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MULTIPLE SOLUTIONS OF SOME ASYMPTOTICALLY LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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In this note we shall apply the additivity property of the Leray-Schauder topological degree in order to assure the existence of multiple solutions in two examples of nonlinear, asymptotically linear elliptic boundary value problems which both attracted remarkable interest in recent years:

I. Positive solutions of nonlinear eigenvalue problems,

II. Nonlinear perturbations of linear problems at resonance.

We do not bother here to state the results in the utmost generality possible. For detailed proofs the interested reader is referred to the forthcoming papers by Ambrosetti-Hess [1] (concerning example I) and Hess [2] (concerning example II).

I. Positive solutions of nonlinear eigenvalue problems

We consider the question of existence of positive solutions of the problem

$$\text{(1)} \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ , \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqq 1)$ with smooth boundary, $\lambda \geqq 0$ is a parameter, and $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying the following conditions

(f1) $f(0) = 0$,

(f2) there exist $m_\infty > 0$ and a bounded function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(s) = m_\infty s + g(s) \quad \forall s \geqq 0$,

(f3) the right-sided derivative $f'_+(0)$ at $s = 0$ exists, and $f'_+(0) < m_\infty$.

Of course (1) admits always the trivial solution $u = 0$. For λ varying in a certain interval we shall prove the existence of positive (i.e. nonnegative and nontrivial) solutions. Let $\lambda_1 > 0$ denote the first eigenvalue of the Dirichlet problem

$$\text{(2)} \qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ ; \end{cases}$$

it is known that the corresponding eigenspace is 1-dimensional and spanned by an eigenfunction ϕ which we may choose to be positive in Ω . Let

$$\lambda_\infty := \frac{\lambda_1}{m_\infty} , \qquad \lambda_0 := \begin{cases} \lambda_1/f'_+(0) & \text{if } f'_+(0) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

If f is strictly monotone increasing, convex and $f'_+(0) > 0$, Amann-Laetsch [3] prove the following:

For $\lambda \in [0,\lambda_\infty] \cup [\lambda_0,+\infty)$, problem (1) has the trivial solution only. It has at least one positive solution for $\lambda \in (\lambda_\infty,\lambda_0)$.

(For a uniqueness result cf. Amann [4], Ambrosetti [5].)

Their statement follows from an abstract theorem on order convex maps in general ordered Banach spaces and uses results from bifurcation theory (note that λ_0 is a bifurcation point from the trivial solution and λ_∞ a bifurcation point from infinity). It seems that their method breaks down if f is allowed to admit also negative values. We are interested in this case here.

Theorem 1. Let (f1) - (f3) be satisfied.

(i) If $\lambda \in (\lambda_\infty,\lambda_0)$, there exists at least one positive solution of problem (1).

(ii) Suppose in addition

$$\liminf_{s\to+\infty} g(s) > 0 . \tag{3}$$

Then for all $\lambda \in [\lambda_\infty,\lambda_0)$, problem (1) admits at least one positive solution. There exists $\varepsilon > 0$ such that for $\lambda \in (\lambda_\infty-\varepsilon,\lambda_\infty)$, problem (1) has at least two positive solutions.

We briefly sketch the proof of Theorem 1, working in the Hilbert space $H = L^2(\Omega)$. Let L be the positive selfadjoint operator induced in H by $-\Delta$, with domain $D(L) = H_0^1(\Omega) \cap H^2(\Omega)$. We extend the function f to $\mathbb{R}$ by setting $f(s) = 0 \ \forall s \leqq 0$, and denote by $F : H \to H$ the Nemytskii operator associated with f : $(Fu)(x) = f(u(x))$ $(x \in \Omega)$ for any function u defined in Ω . By the maximum principle, any solution u of $Lu = \lambda F(u)$ with $\lambda \geqq 0$ is nonnegative in Ω . Thus problem (1) is equivalent to the equation

$$u - \lambda L^{-1}F(u) = 0 \tag{4}$$

in H . Note that $L^{-1} : H \to H$ is compact.

