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A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION

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1. Introduction. We study the global existence, uniqueness and continuous dependence on data of smooth solutions of the initial value problem

$$y_{tt} + \alpha y_t - (\sigma(y_x))_x = g \qquad (0 < t < \infty,\ x \in \mathbb{R}) \ , \tag{1.1}$$

$$y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) \qquad (x \in \mathbb{R}) \ , \tag{1.2}$$

where the subscripts t, x denote partial differentiation, $\alpha > 0$ is a fixed constant, $\sigma : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $y_0, y_1 : \mathbb{R} \to \mathbb{R}$ are given smooth functions. We shall assume throughout that

$$\sigma \in C^2(\mathbb{R}),\ \sigma(0) = 0,\ \sigma'(\xi) \geq \varepsilon > 0 \qquad (\xi \in \mathbb{R};\ \varepsilon > 0) \ ; \tag{σ}$$

the case $\sigma''(\xi) \not\equiv 0$ is of primary interest.

If $\alpha = 0$, $g \equiv 0$ it is known [4], [7] that solutions of the Cauchy problem (1.1), (1.2) will in general develop singularities in the first derivatives even for smooth data, and smooth solutions will not exist for large t. If $\alpha > 0$, $g \equiv 0$ Nishida [10], has established the existence and uniqueness of global smooth solutions of (1.1) for smooth and sufficiently small data (1.2) by a remarkably simple method.

It is the purpose of this note to (i) extend Nishida's method to obtain the global existence and uniqueness of smooth solutions of (1.1), (1.2) with $g \not\equiv 0$, and (ii) study the continuous dependence of solutions of (1.1), (1.2) on the data y_0, y_1, g. The result (i) is implicit in a recent paper of MacCamy [5]; however, his proof of the analogue of the important Lemma 2.3 below is not entirely complete. The result (ii) is new.

We remark that our results (i) and (ii) can be used to obtain a local existence and uniqueness result for smooth solutions of the functional differential equation

$$y_{tt} + \alpha y_t - (\sigma(y_x))_x = G(y) \qquad (0 \leq t \leq T,\ x \in \mathbb{R}) \ , \tag{1.3}$$

subject to the initial condition (1.2), for some $T > 0$. In (1.3) G is a given mapping defined on a suitable function space, and G satisfies a Lipschitz type condition. While limitations of space do not allow us to present this problem in detail, we point out that if $F(g)$ denotes the solution of (1.1), (1.2) on $[0,T] \times \mathbb{R}$, then a solution of (1.3), (1.2) is a fixed point of the composition map K defined by $K(y) = F(G(y))$. Such a fixed point can be found with the aid of our continuous dependence result for smooth solutions of (1.1), (1.2) for sufficiently small data in a manner similar to the method we used with Crandall

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in [1] to solve a functional differential equation in which, however, the underlying problem was an evolution equation of parabolic, rather than hyperbolic type. The details will be presented in a forthcoming joint paper with C. Dafermos.

The Cauchy problem (1.3), (1.2) has arisen in certain applications in heat flow and viscoelastic motion for "materials with memory" studied by MacCamy [5], [6]; the functional G has the form

$$G(y)(t,x) = \Psi(t,x) + \beta y(t,x) + \int_0^t b(t-\tau)y(\tau,x)d\tau \;, \tag{1.4}$$

where Ψ is a real smooth function on $[0,\infty) \times \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |\Psi(t,x)| \in L^1(0,\infty) \cap L^\infty(0,\infty),\; \sup_{x\in\mathbb{R}} |\Psi_x(t,x)| \in L^\infty(0,\infty) \;,$$

$\beta > 0$ is a constant, and $b \in L^1(0,\infty;\mathbb{R})$, the value of G at (t,x) depends on the restriction of $y(\cdot,x)$ to $[0,t]$. In [5], [6] the interest is in the existence of global smooth solutions of the Cauchy problem (1.3), (1.4), (1.2); this is carried out by combining Nishida's method with certain delicate a priori estimates obtained by energy methods. However, the proof in [5], [6] appears to us to be incomplete, because the local existence problem for (1.3), (1.4), (1.2), which can be handled by the method outlined above, is essentially ignored.

In Section 2 we obtain the desired results for a "diagonal" strictly hyperbolic system of first order equations equivalent to (1.1), (1.2); the results for (1.1), (1.2) follow as an easy corollary and these are stated in Section 3. We acknowledge useful discussions with M. G. Crandall, C. Dafermos, and R. J. DiPerna during the preparation of this paper.

