EQUADIFF 4

Jozef Kačur

Nonlinear parabolic boundary value problems with the time derivative in the boundary conditions

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [170]--178.

Persistent URL: http://dml.cz/dmlcz/702217

NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS WITH THE TIME DERIVATIVE IN THE BOUNDARY CONDITIONS

J.Kačur , Bratislava

The subject of this paper is motivated by the nonstationary,nonlinear and mixed boundary value problem for Schrödinger's equation considered in [2-4] .An approximate solution is constructed by solving a corresponding linearized boundary value problem. Construction of the approximate solution is convenient from the numerical point of view. Convergence and some properties of this approximate solution are investigated. Consider the equation

$$(1)\qquad \frac{\partial u}{\partial t} + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + b_0(t,x,u, \nabla u) = 0$$

for $(x,t)\in \Omega\times(0,T)$ where $T<\infty$, $\Omega \subset E^N$ is a bounded domain with Lipschitzian boundary $\delta\Omega$ and $\nabla u \equiv (\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N})$. Let Γ_1,Γ_2 be open disjoint subsets of $\delta\Omega$ and $\Gamma_1\cup\Gamma_2\cup\Lambda = \delta\Omega$ where $\Lambda\subset\delta\Omega$, $mes_{N-1}\Lambda=0$. Together with (1) we consider

$$(2)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial u}{\partial \nu} - b_1(t,x,u) \qquad \text{for } (x,t)\in\Gamma_1\times(0,T) \\ 0 &= -\frac{\partial u}{\partial \nu} - b_2(t,x,u) \qquad \text{for } (x,t)\in\Gamma_2\times(0,T) \end{aligned}$$

where $\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(\nu,x_i)$ and ν is the outward normal to $\delta\Omega$. The initial condition is

$$(3)\qquad u(x,0) = \phi(x)$$

where ϕ is sufficiently smooth in $\bar{\Omega}$.

Our concept of treating the problem (1) -(3) is based on Rothe's method developed in [5-9] .

<u>Notation.</u> We denote $W \equiv W_2^1(\Omega)$ (Sobolev space) , $(u,v)=\int_\Omega u\, v\; dx$,

$(u,v)_{\Gamma_j} = \int_{\Gamma_j} u\, v \; ds$ and $A[u,v] = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx$. By means of $A[u,v]$ for $u,v \in W$ we define the linear operator $A : W \to W^*$ (dual space to W). By $u_B(t)$ (from $L_2(\delta\Omega)$) we denote the trace of $u(t)$ from W for fixed $t \in (0,T)$, by $\|\cdot\|$, $\|\cdot\|_W$, $\|\cdot\|_{\Gamma_1}$ and $\|\cdot\|_{\Gamma_2}$ the norms in the corresponding spaces $L_2(\Omega)$, W, $L_2(\Gamma_1)$ and $L_2(\Gamma_2)$. The letter C will stand for any positive constant.

Assumptions. We assume $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $i,j = 1,\dots,N$ and

(4) $\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j \ge C_E |\xi|^2$;

(5) $|b_j(t,x,\xi) - b_j(t',x,\xi')| \le C(|t-t'| + |t-t'||\xi| + |\xi-\xi'|)$ $\quad j = 0,1,2$;

(6) $\phi \in W_2^2(\Omega)$ and $\frac{\partial \phi}{\partial \nu} = - b_2(0,x,\phi)$ for $x \in \Gamma_2$;

(7) $\left| \frac{\partial b_2(t,x,\xi)}{\partial \xi} \right| \le C \le \frac{C_E}{C_I^2}$ for $(x,t) \in \Gamma_2 \times (0,T)$, $|\xi| < \infty$, where C_I comes from the imbedding inequality $\|v\|_{L_2(\delta\Omega)} \le C_I \|v\|_W$.

We shall be concerned with a weak solution of (1)-(3) which we define in a following way.

