C$ of the fourth section, *i.e.*,

$$\mathscr{C} := \{u \in [\mathbf{J}_\tau]^{<\gamma} \mid u \prec \mathbf{J}_\tau \wedge u \cap \gamma \text{ transitive} \wedge \sup(u \cap \mathrm{On}) = \tau \wedge \gamma \in u\}.$$

Clearly, $\mathscr{C}$ is a club (and hence a club*) set in $[\mathbf{J}_\tau]^{<\gamma}$ as well.

One more thing can be shown in this context. We have tried to translate the old proof and for this we have changed the usual term of stationarity in $[\mathbf{J}_\tau]^{<\gamma}$ to the rather new term of stationarity*. Now, the following question raises: Was this really necessary or is it possible to prove a similar theorem without changing this term. The answer is given in the next statement:

**Lemma 5.4.** *Assuming $\mathsf{Con}(\mathsf{ZF}+0^{\#})$, we cannot drop neither the assumption of restriction C to D in Lemma 4.1 nor we can prove Lemma 5.3 using stationarity rather then stationarity*.*

In the proof of the last lemma we consider a suitable forcing extension $\mathbf{L}^{\mathbb{P}}$ of the constructible universe $\mathbf{L}$ where we can find a stationary set $S \subseteq C$ and $\mu_\alpha > \tau_\alpha$ for each $\alpha \in S$ such that for the canonical upward extensions $\mathfrak{A}_\alpha$ the set $\{\alpha \in S \mid \mathfrak{A}_\alpha$ is not well-founded$\}$ will be also stationary. Moreover, we can show there is a stationary set $S \subseteq C$ such that $\tau_\alpha$ is a cardinal in $\mathbf{L}$ for every $\alpha \in S$. This would be enough because then for every $\alpha$ there must be a $\mu_\alpha > \tau_\alpha$ such that the pseudo-ultrapower $\mathfrak{A}_\alpha$ – this is the extension of the given embedding to the new domain given by $\mu_\alpha$ – cannot be well-founded[16].

To be more exactly, working inside a model of $\mathsf{ZF}+0^{\#}$ let $\gamma$ be the $\gamma$-th Silver insdiscernible. We construct a sequence of forcings $\langle \mathbb{P}_\alpha \mid \alpha \in \mathbb{I} \rangle$ where $\mathbb{I}$ denotes

[16] If there is not such $\mu_\alpha$ for an $\alpha \in S$, then we would be able to extend the given embedding to the whole constructible universe which would mean that we have $0^{\#}$ in the forcing extension $\mathbf{L}^{\mathbb{P}}$ – a contradiction.

the set of all $\alpha \leq \gamma$ which are inaccessible inside $\mathbf{L}$. This sequence will satisfy the following properties to get our contradiction:

- The whole sequence looks very uniformly. Roughly, the $\mathbb{P}_\alpha$ does for $\alpha$ the same as $\mathbb{P}_\gamma$ for $\gamma$, in fact, $\mathbb{P}_\alpha * \dot{\mathbb{P}}_\gamma = \mathbb{P}_\gamma$. Moreover, there will be a uniformly definition.
- For every $\alpha \in \mathbb{I}$, every $\bar{\alpha} \in \mathbb{I} \cap \alpha$ and every $i < \omega$, $\mathbf{L}^{\mathbb{P}_\alpha} \models \mathrm{cf}(\bar{\alpha}^{(+i)\mathbf{L}}) = \omega$.

We will now define a set which will turn out be the crucial idea to achieve our goal. For $\alpha \in \mathbb{I}$ set $\tau_\alpha := \alpha^{(+\omega)\mathbf{L}}$, $\tau := \gamma^{(+\omega)\mathbf{L}}$, and finally

$$S := \{\mathrm{rng}(\sigma) \mid \exists \alpha \in \mathbb{I}\ \exists \sigma \in \mathbf{L}[G_\gamma]\ (\sigma : \mathbf{L}_{\tau_\alpha} \to \mathbf{L}_\tau \text{ elementary} \wedge \alpha = \mathrm{crit}(\sigma))\}.$$

Having such sequence of forcings we can prove that $S$ is indeed stationary inside $\mathbf{L}^{\mathbb{P}_\gamma}$. For, fix $\mathbb{P}_\alpha$-generic filters $G_\alpha$ for every $\alpha \in \mathbb{I}$.