Finally, we mention related work of Matsumara [8], [9] received after the completion of this paper; the author generalizes Nishida's results for (1.1), (1.2) with $g \equiv 0$ from one space dimension to quasilinear hyperbolic equations in several space dimensions, and he obtains global existence of weak solutions and results concerning their decay (Nishida's method does not apply in this case).

2. Equivalent Systems and Preliminary Results. We assume that σ in (1.1) satisfies assumptions (σ). In addition, assume that g, and the initial functions y_0, y_1 in (1.1), (1.2) satisfy:

(g) $\quad g,\, g_x \in C([0,\infty) \times \mathbb{R}),\; g(t) = \sup_{x\in\mathbb{R}} |g(t,x)| \in L^\infty(0,\infty) \cap L^1(0,\infty)$,

$$g_1(t) = \sup_{x\in\mathbb{R}} |g_x(t,x)| \in L^\infty(0,\infty) \;,$$

(I) $\quad y_0 \in \beta^2(\mathbb{R}),\; y_1 \in \beta^1(\mathbb{R})$,

where β^m denotes the set of real functions with continuous and bounded derivatives up to and including order m.

Following Nishida [10] we reduce the Cauchy problem (1.1), (1.2) to the equivalent system (2.3) below. Putting $y_x = v$ and $y_t = w$ in (1.1), (1.2) yields the equivalent Cauchy problem

$$(2.1)\qquad \begin{cases} v_t - w_x = 0,\ w_t - \sigma'(v)v_x + \alpha w = g & (0 < t < \infty,\ x \in \mathbb{R}) \\ v(0,x) = y_0'(x),\ w(0,x) = y_1(x) & (x \in \mathbb{R}) \quad . \end{cases}$$

The eigenvalues of the matrix of (2.1)

$$\begin{pmatrix} 0 & 1 \\ -\sigma'(v) & 0 \end{pmatrix}$$

are $\lambda = -\sqrt{\sigma'(v)}$, $\mu = \sqrt{\sigma'(v)}$; by assumptions (σ), λ and μ are real and distinct so that (2.1) is a strictly hyperbolic problem in the region $\{(v,w) : v \in \mathbb{R}, w \in \mathbb{R}\}$. To diagonalize (2.1) introduce the Riemann invariants

$$(2.2)\qquad r = w + \phi(v),\ s = w - \phi(v),\ \phi(v) = \int_0^v \sqrt{\sigma'(\xi)}\, d\xi \quad ;$$

by (σ) the mapping $(v,w) \to (r,s)$ defined by (2.2) is one to one from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R} \times \mathbb{R}$. A simple calculation shows that (2.1) is equivalent to the Cauchy problem for the diagonal, strictly hyperbolic system

$$(2.3)\qquad \begin{cases} r_t + \lambda r_x + \frac{\alpha}{2}(r+s) = g & \\ s_t + \mu s_x + \frac{\alpha}{2}(r+s) = g & (0 < t < \infty;\ x \in \mathbb{R}) \\ r(0,x) = r_0(x),\ s(0,x) = s_0(x) & (x \in \mathbb{R}) \ , \end{cases}$$

where by (2.2) $\lambda = \lambda(r-s)$, $\mu = \mu(r-s) \in C'(\mathbb{R})$, and where by (2.1)

$$(2.4)\qquad r_0(x) = y_1(x) + \phi(y_0'(x)),\ s_0(x) = y_1(x) - \phi(y_0'(x)) \qquad (x \in \mathbb{R}) \quad ;$$

by assumptions (σ) and I, the initial data $r_0, s_0 \in \beta^1(\mathbb{R})$. It is also seen that if r, s is a smooth (β^1) solution of the problem (2.3) for $(t,x) \in \Omega \subseteq ([0,\infty) \times \mathbb{R})$, then y, determined by the relations $y_t = w(r,s)$, $y_x = v(r,s)$ (where v, w are uniquely determined by (2.2)), will be a smooth (β^2) solution of the Cauchy problem (1.1), (1.2) and conversely; we shall therefore deduce our results for (1.1), (1.2) from (2.3).