Definition. The function $u \in L_\infty(\langle 0,T\rangle, W) \cap C(\langle 0,T\rangle, L_2(\Omega))$ is a weak solution of (1)-(3) if

i) $u(0) = \phi$

ii) $\frac{du}{dt} \in L_\infty(\langle 0,T\rangle, L_2(\Omega))$; $\frac{du_B}{dt} \in L_\infty(\langle 0,T\rangle, L_2(\Gamma_2))$;

iii) the identity

(8) $$\left(\frac{du(t)}{dt}, v\right) + A[u(t), v] + \left(\frac{du_B(t)}{dt}, v\right)_{\Gamma_1} + (b_0(t,x,u(t),\nabla u(t)), v) + \sum_{j=1,2} (b_j(t,x,u_B(t)), v)_{\Gamma_j} = 0$$

holds for all $v \in W$ and a.e. $t \in (0,T)$.

Clearly, if a weak solution $u(t)$ is sufficiently smooth, then it satisfies (1)-(3) in the classical sense.

We define an approximate solution $u_n(t)$ (see (10)) of (1)-(3) in the following way. Let n be a positive integer, $h = \frac{T}{n}$, $t_i = ih$ and $u_i \in W$ $i = 1,\dots,N$ solutions of the linear elliptic problems

(1') $$\frac{u - u_{i-1}}{h} + A u + b_0(t_i,x,u_{i-1},\nabla u_{i-1}) = 0$$

$$u + h\,\frac{\partial u}{\partial \nu} = u_{i-1} - h\, b_1(t_i, x, u_{i-1}) \quad \text{on} \quad \Gamma_1$$

(2)

$$\frac{\partial u}{\partial \nu} = -\, b_2(t_i, x, u_{i-1}) \qquad \text{on} \quad \Gamma_2$$

where $u_0 = \phi$. Precisely, successively for $i = 1,\dots,N$ the elements $u_i \in W$ satisfy the identities

$$(9)\quad \Big(\frac{u_i - u_{i-1}}{h}, v\Big) + A\,[u_i, v] + (b_0(t_i, x, u_{i-1}, \nabla u_{i-1}), v) +$$

$$+ \Big(\frac{u_{i,B} - u_{i-1,B}}{h}, v\Big)_{\Gamma_1} + \sum_{j=1,2} (b_j(t_i, x, u_{i-1,B}), v)_{\Gamma_j} = 0$$

for all $v \in W$. Existence and uniqueness of u_i is well known. Now,we define

$$(10)\quad u_n(t) = u_{i-1} + (t - t_{i-1})\, h^{-1}(u_i - u_{i-1}) \quad \text{for } t_{i-1} \le t \le t_i \ , \ i = 1,\dots,N.$$

Theorem 1. Under the assumptions (4)-(7) there exists the unique weak solution $u \in L_\infty(\langle 0,T\rangle, W \cap W^2_{2,loc}(\Omega))$ of (1)-(3) and $u_n(t) \to u(t)$ in $L_2(\Omega)$ uniformly for $t \in \langle 0,T\rangle$.

Remark 1. Theorem 1 implies that $u(t)$ satisfies (1) for a. e. (x,t) from $\Omega \times \langle 0,T\rangle$ in the classical sense.

Before proving Theorem 1, we prove some a priori estimates for $u_n(t)$.

Lemma 1. There exist C_1, C_2 and n_0 such that

$$(11)\quad \Big\|\frac{u_i - u_{i-1}}{h}\Big\|^2 + \Big\|\frac{u_{i,B} - u_{i-1,B}}{h}\Big\|^2_{\Gamma_1} + \frac{1}{h}\|u_i - u_{i-1}\|^2_W \le C_1 + C_2\, h \sum_{j=1}^{i} \|u_j\|^2_W$$

holds for all $n \ge n_0$, $i = 1,\dots,n$.