We now apply topological degree arguments to prove the existence of nontrivial solutions of (4). For $R > 0$ let B_R denote the open ball in H around 0 , with radius R .

<u>Lemma 1</u>. <u>To each</u> $0 \leqq \lambda < \lambda_0$ <u>there exists</u> $R_1 = R_1(\lambda) > 0$ <u>such that</u> $\deg(I-\lambda L^{-1}F, B_{R_1}, 0) = 1$.

<u>Proof</u>. The assumptions (f1) - (f3) imply the existence of a constant $c > 0$ such that $|f(s)| \leqq c|s|$ $\forall s \in \mathbb{R}$. Let $0 \leqq \lambda < \lambda_0$. By the homotopy invariance of the Leray-Schauder degree, Lemma 1 is proved if we can show that for some $R_1 > 0$,

$$u - t\lambda L^{-1}F(u) \neq 0 \quad \forall \|u\| = R_1 \ , \quad \forall t \in [0,1] \ .$$

This is established indirectly, using measure-theoretic arguments and the variational characterisation of the first eigenvalue λ_1 of L .

<u>Lemma 2</u>. <u>Let</u> $\lambda > \lambda_\infty$. <u>Then</u> $\deg(I-\lambda L^{-1}F, B_{R_2}, 0) = 0$ <u>for all</u> $R_2 = R_2(\lambda)$ <u>sufficiently large</u>.

In particular, if $\lambda_\infty < \lambda < \lambda_0$, we can choose $R_2(\lambda) > R_1(\lambda)$. Then, by the additivity property of the degree, $\deg(I-\lambda L^{-1}F, B_{R_2} \setminus \bar{B}_{R_1}, 0) = -1$, and consequently there is a (nontrivial) solution of (4) in $B_{R_2} \setminus \bar{B}_{R_1}$. This proves Theorem 1(i).

<u>Idea of proof of Lemma 2</u>. One shows, again indirectly, that to $\lambda > \lambda_\infty$, there exists $K = K(\lambda) > 0$ such that

$$u - \lambda L^{-1}F(u) = \tau\phi \ , \text{ with } \tau \geqq 0 \Longrightarrow \|u\| < K \ . \tag{5}$$

Since $I - \lambda L^{-1}F$ is a bounded mapping, it then follows that there is a constant $a > 0$ having the property that

$$u - \lambda L^{-1}F(u) \neq a\phi \quad , \quad \forall u \in \bar{B}_K \ .$$

As, by (5) ,

$$u - \lambda L^{-1}F(u) \neq ta\phi \quad \forall \|u\| = K \ , \quad \forall t \in [0,1] \ ,$$

this implies that

$$\begin{aligned} 0 &= \deg(I-\lambda L^{-1}F-a\phi, B_K, 0) \\ &= \deg(I-\lambda L^{-1}F, B_K, 0) \ . \end{aligned}$$

The proof of Lemma 2 was inspired by a related argument used in Brown-Budin [6]; in our case however we are able to prove an a priori estimate of the form (5) only for this particular function ϕ .

<u>Lemma 3</u>. <u>If in addition</u> (3) <u>holds, the assertion of Lemma 2 remains valid also for</u> $\lambda = \lambda_\infty$.

Since the degree is invariant in connected components of

$H \setminus (I-\lambda_\infty L^{-1}F)(\partial B_{R_2(\lambda_\infty)})$, there exists $\varepsilon > 0$ such that

$\deg(I-\lambda L^{-1}F, B_{R_2(\lambda_\infty)}, 0) = 0$, $\forall\ \lambda \in (\lambda_\infty - \varepsilon, \lambda_\infty)$.

Lemma 4. For $0 < \lambda < \lambda_\infty$ we have
$\deg(I-\lambda L^{-1}F, B_{R_3}, 0) = 1$, provided $R_3 = R_3(\lambda)$ is sufficiently large.