Stationary subsets of $\mathscr{C}$ can also be characterised in a different way which turns out to be very useful for our matters here. In fact, we can show that a subset $S \subseteq C$ is stationary if and only if for every algebra $\langle \mathbf{L}_\tau \mid f_i(i < \omega)\rangle$ there is a set in $S$ closed under it[17]. Therefore, let us argue with an algebra $\mathfrak{B} \in \mathbf{L}[G_\gamma]$ on $\mathbf{L}_\tau$. Then we know using acceptability that we can find a constructible predicate $B$ such that we have full information about $\mathfrak{B}$ inside $\langle \mathbf{L}_\tau[G_\gamma], B\rangle$. This will help us later on.

All we have to do is to find a suitable $\sigma$ such that its range lies in $S$. Therefore, let us start to construct many embeddings and see what we can do with them. For, using the Silver indiscernibles we can find for each $\alpha \in \mathbb{I}$ a non-trivial elementary embedding $\tilde{\pi}_\alpha$ from $\mathbf{L}$ into itself shifting the cardinal successor stages of $\alpha$: for $i < \omega$ let $\tilde{\pi}_\alpha(\alpha^{(+i)\mathbf{L}}) = \gamma^{(+i)\mathbf{L}}$. These maps form a direct limit such that we can pick a large enough $\alpha$ with $B \in \mathrm{rng}(\tilde{\pi}_\alpha)$, say $\tilde{\pi}_\alpha(\bar{B}) = B$.

Consider now only this chosen $\alpha$ and set $\pi := \tilde{\pi}_\alpha \restriction \mathbf{L}_{\tau_\alpha}$. Then we know $\pi : \langle \mathbf{L}_{\tau_\alpha}, \bar{B}\rangle \to \langle \mathbf{L}_\tau, B\rangle$ is still an elementary embedding.

If we now extend $\pi$ on both sides using the appreciated generic filter we get $\sigma : \langle \mathbf{L}_{\tau_\alpha}[G_\alpha], \bar{B}\rangle \to \langle \mathbf{L}_\tau[G_\gamma], B\rangle$ which is an elementary embedding as well because of the first property of the forcing sequence.

In fact, this map $\sigma$ almost satisfy the properties of the defined set $S$ and, moreover, its range is closed under the given algebra $\mathfrak{B}$. The only thing missing is to find such map in $\mathbf{L}[G_\gamma]$ rather than in $\mathbf{V}$ what we have done so far because we were using the Silver indiscernibles. This is where we now use the second property of our forcing. In fact, we show that we can find a countable sequence $\langle x_i^\alpha \mid i < \omega\rangle$ inside $\mathbf{L}[G_\gamma]$ such that $\mathbf{L}_{\tau_\alpha}$ can be obtained by a Skolem hull of $\alpha \cup \{x_i^\alpha \mid i < \omega\}$. But then we are able to approximate $\langle \mathbf{L}_{\tau_\alpha}[G_\alpha], \bar{B}\rangle$ in countable many steps, say $L_\alpha^n$ for $n < \omega$. Considering $\langle \sigma_\alpha \restriction L_\alpha^n \mid n < \omega\rangle$ we can show the existence of a map with the same properties but now in $\mathbf{L}[G_\gamma]$ using the absoluteness of the well-foundedness of a suitable relation.

[17] Actually, this would not be equivalent if we consider arbitrary subsets of $[\mathbf{L}_\tau]^{<\gamma}$.

This was all we tried to do. Now we have an embedding $\sigma$ in $\mathbf{L}[G_\gamma]$ such that its range lies in $S$ and is closed under $\mathfrak{B}$. This means $S$ is stationary.

It remains to show the existence of such forcing sequence. The idea is the following: Considering the second property of the desired forcings we might think of Namba forcing. It would give $\omega_2$ a countable cofinality without changing $\omega_1$ and other nice properties. So, the only thing we have to do is to iterate this idea. Of course, there is new problem—applying Namba forcing once the old $\omega_3$ will have a cofinality $\omega_1$. Which makes it impossible to shoot a countable cofinal sequence into this ordinal without changing $\omega_1$.