The proof of (11) is based on the identity (9), suitable application of Young's inequality $(ab \le 2^{-1}(\varepsilon a)^2 + (2\varepsilon)^{-2} b^2)$ and the assumptions (4)-(7). We point out the basic steps of the proof.Subtracting(9)for $i = j$, $i = j-1$ and putting $v = u_j - u_{j-1}$ successively we obtain the recurrent inequality

$$(12)\quad (1 - C_1 h)\Big(\Big\|\frac{u_j - u_{j-1}}{h}\Big\|^2 + \Big\|\frac{u_{i,B} - u_{i-1,B}}{h}\Big\|^2_{\Gamma_1} + \frac{C}{h}\|u_i - u_{i-1}\|^2_W\Big) \le$$

$$\le (1 + C_2 h)\Big(\Big\|\frac{u_{j-1} - u_{j-2}}{h}\Big\|^2 + \Big\|\frac{u_{j-1,B} - u_{j-2,B}}{h}\Big\|^2_{\Gamma_1} + \frac{C}{h}\|u_{i-1} - u_{i-2}\|^2_W\Big) +$$

$$+ C_3 h \sum_{i=1}^{j} \|u_i\|^2_W + C_4 h$$

where (4), (6) and (7) has been used. Similarly, from (9) for $i=1$ and $v=\frac{u_1-\phi}{h}$ we obtain

$$(13)\quad \left\| \frac{u_1-\phi}{h} \right\|^2 + \left\| \frac{u_{1,B}-\phi}{h} \right\|^2_{\Gamma_1} + \frac{C}{h} \| u_1-\phi \|^2_W \leq C(\phi)$$

where (6) has been used. The constant $C(\phi)$ depends on $\| \phi \|_{W_2^2(\Omega)}$. There exist $\delta>0$, $K>0$ such that $(1-C_1h)^i \geq \delta$ and $(1+C_2h)^i \leq K$ hold for all $n \geq n_0$ and $i=1,\dots,n$. Thus, from (12) and (13) Lemma 1 follows.

The estimate (11) implies

$$(14)\quad \|u_i\|^2 \leq C_1 + C_2 h \sum_{j=1}^{i} \|u_j\|^2_W , \quad \|u_{i,B}\|^2_{\Gamma_1} \leq C_1 + C_2 h \sum_{j=1}^{i} \|u_j\|^2_W$$

for all $n \geq n_0$ and $i=1,\dots,n$.

<u>Lemma 2.</u> Let $\varepsilon>0$. There exist $C_1(\varepsilon)$, $C_2(\varepsilon)$ such that

i) $A[u_i, u_i] \leq C_1(\varepsilon) + C_2(\varepsilon) \sum_{j=1}^{i} h \|u_j\|^2_W + \varepsilon \|u_{i-1}\|^2_W$;

ii) $|(b_2(t_i,x,u_{i-1,B}), u_i)_{\Gamma_2}| \leq C_1(\varepsilon) + C_2(\varepsilon) h \sum_{j=1}^{i} \|u_j\|^2_W + \varepsilon \|u_{i-1}\|^2_W$.

From (9) for $v \in C_0^\infty(\Omega)$ and (5) we conclude

$$(15)\quad |A[u_i, v]| \leq \left\| \frac{u_i-u_{i-1}}{h} \right\| \|v\| + C_1 + C_2 \|u_{i-1}\|_W \|v\| .$$

The estimate (15) takes place also for $v \in L_2(\Omega)$ and hence from (11), (14) and (15) for $v=u_i$ Assertion i) follows. Similarly, from (9), (11), (14) and Assertion i) we obtain Assertion ii).

<u>Lemma 3.</u> There exist C and n_0 such that the estimates

i) $\left\| \frac{u_i-u_{i-1}}{h} \right\| \leq C$, $\left\| \frac{u_{i,B}-u_{i-1,B}}{h} \right\|_{\Gamma_1} \leq C$;

ii) $\|u_i\|_W \leq C$;

iii) $\|u_i-u_{i-1}\|^2_W \leq \frac{C}{n}$

hold for all $n \geq n_0$, $i=1,\dots,n$.