The following local result for (2.3) is known [2; Sec. 8], [3, Theorem VI]:

<u>Lemma 2.1. Let</u> $r_0, s_0 \in \beta^1(\mathbb{R})$, <u>let assumptions</u> (σ) <u>hold, and assume that</u> $g, g_x \in \beta^0$ <u>for</u> $(t,x) \in [0,T] \times \mathbb{R}$, <u>where</u> $T > 0$. <u>Then there exists a number</u> $0 < T_1 \le T$ <u>such that the Cauchy problem</u> (2.3) <u>has a unique smooth solution</u> $r, s \in \beta^1([0,T_1] \times \mathbb{R})$.

The objective of the next two lemmas is to obtain apriori estimates on r, s, r_x, s_x (and hence by (2.3) on r_t, s_t), independent of T, which enable us to continue the local β^1-solution in t by a standard method.

<u>Lemma 2.2. Let the assumptions of Lemma 2.1 hold. In addition, assume that</u> $g(t) = \sup_{x \in \mathbb{R}} |g(t,x)| \in L^1(0,\infty)$. <u>Define the a priori constant</u> $M_0 > 0$ <u>by</u>

$$M_0 = r_0 + s_0 + 2\int_0^\infty g(\xi)d\xi,\ r_0 = \sup_{x \in \mathbb{R}} |r_0(x)|,\ s_0 = \sup_{x \in \mathbb{R}} |s_0(x)| \ .$$

For as long as the β^1-solution r, s of (2.3) exists one has

(2.5) $$\sup_{x\in\mathbb{R}} |r(t,x)| \le M_0, \quad \sup_{x\in\mathbb{R}} |s(t,x)| \le M_0 .$$

Sketch of Proof. The proof is similar to that of [10, Lemma 1], [5, Lemma 6.2]. Define the λ and μ characteristics of (2.3) respectively by

(2.6) $$x = x_1(t,\beta) = \beta + \int_0^t \lambda d\tau, \quad x = x_2(t,\gamma) = \gamma + \int_0^t \mu d\tau \quad (\beta,\gamma \in \mathbb{R}) ,$$

where $\lambda = \lambda[r(t,x_1(t,\beta)) - s(t,x_1(t,\beta))]$, $\mu = \mu[r(t,x_2(t,\gamma)) - s(t,x_2(t,\gamma))]$.

Let $' = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}$, $` = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}$ denote differentiation along the $\lambda$ and $\mu$ characteristics respectively, thus $r'(t,x_1) = \frac{d}{dt} r(t,x_1(t,\beta))$, $s`(t,x_2) = \frac{d}{dt} s(t,x_2(t,\gamma))$. Equations (2.3) become the ordinary differential equations

(2.7) $$\begin{cases} \dfrac{d}{dt} r(t,x_1(t,\beta)) + \dfrac{\alpha}{2} (r(t,x_1(t,\beta)) + s(t,x_1(t,\beta))) = g(t,x_1(t,\beta)) \\ \dfrac{d}{dt} s(t,x_2(t,\gamma)) + \dfrac{\alpha}{2} (r(t,x_2(t,\gamma)) + s(t,x_2(t,\gamma))) = g(t,x_2(t,\gamma)) \; ; \end{cases}$$

note that solutions of (2.7) will exist for as long as the slopes λ, μ of the characteristics $x_1(t,\beta)$ and $x_2(t,\gamma)$ remain bounded. Put

$$R(t) = e^{\frac{\alpha}{2}t} \left[\sup_{x\in\mathbb{R}} |r(t,x)| + \sup_{x\in\mathbb{R}} |s(t,x)|\right]$$

$$r_0 = \sup_{x\in\mathbb{R}} |r_0(x)|, \quad s_0 = \sup_{x\in\mathbb{R}} |s_0(x)| .$$

Integrate each of the equations (2.9) using $r(0,x_1(0,\beta)) = r_0(\beta)$, $s(0,x_2(0,\gamma)) = s_0(\gamma)$ (see (2.3), (2.6)), add the resulting equations and take absolute values; a standard argument yields the inequality

(2.8) $$R(t) \le r_0 + s_0 + \frac{\alpha}{2}\int_0^t R(\xi)d\xi + 2\int_0^t e^{\frac{\alpha}{2}\xi} g(\xi)d\xi .$$

Gronwall's inequality applied to (2.8) gives

(2.9) $$R(t) \le (r_0+s_0)e^{\frac{\alpha}{2}t} + 2e^{\frac{\alpha}{2}t} \int_0^t g(\xi)d\xi ,$$

and thus finally

(2.10) $$\sup_{x\in\mathbb{R}} |r(t,x)| + \sup_{x\in\mathbb{R}} |s(t,x)| \le M_0$$

and the proof is complete.