The solution will be to collapse at the beginning the first $\omega$ infinite cardinals to $\omega_2$, in fact, because of technical reasons to get better properties of the forcing we will collapse the first $\omega + 1$ cardinals, and then apply Namba forcing and we will get our desired countable cofinal subsets for each of the former cardinals considered so far. Then the job for $\omega_2$ is done and we only have to do the same for all $\alpha$ in $\mathbb{I}$.

Nevertheless, to be able to go on in a suitable forcing iteration behind limit stages is still a problem. One solution here is to take Shelah's RCS-iteration. This stands for *R*evised *C*ountable *S*upport forcing which Shelah has introduced in [Shelah98]. Fortunately, Namba forcing has the property which is necessary to apply this iteration strategy. Nevertheless, there are new problems to solve. One of those is the fact that this iteration strategy is already defined on odd stages by a suitable Levy-collapse which forces us to apply the Namba forcing immediately after our collapse we have discussed before. But this means we have to use a star product of two forcings but the property which forcings must have to be allowed in the RCS-iteration is in general not closed under this kind of operation. Fortunately, we are able to show this for our special forcing product.

Therefore, we may define $\mathbb{P}_\alpha$ for $\alpha \in \mathbb{I}$ as the so-called *revised limit* of the RCS-iteration $\langle \mathbb{P}_j, \dot{\mathbb{Q}}_j \mid j < \alpha \rangle$ defined by

$$\mathbb{Q}_j := \begin{cases} Coll\left(\omega_2, \omega_2^{+(\omega+1)\mathbf{L}}, \omega_2\right) * Nm & : \text{if } j = \omega_2 \text{ or } j \in \mathbb{I}, \\ Levy\left(2^{|\mathbb{P}_k| + |k|}, \omega_1\right) & : \text{if } j = 2k + 1, \\ \langle 1, 0 \rangle & : \text{else.} \end{cases}$$

Here, by *Coll* is meant the usual collapse forcing, *Nm* stands for Namba forcing, *Levy* for the Levy collapse, and by $\langle 1,0 \rangle$ we mean the trivial partial order. It turns out to be a very friendly forcing: $\mathbb{P}_\alpha$ will not change the reals, $\omega_1$ will not be changed, GCH will be preserved, and it will have the $\alpha$-chain-condition. Moreover, $\alpha$ will be the new $\omega_2$ such that by applying $\mathbb{P}_\gamma = \mathbb{P}_\alpha * \dot{\mathbb{P}}_\gamma$ we shoot countable sequences step by step into every $\alpha^{(+i)\mathbf{L}}$ for $\alpha \in \mathbb{I} \cap \gamma$ and $i < \omega$. So, after all, we have got a forcing which has the desired properties.

## 6. What really has been done

We have considered so far extensions of embeddings, speaking about initial segments of **L**. Thinking about this in a very general way we know that there is a bit more we could do. In fact, to avoid details we have not talked about fine structure yet. One important feature dealing with the **J**-structure is the possibility to go one step further considering the preserving properties of the pseudo-ultrapower map. If we gave a bit more functions into the construction, then we would get a slightly different ultrapower map which is better preserving, in fact, some part of the so-called $\Sigma^*$-formulae[18] would be also preserved. This is the point where fine structure comes into play and where on the other hand one also can see the real power of this pseudo-ultrapower construction. The variant of the construction is called *canonical fine structure upward extension.* In this case we really have strong properties of the ultrapower map which we are interested in but we will not talk about here.

Using the proof ideas of the lemmas we have seen in the sections before we now get similar statements for the fine structure pseudo-ultrapower as we shall see next.

**Definition 6.1.** Say that $\bar{\alpha}$ *looks very nice in* $\alpha$ if

(a) $\alpha \leq \alpha$.

(b) If $\bar{\alpha} < \alpha$, then $\mathrm{cf}(\bar{\alpha}) > \omega$.