Proof. From (9) for $v=u_i$, Lemma 1-2 and (4) we obtain

(16) $C \|u_i\|_W^2 \le C_1(\varepsilon)+C_2(\varepsilon)\sum_{j=1}^{i} h \|u_j\|_W^2 + \varepsilon \|u_{i-1}\|_W^2$.

The estimate

(17) $\|u_{i-1}\|_W^2 \le 2\|u_i\|_W^2 + 2\|u_i-u_{i-1}\|_W^2 \le 2\|u_i\|_W^2 + C_1 + C_2\sum_{j=1}^{i} h \|u_i\|_W^2$

take place because of Lemma 1. The estimates (16) and (17) imply

$$\|u_i\|_W^2 \le C_1 + C_2\sum_{j=1}^{i} h \|u_j\|_W^2$$

from which we obtain (see e.g.[8]) $\|u_i\|_W \le C$ and hence Lemma 3 follows. From the regularity results (in the interior of the domain Ω) for solutions of linear elliptic equations and Lemma 3 we obtain easily

(18) $\|u_i\|_{W_2^2(\Omega')} \le C(\Omega')$ for all n, $i=1,\dots,n$

where Ω' is an arbitrary subdomain of Ω, with $\overline{\Omega}' \subset \Omega$.

<u>Proof of Theorem 1.</u> Lemma 3 implies

(19) $\|u_n(t)-u_n(t')\| \le C|t-t'|$, $\|u_{n,B}(t)-u_{n,B}(t')\|_{\Gamma_1} \le C|t-t'|$

(20) $\left\|\frac{d^- u_n(t)}{dt}\right\| \le C$, $\left\|\frac{d^- u_{n,B}\, t}{dt}\right\|_{\Gamma_1} \le C$;

(21) $\|u_n(t)\|_W \le C$;

(22) $\|u_n(t)\|_{W_2^2(\Omega')} \le C$;

for all n, where $\frac{d^-}{dt}$ is the left hand derivative. From the compactness of the imbedding $W \to L_2(\Omega)$ and by the method of diagonalization we find outthat $u_n(t) \to u(t)$ in $L_2(\Omega)$ for all rational points t of $\langle 0,T\rangle$ (here $\{u_n(t)\}$ is a suitable subsequence ofthe original $\{u_n(t)\}$). Hence using (19) we obtain that there exist $u : \langle 0,T\rangle \to L_2(\Omega)$ such that $u_n(t) \to u(t)$ for all $t \in \langle 0,T\rangle$. Using the Borel covering theorem we find out that this convergence is uniform in $\langle 0,T\rangle$. Reflexivity of W, (21) and (22) imply $u \in L_\infty(\langle 0,T\rangle, W \cap W^2_{2,loc}(\Omega))$. Then,similarly we conclude $u_{n,B} \to u_B$ in the norm of the space $C(\langle 0,T\rangle, L_2(\Gamma_1))$. Hence and from (19) we obtain

(23) $\|u(t)-u(t')\| \le C|t-t'|$, $\|u_B(t)-u_B(t')\|_{\Gamma_1} \le C|t-t'|$.

Thus, applying the result of Y.Komura (see e.g.[1]) from (23)we obtain $\frac{du}{dt} \in L_\infty(\langle 0,T\rangle, L_2(\Omega))$ and $\frac{du_B}{dt} \in L_\infty(\langle 0,T\rangle, L_2(\Gamma_1))$. Let us denote