This is a simple consequence of the asymptotic behavior of the function f .

Suppose now (3) holds, and let $\lambda \in (\lambda_\infty - \varepsilon, \lambda_\infty)$. By Lemma 1, there is $R_1 > 0$ such that $\deg(I-\lambda L^{-1}F, B_{R_1}, 0) = 1$. Further, Lemma 3 implies that $\deg(I-\lambda L^{-1}F, B_{R_2(\lambda_\infty)}, 0) = 0$. Hence there exists a nontrivial solution of (4) in $B_{R_2(\lambda_\infty)} \setminus \bar{B}_{R_1}$. Moreover, by Lemma 4 we find $R_3 > R_2(\lambda_\infty)$ such that
$\deg(I-\lambda L^{-1}F, B_{R_3}, 0) = 1$. It follows that $\deg(I-\lambda L^{-1}F, B_{R_3} \setminus \bar{B}_{R_2(\lambda_\infty)}, 0) = 1$ and thus the existence of a second nontrivial solution is guaranteed. Theorem 1(ii) is proved.

II. Nonlinear perturbations of linear problems at resonance

The problem

$$(6) \qquad \begin{cases} (-\Delta-\lambda_k)u + g(u) = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} ,$$

is investigated. Here λ_k denotes the k-th eigenvalue of the Dirichlet problem (2), $g : \mathbb{R} \to \mathbb{R}$ is a continuous function having limits $g_\pm := \lim_{s\to\pm\infty} g(s)$ $(\in \mathbb{R})$ and h is a given element in $H = L^2(\Omega)$. Let again L be the realization in H of the operator $-\Delta$ with Dirichlet boundary conditions, and denote by $G : H \to H$ the Nemytskii operator associated with g . Supposing in addition that

$$(7) \qquad g_- < g(s) < g_+ \quad \forall s \in \mathbb{R} ,$$

it is well-known that condition

$$(LL) \qquad (h,v) < \int_\Omega (g_+ v^+ - g_- v^-) \quad \forall v \in N(L-\lambda_k),\ v \neq 0 ,$$

is both necessary and sufficient for solvability of problem (6) (e.g. [7-12]). Moreover it follows that the range $R((L-\lambda_k)+G)$ is open in H .

We consider here problem (6) under conditions which are in a certain sense opposite to hypothesis (7) and imply a closed range of $(L-\lambda_k) + G$.

Let H be decomposed as $H = N(L-\lambda_k) \oplus R(L-\lambda_k)$. We set $H_1 := N(L-\lambda_k)$, $H_2 := R(L-\lambda_k)$, and denote by P_i $(i = 1,2)$ the orthogonal projection onto H_i . Let $h_i := P_i h$ $(i = 1,2)$.

We now suppose without loss of generality that $g_- \leqq 0 \leqq g_+$. Let $\mathcal{S} \subset H_1$ be the nonempty, closed, convex set defined by

$$\mathcal{S} := \{h_1 \in H_1 : (h_1, v) \leqq \int_\Omega (g_+ v^+ - g_- v^-) \quad \forall v \in H_1\} \quad .$$

The following condition is imposed on g :

(g1) There exists $\delta \geqq 0$ such that

$$\begin{cases} g(s) \geqq g_+ & \forall s \geqq \delta \\ g(s) \leqq g_- & \forall s \leqq -\delta \end{cases} \quad .$$

Let

$$\gamma_\pm := \liminf_{s \to \pm\infty} (g(s) - g_\pm) s \qquad (\geqq 0) \quad .$$

<u>Theorem 2. Suppose that either</u>

(a) $k = 1$ <u>(perturbation in the first eigenvalue) and both</u> $\gamma_- > 0$, $\gamma_+ > 0$, <u>or</u>