(c) If $\bar{\alpha} < \alpha$, then

$$\forall \xi < \bar{\alpha}\ \exists \tau \leq \bar{\alpha}(\xi < \tau \wedge \mathrm{cf}(\tau) > \omega \wedge \mathbf{J}_\alpha \models \tau \text{ is a successor cardinal}).$$

Clearly, if $\bar{\alpha}$ looks very nice in $\alpha$, $\bar{\alpha}$ is a cardinal in $\mathbf{J}_\alpha$. With the same proof as the one of Lemma 3.2 we show the following statement:

**Lemma 6.2.** *Let $\bar{\alpha}$ looks very nice in $\alpha$. Then the canonical fine structure upward extension exists, i.e., the pseudo-ultrapower $\mathfrak{A}$ is well-founded.*

Actually, we are able to prove a much general statement. This first criterion will also work for the general relatively constructed hierchary $\mathbf{J}^A_\alpha$ (so far we have considered the special case where $A$ is the empty set). In fact, there is no restriction to use a general predicate $A$ such that we might start from a very general embedding $\bar{\pi} : \mathbf{J}^A_{\bar{\alpha}} \to \mathbf{J}^B_{\bar{\beta}}$ to get a $\pi : \mathbf{J}^A_\alpha \to \mathbf{J}^B_\beta$ in case of a well-founded pseudo-ultrapower. But, in the following we will forget the additional predicate $A$ because for the statements coming next the translations to the general situation are much harder and are only possible for very special $A$.

Using the techniques of the fourth section we are also able to show the fine structure variant of the Frequent Extension Lemma 4.1:

[18] The $\Sigma^*$-hierarchy of formulae is a fully different one compared to the usual Levy hierchary. Moreover, it was designed for the application of fine structure in core model theory.

**Lemma 6.3.** *Let $S \subseteq D$ be stationary in $\gamma$. For $\alpha \in S$ let $\mu_\alpha > \tau_\alpha$ be arbitrary chosen such that $\tau_\alpha$ is a cardinal in $\mathbf{J}_{\mu_\alpha}$. In addition, let $\tilde{\sigma}_\alpha : \mathbf{J}_{\mu_\alpha} \to \mathfrak{A}_\alpha$ the canonical fine structure upward extension of $\sigma_\alpha$. **Then** there is a club set $C' \subseteq \gamma$ such that the pseudo-ultrapower $\mathfrak{A}_\alpha$ is well-founded for every $\alpha \in S \cap C'$, i.e., the set $\{\alpha \in S \mid \mathfrak{A}_\alpha$ is **not** well-founded$\}$ is **not** stationary.*

In fact, we also get the new version of the Frequent Extension Lemma 5.3:

**Lemma 6.4.** *Let $\mathscr{S} \subseteq \mathscr{C}$ be stationary* in $[\mathbf{J}_\tau]^{<\gamma}$. Choose for every $u \in \mathscr{S}$ a $\mu_u \geq \tau_u$ such that $\tau_u$ is a cardinal in $\mathbf{J}_{\mu_u}$. Let $\tilde{\sigma}_u : \mathbf{J}_{\mu_u} \to \mathfrak{A}_u$ be the canonical fine structure upward extension of $\sigma_u$. **Then** there is a uncountable closed and unbounded (club*) set $\mathscr{C}'$ such that the elements of the stationary* set $\mathscr{S} \cap \mathscr{C}'$ are only indices of well-founded pseudo-ultrapowers, i.e., the set $\mathscr{S}' := \{u \in \mathscr{S} \mid \mathfrak{A}_u$ is **not** well-founded$\}$ is **not** stationary* in $[\mathbf{J}_\tau]^{<\gamma}$.*

Moreover, Lemma 5.4 goes also through now considering the fine structure upward extension and shown the same limitation that we cannot drop any of the two main restrictions.

**Lemma 6.5.** *Assuming $\mathsf{Con}(\mathsf{ZF} + 0^{\#})$, we cannot drop neither the assumption of restriction C to D in Lemma 6.3 nor we can prove Lemma 6.4 using stationarity rather stationarity*.*

The proofs of all fouir lemmas stated in this section use the same ideas as the lemmas for the $\Sigma_0$-case before.

## 7. A typical Example

We now want to describe one typical application to get an idea how one can use such lemmas. For, we will give a sketch of a proof of the well-known *Covering Lemma* due to Jensen[19].

**Lemma 7.1.** *If $0^{\#}$ does not exist, then for an arbitrary uncountable subset $X$ of ordinals there is a subset $Y \in \mathbf{L}$ of ordinals with the same (real) cardinality as $X$ such that $X \subseteq Y$.*

The proof given in detail seems to be rather long and technical but to catch the idea should be possible much easier; however, we cannot avoid all fine structure details here, though. Towards a contradiction we assume that the statement fails. For, choose a minimal $\tau \in \mathrm{On}$ which witnesses a counterexample $X \subsetneq \tau$. Since $\tau$ is minimal chosen it must be a cardinal within $\mathbf{L}$. Moreover, $|X| < |\tau|$, since otherwise we could consider $Y := \tau$.