$x_n(t)=u_i$ for $t_{i-1}< t \le t_i$, $i=1,\dots,n$, $x_n(0)=u_0$,

$b_{j,n}(t,x,\xi)= b_j(t_i,x,\xi)$ for $t_{i-1} < t \le t_i$, $i=1,\dots,n$, $b_{j,n}(0,x,\xi)= b_j(0,x,\xi)$ where $\xi \in E^{N+1}$, $x \in \Omega$ for $j=0$ and $\xi \in E^1$, $x \in \Gamma_j$ for $j=1,2$. Using our notation the identity (9) can be rewitten into the form

$$(24)\ \left(\frac{d^- u_n\ t}{dt}, v\right)+\left(\frac{d^- u_{n,B}\ t}{dt}, v\right)_{\Gamma_1}+ A\left[x_n(t), v\right] +$$
$$+\left(b_{o,n}(t,x,x_n(t-\tfrac{T}{n}),\nabla x_n(t-\tfrac{T}{n})), v\right) + \sum_{j=1,2}\left(b_{j,n}(t,x,x_{n,B}(t-\tfrac{T}{n})), v\right)_{\Gamma_j} = 0$$

for all $t\in(\frac{T}{n},T)$ and n . Integrating (24) over $\langle\frac{T}{n},t\rangle$ and taking limit for $n \to \infty$ we obtain

$$(25)\ (u(t), v)+(u_B(t), v)_{\Gamma_1} -(\phi, v) - (\phi, v)_{\Gamma_1} + \int_0^t \{ A[u(s), v] +$$
$$+(b_0(s,x,u(s),\nabla u(s)), v) + \sum_{j=1,2}(b_j(s,x,u_B(s)), v)\}\, ds=0$$

for all $v\in W$ since we have the a priori estimates

$\|x_n(t)\|_W \le C$, $\|x_n(t)\|_{W_2^2(\Omega^-)} \le C(\Omega^-)$, $\|x_n(t)- u_n(t)\|\le \frac{C}{n}$ and

$\|x_n(t- \frac{T}{n}) - x_n(t)\|^2_W \le \frac{C}{n}$. From (25)we find out that $u(t)$ is a weak solution of (1)-(3). If $u_1(t)$, $u_2(t)$ are two solution of (1)-(3)then $u(t)= u_1(t) - u_2(t)$ satisfies

$$(26)\ \left(\frac{du}{dt}, v\right) + \left(\frac{du_B}{dt}, v\right)_{\Gamma_1}+ A\left[u, v\right] - C_1\|u\|\|v\| - C_2\|u\|_{\Gamma_1}\|v\|_{\Gamma_1} -$$
$$- C\|u\|_{\Gamma_2}\|v\|_{\Gamma_2} \le 0$$

for all $v\in W$ (C is from (7)). Substituing $u = v\exp(\lambda t)$ into (26) for sufficient large λ and using (7) we obtain

$$\frac{d}{dt}\|v\|^2+ \frac{d}{dt}\|v_B\|^2_{\Gamma_1} \le 0$$

which implies uniqueness since $v(0)=v_B(0) = 0$. From the uniqueness

we conclude that the original sequence $\{u_n(t)\}$ converges to $u(t)$ in $C(\langle 0,T\rangle, L_2(\Omega))$ and the proof is complete.
Due to the a priori estimates for $u_n(t)$ and $x_n(t)$ the stronger regularity results for $u(t)$ can be proved.
Let X be a reflexive Banach space with its dual space X^* and the pairing $(\cdot\,,\,\cdot)$. If $(w(t), v) \in C^1((0,T))$ and $\frac{d}{dt}(w(t), v) = (g(t), v)$ $(g(t) \in X)$ holds for all $v \in X^*$ then w is weakly differentiable in X (with respect to $t \in (0,T)$) and we denote $w'(t) = g(t)$.
Lemma 4. Let $u(t)$ be as in Theorem 1. Then

1) The function $u(\cdot)$ is weakly continuous in W and the estimates

$$\|u(t)\|_W \le C \quad , \quad \|u(t)\|_{W_2^2(\Omega')} \le C(\Omega')$$

hold for all $t \in (0,T)$;