Lemma 2.3. Let the assumptions of Lemma 2.1 and (g) be satisfied. Define the constant $D_1 > 0$ by

$$D_1 = r_0 + s_0 + \sup_{x\in\mathbb{R}} |r_0'(x)| + \sup_{x\in\mathbb{R}} |s_0'(x)| + \|g\|_{L^1(0,\infty)} + \|g\|_{L^\infty(0,\infty)} + \|g_1\|_{L^\infty(0,\infty)} .$$

For as long as the β^1*-solution* r, s *of* (2.3) *exists and if* D_1 *is sufficiently small, there exists a constant* $M_1 = M_1(D_1) > 0$ *where* $M_1(D_1) \to 0$ *as* $D_1 \to 0$, *such that*

$$\sup_{x\in\mathbb{R}} |r_x(t,x)| \le M_1, \quad \sup_{x\in\mathbb{R}} |s_x(t,x)| \le M_1 . \tag{2.11}$$

Sketch of Proof. (Compare [10, Lemma 2], [5, Lemma 6.3].) Differentiate the first equation in (2.3) obtaining (recall $\lambda = \lambda(r-s)$)

$$r_{xt} + \lambda r_{xx} = -\lambda_r r_x^2 - \lambda_s r_x s_x - \frac{\alpha}{2}(r_x+s_x) + g_x . \tag{2.12}$$

We remark that although Lemma 2.1 does not assert the existence of r_{xx} and r_{xt}, note that the left side of (2.12) is r_x' and this does exist for as long as the β^1-solution r, s of (2.3) exists. This observation also justifies the validity of equations (2.12)-(2.18) which follow. Since $\mu = -\lambda$ the second equation in (2.3) gives

$$s_x = \frac{s'}{2\lambda} + \frac{\alpha}{4\lambda}(r+s) - \frac{g}{2\lambda} \qquad \left(' = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}\right) . \tag{2.13}$$

Define

$$h = \frac{1}{2}\log(-\lambda(r-s)) . \tag{2.14}$$

Differentiating h along the λ-characteristic and using $\lambda_s = -\lambda_r$ gives

$$h' = \frac{\lambda_r}{2\lambda}\left(-\frac{\alpha}{2}(r+s) + g - s'\right) . \tag{2.15}$$

Substitution of (2.13), (2.15) into (2.12) yields

$$r_x' + \left(\frac{\alpha}{2} + \lambda_r r_x + h'\right) r_x = -\frac{\alpha}{4\lambda}s' - \frac{\alpha^2}{8\lambda}(r+s) + \frac{\alpha}{4\lambda}g + g_x ,$$

or equivalently

$$(e^h r_x)' + \left(\frac{\alpha}{2} + \lambda_r r_x\right)e^h r_x = \left(-\frac{\alpha}{4\lambda}s' - \frac{\alpha^2}{8\lambda}(r+s) + \frac{\alpha}{4\lambda}g + g_x\right)e^h . \tag{2.16}$$

Define the function z by

$$z(r-s) = \int_0^{r-s} \frac{\alpha}{4\lambda(\xi)} e^{h(\xi)} d\xi ; \tag{2.17}$$

then $z' = -\frac{\alpha^2}{8\lambda} e^h(r+s) + \frac{\alpha}{4\lambda}e^h g - \frac{\alpha}{4\lambda}e^h s'$ and (2.16) becomes

$$(e^h r_x)' + \left(\frac{\alpha}{2} + \lambda_r r_x\right)e^h r_x = z' + e^h g_x . \tag{2.18}$$

To integrate (2.18) along the λ-characteristic put

$$\left\{\begin{aligned} k(t) &= \frac{\alpha}{2} + \lambda_r(r(t,x_1(t,\beta)) - s(t,x_1(t,\beta)))r_x(t,x_1(t,\beta)) \\ \rho(t) &= r_x(t,x_1(t,\beta))\exp[h(t,x_1(t,\beta))] \\ p(t) &= z'(t,x_1(t,\beta)) + g_x(t,x_1(t,\beta))\exp[h(t,x_1(t,\beta))] . \end{aligned}\right. \tag{2.19}$$

Then

$$(2.20)\qquad \rho(t) = \rho(0)\exp[-\int_0^t k(\tau)d\tau] + \int_0^t p(\xi)\exp[-\int_\xi^t k(\tau)d\tau]d\xi .$$