(b) $k > 1$ <u>and at least one of</u> γ_- , γ_+ <u>is positive</u>. <u>Then to</u> each $h_2 \in H_2$ <u>there exists an open set</u> $\mathcal{S}_{h_2}$ in H_1 , $\mathcal{S}_{h_2} \supset \mathcal{S}$, <u>such that</u>

(i) <u>if</u> $h_1 \in \mathcal{S}_{h_2}$, <u>then</u> (6) <u>admits at least one solution for</u> $h = h_1 + h_2$;

(ii) <u>if</u> $h_1 \in \mathcal{S}_{h_2} \setminus \mathcal{S}$, <u>then</u> (6) <u>admits at least two solutions</u> <u>for</u> $h = h_1 + h_2$.

Employing the strong maximum principle, assumption (a) can be slightly weakened (cf. Fučik-Hess [13]). Theorem 2 generalizes some results of Ambrosetti-Mancini [14,15]. A consequence is

<u>Theorem 3. Under either assumption of Theorem 2, the range of</u> $(L-\lambda_k) + G$ <u>is closed in</u> H .

It follows that assertion (i) of Theorem 2 remains valid for $h_1 \in \overline{\mathcal{S}_{h_2}}$. In order that $R((L-\lambda_k) + G)$ is closed in H , one can show by examples that in general conditions stronger than (g1) are needed.

<u>Idea of proof of Theorem 2.</u>

(i) Suppose $h_2 \in H_2$ is fixed, and let $h_1 \in \mathcal{S}$. Obviously the equation

$$(L-\lambda_k) u + G(u) = h$$

is equivalent to the equation

$$u + ((L-\lambda_k) + P_1)^{-1} (G(u) - P_1 u - h) = 0 \ .$$

Introduce the homotopy

$\mathcal{H}(t,u) := u + t((L-\lambda_k) + P_1)^{-1}(G(u) - P_1u - h)$, $t \in [0,1]$, $u \in H$.
It can be shown that there exists a rectangle $B = \{u \in H: \|P_1u\| < c_1, \|P_2u\| < c_2\}$ in H such that $\mathcal{H}(t,u) \neq 0$ $\forall t \in [0,1]$, $\forall u \in \partial B$ and thus

$$\deg(\mathcal{H}(1,.), B, 0) = \deg(I, B, 0) = 1 .$$

Moreover to each $h_1 \in \mathcal{S}$ there is an open neighborhood $\mathcal{U}(h_1) \subset H_1$ such that the degree remains 1 for h_1 replaced by $\tilde{h}_1 \in \mathcal{U}(h_1)$. Hence $\mathcal{S}_{h_2} := \bigcup_{h_1 \in \mathcal{S}} \mathcal{U}(h_1)$ suffices.

(ii) Let still $h_2 \in H_2$ be fixed, and suppose now $h_1 \in \mathcal{S}_{h_2} \setminus \mathcal{S}$. Then $\exists \tilde{v} \in H_1$:

$$(h_1, \tilde{v}) > \int_\Omega (g_+\tilde{v}^+ - g_-\tilde{v}^-) \quad (\geq 0)$$

and consequently $h_1 \neq 0$. Since $R(G)$ is bounded in H , $(1+K)h \notin R((L-\lambda_k) + G)$ for sufficiently large K . Set

$$\mathcal{K}(t,u) := u + ((L-\lambda_k) + P_1)^{-1}(G(u) - P_1u - (1+t)h),$$

$t \in [0,K]$, $u \in H$. We can find a rectangle $C \supset \bar{B}$ (where B is a rectangle as obtained in the proof of part (i)) such that $\mathcal{K}(t,u) \neq 0$ $\forall t \in [0,K]$, $\forall u \in \partial C$. Hence

$$\begin{aligned}\deg(\mathcal{H}(1,.),C,0) &= \deg(\mathcal{K}(0,.),C,0)\\ &= \deg(\mathcal{K}(K,.),C,0) = 0 .\end{aligned}$$

By the additivity property of the degree, the existence of a further solution in $C \setminus \bar{B}$ follows.

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