2) The functions $A[u(t), v]$, $(b_0(t,x,u(t),\nabla u(t)), v)$ and $(b_j(t,x,u_B(t)), v)_{\Gamma_j}$ $(j=1,2)$ are continuous in $(0,T)$ for all $v \in W$;

3) The functions $u(t)$, $u_B(t)$ are weakly derivable in $L_2(\Omega)$, $L_2(\Gamma_1)$ respectively and $u'(\cdot)$, $u_B'(\cdot)$ are wekly continuous in the corresponding spaces. The estimate

$$(27) \quad \|u'(t)\| + \|u_B'(t)\|_{\Gamma_1} \le C$$

holds for all $t \in (0,T)$;

4) $u' = \frac{du}{dt}$, $u_B' = \frac{du_B}{dt}$ hold for a.e. $t \in (0,T)$ ($\frac{d}{dt}$ is the strong derivative) ;

5) The identity (8) (with u', u_B' instead of $\frac{du}{dt}$, $\frac{du_B}{dt}$) holds for all $t \in (0,T)$.

Assertion 1) is a consequence of the a priori estimates for $u_n(t)$, reflexivity of the spaces W, $W_2^2(\Omega')$,(23) and the uniqueness of $u(t)$. Assertion 2) is a consequence of Assertion 1) .From Assertion 2) and (25) we conclude $(u(t), v) \in C^1((0,T))$ for all $v \in L_2(\Omega)$ and $(u_B(t),v)_{\Gamma_1} \in C^1((0,T))$ for all $v \in L_2(\Gamma_1)$ which implies the existence of $u'(t)$, $u_B'(t)$. Hence and from (25) Assertion 5) follows. Due to (8) and (24) we conclude $\left(\frac{d^- u_n(t)}{dt}, v\right) \to (u'(t), v)$ for all $v \in L_2(\Omega)$, $t \in (0,T)$ and $\left(\frac{d^- u_{n,B}(t)}{dt}, v\right) \to (u'(t), v)$ for all $v \in L_2(\Gamma_1)$, $t \in (0,T)$.Thus the a priori estimates of $\frac{d^- u_n(t)}{dt}$, $\frac{d^- u_{n,B}(t)}{dt}$ imply (27). The identity(8) and Assertion 2) imply the weak continuity of $u'(\cdot)$ and $u_B'(\cdot)$ in

$L_2(\Omega)$, $L_2(\Gamma_1)$ respectively and Assertion 3) is proved.Asertion 4) is the well known result (see e.g. [6]) .

Using Theorem 1 ,Lemma 4 and a priori estimates for $u_n(t)$, $x_n(t)$ we can prove a stronger convergence results.

Theorem 2. Suppose (4)-(7).Let $u(t)$, $u_n(t)$ and $x_n(t)$ be as in Theorem 1.Then

i) $x_n(t) \to u(t)$ in the norm of the space W uniformly in $t \in \langle 0,T\rangle$;

ii) $u_n \to u$ in the norm of the space $C(\langle 0,T\rangle, W)$;

iii) $u(t)$ is a Hölder continuous function from $\langle 0,T\rangle \to W$. The estimate

$$\|u(t) - u(t')\|_W^2 \leq C\,|t-t'|$$

holds for all $t,t' \in \langle 0,T\rangle$.

For the proof we subtract (8) and (24) for $v = x_n(t) - u(t)$.Then, using Lemma 4 and a priori estimates for $x_n(t)$, $u_n(t)$ and $u(t)$ we estimate

$$(28)\quad C_E\,\|x_n(t) - u(t)\|_W^2 \leq C_1\,\|x_n(t) - u(t)\| + C_2\,\|x_{n,B}(t) - u_B(t)\|_{\Gamma_1} +$$

$$+ C_3 h\,\|x_{n,B}(t-\tfrac{T}{n}) - u_B(t)\|_{\Gamma_2} + C\,\|x_{n,B}(t-\tfrac{T}{n}) - u_B(t)\|_{\Gamma_2}\,\|x_{n,B}(t) - u_B(t)\|_{\Gamma_2}$$