Suppose we can show that for any solution r, s of (2.3)

$$(2.21)\qquad |\lambda_r r_x| \le \frac{\alpha}{4} .$$

Then $k(t) = \frac{\alpha}{2} + \lambda_r(\cdot)r_x(\cdot) \ge \frac{\alpha}{4}$ and by an easy calculation

$$(2.22)\qquad |\rho(t)| \le |\rho(0)| + 3 \sup_{0\le\tau\le t} |z(\tau,x_1(\tau,\beta))| + \frac{4}{\alpha} \sup_{0\le\tau\le t} |g_x(\tau,x_1(t,\beta))\exp h(\tau,x_1(\tau,\beta)| .$$

We next show that (2.21) holds for any solution r, s of (2.3), provided the constant $D_1 > 0$ is sufficiently small. Indeed, at $t = 0$ and for any $\beta \in \mathbb{R}$ $\lambda_r(r(0,\beta) - s(0,\beta))r_x(0,\beta)$ satisfies

$$(2.23)\qquad |r_0'(\beta)\lambda_r(r_0(\beta) - s_0(\beta))| \le \sup_{x\in\mathbb{R}} |r_0'(x)\lambda_r(r_0(x) - s_0(x))| < \frac{\alpha}{4}$$

provided $D_1 > 0$ is sufficiently small. By (2.19) $\lambda_r r_x = \lambda_r e^{-h}\rho$ and by (σ), (2.14) and Lemma 2.2 $|\lambda_r e^{-h}|$ is uniformly bounded in (t,x) by a constant $K_1(D_1) > 0$, where $K_1(D_1) \to 0$ as $D_1 \to 0$, for any solution r, s of (2.3). By (2.14), (2.17), (2.22) and Lemma 2.2 the quantity ρ is uniformly bounded for any solution r, s of (2.3) as follows:

$$(2.24)\qquad |\rho(t)| \le |\rho(0)| + 3 \sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} |z[r(\tau,x) - s(\tau,x)]| + \frac{4}{\alpha} \sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} \sqrt{-\lambda[r(\tau,x) - s(\tau,x)]}\, |g_x(\tau,x)|$$

$$\le K_2(D_1), \text{ where } K_2(D_1) \to 0 \text{ as } D_1 \to 0 .$$

Therefore $|\lambda_r r_x| \le K_1(D_1)K_2(D_1)$ uniformly in (t,x) for any solution r, s of (2.3). The assertion (2.21) holds at $t = 0$, for D_1 sufficiently small (by (2.23)). Choosing D_1 smaller if necessary so that $K_1(D_1)K_2(D_1) \le \frac{\alpha}{4}$, we conclude that (2.21) continues to hold for as long as the solution r, s exists. Returning to (2.22), (2.24) and using $r_x = \rho e^{-h}$ establishes the estimate for r_x in (2.11); a similar argument yields the estimate for s_x and completes the proof of Lemma 2.3.

The a priori estimates of Lemmas 2.2 and 2.3, together with $\|g\|_{L^\infty(0,\infty)} < \infty$ yield uniform a priori estimates for r_t, s_t, for any solution r, s of (2.3), provided $D_1 > 0$ is sufficiently small. Then Lemmas 2.1, 2.2, 2.3 and a standard continuation argument give the first part of the following global result for the Cauchy problem (2.3).

Theorem 2.1. Let the assumption (σ), (g) be satisfied, and let the initial data r_0, $s_0 \in \beta^1(\mathbb{R})$. If the constant D_1 (see Lemma 2.3) is sufficiently small, then the Cauchy problem (2.3) has a unique β^1-solution r, s for $0 \leq t < \infty$, $x \in \mathbb{R}$ and the a priori estimates (2.5), (2.11) are satisfied for $0 \leq t < \infty$.

Let the above assumptions be satisfied by initial data r_0, s_0 and $\bar{r}_0$, $\bar{s}_0$ and forcing functions g, $\bar{g}$; denote by r, s and $\bar{r}$, $\bar{s}$ the corresponding β^1-solutions of (2.3) on $[0,\infty) \times \mathbb{R}$. Define

$$\zeta(t) = \sup_{x\in\mathbb{R}} |r(t,x) - \bar{r}(t,x)| + \sup_{x\in\mathbb{R}} |s(t,x) - \bar{s}(t,x)| .$$

Then there exists a constant $M_2 = M_2(\sigma,M_0) > 0$ such that

$$(2.25) \quad \zeta(t) \leq e^{2M_2M_1t}(\zeta(0) + \int_0^t e^{-2M_2M_1\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)| d\tau) \quad (0 \leq t < \infty) ,$$

where M_0, M_1 are the bounds in (2.5), (2.11) respectively.