[19] *cf.* [DevJen75].

We now define recursively a $H \prec \mathbf{L}_\tau$ such that $|X| = |H|$, covering $X$ and having suitable closure properties. An argument using the condensation property will give us a $\bar{\tau}$ such that $\mathbf{L}_{\bar{\tau}}$ and are isomorphic via a morphism $\sigma$. Because of the definition of $H$ we have $\bar{\tau} < \tau$.

Now, if $\bar{\tau}$ is a cardinal in $\mathbf{L}$, then we can extend our map $\sigma$ to the whole constructible universe. Knowing this extension were well-founded we would have shown the existence of $0^{\#}$ – obviously a contradiction. This is the first application of our criteria.

If $\bar{\tau}$ is not a cardinal in $\mathbf{L}$, then there is a minimal $\bar{\beta}$ such that the projectum[20] of $\mathbf{J}_{\bar{\beta}}$ is smaller than $\bar{\tau}$. But $\bar{\tau}$ looks very nice in $\bar{\beta}$ such that the canonical fine structure upward extension $\tilde{\sigma}$ of $\sigma$ is well-founded using the second criterion:

$$\begin{tikzcd}
\mathbf{J}_{\bar{\beta}} \arrow[r, "\tilde{\sigma}"] & \mathbf{J}_{\beta} \\
\mathbf{J}_{\bar{\tau}} \arrow[u, "\subseteq"] \arrow[r, "\sigma", "\Sigma_0"'] & H \arrow[u, "\subseteq"']
\end{tikzcd}$$

Moreover, we know that we can reach the whole $\mathbf{J}_{\bar{\beta}}$ taking the $\Sigma_1$-Skolem hull of the greatest projectum which falls under $\bar{\tau}$ using only one more parameter, the so-called standard parameter for $\mathbf{J}_{\bar{\beta}}$. To be exactly, for $n < \omega$ where $\omega\varrho_{\bar{\beta}}^{n+1} < \bar{\tau} \leq \omega\varrho_{\bar{\beta}}^{n}$ we have $\mathbf{J}_{\bar{\beta}} = \tilde{h}_{\beta}\left(\omega\varrho_{\bar{\beta}}^{n+1} \times \{p_{\beta}\}\right)$, since standard parameters in $\mathbf{L}$ are always so-called very good parameters. Define the $\tilde{Y} := \tilde{h}_{\beta}^{n+1}\left(\sigma\left(\omega\varrho_{\bar{\beta}}^{n+1}\right) \times \{\sigma(p_{\bar{\beta}})\}\right)$. Then $\tilde{Y}$ is constructible, $|\tilde{Y}|^{\mathbf{L}} < \tau$ and $X \subseteq \tilde{Y}$. It might be that this covering of $X$ is still to large in cardinality but then we can get a contradiction as follows: Since $\tau$ are minimal chosen we find a bijection $g : \delta \leftrightarrow \tilde{Y}$ within $\mathbf{L}$. Set $\bar{X} := g^{-1}{}''X$. Since $\bar{X} \subseteq \delta < \tau$ there is a covering $\bar{Y}$ of $\bar{X}$. But with $\bar{Y}$ for $\bar{X}$ also $g''\bar{Y}$ will have the properties of the theorem for $X$ – a contradiction.

We can follow exactly this strategy in the case of an uncountable cofinality of $\tau$ using the first criterion (Lemma 6.2). In the other case we have to work a bit more for proving the well-foundedness of the pseudo-ultrapower using the Frequent Extension Lemma (Lemma 6.3 or Lemma 6.4).

## References


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[20] To be precisely, we have to speak about the so-called *projecta* of a level $\mathbf{J}_\alpha$. These are one of the basic tools in fine structure theory. An introduction can be found in [Jensen72]. Roughly, it is the smallest ordinal $\delta \leq \alpha$ where we can define a subset (inside $\mathbf{J}_\alpha$) which is not a member of $\mathbf{J}_\alpha$. So, it speaks about new definable subsets of the given structure. This we would call $\rho_\alpha = \rho_\alpha^1$. Iterating this idea we get $\rho_\alpha^n$.

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