where C is from (7). Due to (7) and Lemma 3 (iii)) we obtain

$$C\,\|x_{n,B}(t-\tfrac{T}{n}) - u_B(t)\|_{\Gamma_2}\,\|x_{n,B}(t) - u_B(t)\|_{\Gamma_2} \leq C\,C_I^2\,(\|x_n(t) - u(t)\|_W^2 +$$

$$+ \|x_{n,B}(t-\tfrac{T}{n}) - x_{n,B}(t)\|_W\,\|x_{n,B}(t) - u_B(t)\|_W) \leq C\,C_I^2\,(\|x_n(t) - u(t)\|_W^2 +$$

$$+ C\,\sqrt{h}\,).$$

Hence, from Theorem 1,(7) and (28) Assertion i) follows. Assertion ii) follows from Assertion i) Lemma 3 (iii)) and the estimate

$$\|u_n(t) - u(t)\|_W \leq 2\,\|x_n(t) - u(t)\|_W^2 \quad 2\,\|x_n(t) - u_n(t)\|_W^2 \leq$$

$$\leq 2\,\|x_n(t) - u(t)\|_W^2 + C_1\,\sqrt{h}\,.$$

Finally, subtracting (8) for $t = t$ and $t = t'$ and putting $v = u(t) - u(t')$ we obtain

$$\|u(t) - u(t')\|_W^2 \leq C_1\,|t-t'| + C_2\,\|u(t)\|_W\,|t-t'| + C\,\|u(t) - u(t')\|_{\Gamma_2}^2$$

where (23) and Lemma 4 has been used (C is from (7)).Hence and from the estimate $C \|u(t)-u(t')\|^2_{\Gamma_2} \le C\, C_I^2 \|u(t)-u(t')\|^2_W$ Assertion iii) follows.

Remark 2. In [1] a similar result is proved for the case of A being a nonlinear, monotone and coercive operator and $b_j(t,x,\xi)$ $j=0,1,2$ being monotone in ξ. Howewer in that case u_i $(i=1,\dots,n)$ are the solutions of a corresponding nonlinear elliptic boundary value problems.

Remark 3. All results hold true if either Γ_1 or Γ_2 is empty.

References

[1] J.Kačur : Nonlinear parabolic equations with the mixed nonlinear and nonstationary boundary conditions.Mathematica Slovaca, to appear.

[2] V.V.Barkovskij and V.L.Kulčickij : Generalized solutions of some mixed boundary value problems for Schrödinger's equation. Linear and nonlinear boundary value problems, AN USSR , Kiev, 1971 in Russian .

[3] V.L.Kulčickij : Regularity of solutions of some mixed boundary value problems for Schrödinger's equation.Linear and nonlinear boundary value problems, AN USSR,Kiev, 1971 in Russian .

[4] L.V.Ljubič : Solution of the heat conduction problem in a right dihedral angle with time derivatives in the boundary condition.Dokl.Akad.Nauk Ukrain.SSR Ser.A 1976,no.8,691-693 in Ukrainian .

[5] K.Rektorys : On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in the space variables. Czech.Math.J., 21 96 ,1971, 318-339.

[6] J.Kačur : Method of Rothe and nonlinear parabolic equations of arbitrary order. Czech.Math.J.,to appear.

[7] J.Nečas : Application of Rothe's method to abstract parabolic equations.Czech.Math.J.,Vol.24 99 ,1974,No3,496-500 .

[8] J.Kačur : Application of Rothe's method to nonlinear evolution equations.Mat.Časopis Sloven.Akad.Vied,25,1975,No 1,63-81.

[9] J.Kačur and A.Wawruch : On an approximate solution for quasilinear parabolic equations. Czech.Math.J.,27, 102 1977,220-241.

Author's address : Institute of applied mathematics,Mlýnska dolina, 816 31 Bratislava, Czechoslovakia