Remark. The continuous dependence result (2.25) also holds for local solutions r, s and $\bar{r}$, $\bar{s}$ on $[0,T_1] \times [0, \ )$ of the Cauchy problem (2.3) (see Lemma 2.1), but only for $0 \leq t \leq T_1$.

Proof of Theorem 2.1. It remains only to prove (2.25). If r, s and $\bar{r}$, $\bar{s}$ are β^1-solutions of (2.3) for the situation in the theorem, one has

$$(2.26) \quad \begin{cases} (r-\bar{r})_t + \lambda r_x - \bar{\lambda}\bar{r}_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} \\ (s-\bar{s})_t + \mu s_x - \bar{\mu}\bar{s}_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} \end{cases} \quad \begin{matrix} 0 < t < \infty \\ x \in \mathbb{R} \end{matrix}$$

subject to the initial conditions

$$(2.27) \quad r(0,x) - \bar{r}(0,x) = r_0(x) - \bar{r}_0(x), \; s(0,x) - \bar{s}(0,x) = s_0(x) - \bar{s}_0(x) \quad (x \in \mathbb{R})$$

where $\lambda = \lambda(r-s)$, $\mu = \mu(r-s)$, $\bar{\lambda} = \lambda(\bar{r}-\bar{s})$, $\bar{\mu} = \mu(\bar{r}-\bar{s})$. But

$$\lambda r_x - \bar{\lambda}\bar{r}_x = \lambda(r-\bar{r})_x + (\lambda-\bar{\lambda})\bar{r}_x$$

$$\mu s_x - \bar{\mu}\bar{s}_x = \mu(s-\bar{s})_x + (\mu-\bar{\mu})\bar{s}_x ;$$

Therefore (2.26) can be written as

$$(2.28) \quad \begin{cases} (r-\bar{r})_t + \lambda(r-\bar{r})_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} - (\lambda-\bar{\lambda})\bar{r}_x \\ (s-\bar{s})_t + \mu(s-\bar{s})_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} - (\mu-\bar{\mu})\bar{s}_x . \end{cases}$$

Recalling the definitions of λ, μ and using (σ) and the mean value theorem one has

$$\lambda - \bar{\lambda} = \lambda(r-s) - \lambda(\bar{r}-\bar{s}) = \frac{d\lambda}{d\xi}(\bar{r} - \bar{s} + \theta_1[r - s - (\bar{r}-\bar{s})])(r - s - (\bar{r}-\bar{s}))$$

$$\mu - \bar{\mu} = \mu(r-s) - \mu(\bar{r}-\bar{s}) = \frac{d\mu}{d\xi}(\bar{r} - \bar{s} + \theta_2[r - s - (\bar{r}-\bar{s})])(r - s - (\bar{r}-\bar{s})) ,$$

for some $0 < \theta_1, \theta_2 < 1$. Therefore, (2.28) becomes

$$(2.29)\quad \begin{cases} (r-\bar{r})' + \frac{\alpha}{2}(r-\bar{r}) = -\frac{\alpha}{2}(s-\bar{s}) - \frac{d\lambda}{d\xi}(\cdot)(r-\bar{r}-(s-\bar{s}))\bar{r}_x + g - \bar{g} \\ (s-\bar{s})^{\backprime} + \frac{\alpha}{2}(s-\bar{s}) = -\frac{\alpha}{2}(r-\bar{r}) - \frac{d\mu}{d\xi}(\cdot)(r-\bar{r}-(s-\bar{s}))\bar{s}_x + g - \bar{g}\,, \end{cases}$$

where $' = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}$, $^{\backprime} = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}$. We next note that assumption (σ) and Lemmas 2.2 and 2.3 imply the existence of a constant $M_2 = M_2(\sigma, M_0)$ such that

$$(2.30)\qquad \left|\frac{d\lambda}{d\xi}(\cdot)\bar{r}_x\right| \le M_2 M_1, \quad \left|\frac{d\mu}{d\xi}(\cdot)\bar{s}_x\right| \le M_2 M_1\,,$$

uniformly in $(t,x) \in [0,\infty) \times \mathbb{R}$ where M_1 is the bound in Lemma 2.3. Integrating the first equation in (2.29) along any λ-characteristic and the second along any μ-characteristic and making simple estimates one obtains the pair of inequalities

$$e^{\frac{\alpha}{2}t} \sup_{x\in\mathbb{R}} |r(t,x) - \bar{r}(t,x)| \le \sup_{x\in\mathbb{R}} |r_0(x) - \bar{r}_0(x)| + \int_0^t e^{\frac{\alpha}{2}\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)|d\tau$$
$$+ \int_0^t e^{\frac{\alpha}{2}\tau} (\tfrac{\alpha}{2} + M_2M_1) \sup_{x\in\mathbb{R}} |s(\tau,x) - \bar{s}(\tau,x)|d\tau + M_1M_2 \int_0^t e^{\frac{\alpha}{2}\tau} \sup_{x\in\mathbb{R}} |r(\tau,x) - \bar{r}(\tau,x)|d\tau\,,$$

$$e^{\frac{\alpha}{2}t} \sup_{x\in\mathbb{R}} |s(t,x) - \bar{s}(t,x)| \le \sup_{x\in\mathbb{R}} |s_0(x) - \bar{s}_0(x)| + \int_0^t e^{\frac{\alpha}{2}\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)|d\tau$$
$$+ \int_0^t e^{\frac{\alpha}{2}\tau} (\tfrac{\alpha}{2} + M_2M_1) \sup_{x\in\mathbb{R}} |r(\tau,x) - \bar{r}(\tau,x)|d\tau + M_2M_1 \int_0^t e^{\frac{\alpha}{2}\tau} \sup_{x\in\mathbb{R}} |s(\tau,x) - \bar{s}(\tau,x)|d\tau\,.$$

Adding these inequalities one obtains on using the definition of ζ

$$(2.31)\qquad \zeta(t)e^{\frac{\alpha}{2}t} \le \zeta(0) + 2\int_0^t e^{\frac{\alpha}{2}\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)|d\tau$$
$$+ \int_0^t (\tfrac{\alpha}{2} + 2M_1M_2)e^{\frac{\alpha}{2}\tau} \zeta(\tau)d\tau \qquad (0 \le t < \infty)\,.$$

Finally, applying Gronwall's inequality to (2.31) yields the result (2.25) completing the proof.

3. Global Existence, Uniqueness, and Continuous Dependence for the Cauchy Problem (1.1), (1.2).

As an immediate consequence of Theorem 2.1 and of the equivalence of the Cauchy problems (1.1), (1.2) and (2.3), (2.4) we obtain the main result of this paper.

Theorem 3.1. *Let the assumptions* (σ), (g), (I) *be satisfied. Define the constant*

$$(3.1)\quad D = \sup_{x\in\mathbb{R}} |y_0'(x)| + \sup_{x\in\mathbb{R}} |y_1(x)| + \sup_{x\in\mathbb{R}} |y_0''(x)| + \|g\|_{L^1(0,\infty)} + \|g\|_{L^\infty(0,\infty)} + \|g_1\|_{L^\infty(0,\infty)}\,.$$

If D *is sufficiently small, then the Cauchy problem* (1.1), (1.2) *has a unique* β^2*-solution* y *on* $[0,\infty) \times \mathbb{R}$, *and the solution* y *satisfies the a priori estimates:*

$$(3.2)\quad \sup_{x\in\mathbb{R}}|y_x(t,x)|,\ \sup_{x\in\mathbb{R}}|y_t(t,x| \le \sup_{x\in\mathbb{R}}|y_0'(x)| + \sup_{x\in\mathbb{R}}|y_1(x)| + 2\int_0^\infty g(\xi)d\xi = M_0$$
$$(0 \le t < \infty),$$

<u>and there exists a constant</u> $M_1 = M_1(D) > 0$ (which $\to 0$ as $D \to 0$) <u>such that</u>

$$(3.3)\quad \sup_{x\in\mathbb{R}}|y_{xx}(t,x)|,\ \sup_{x\in\mathbb{R}}|y_{xt}(t,x)|,\ \sup_{x\in\mathbb{R}}|y_{tt}(t,x)| \le M_1 \qquad (0 \le t < \infty).$$

<u>Let, in addition, the assumptions</u> (g) <u>and</u> (I) <u>be satisfied also by functions</u> $\bar{g}$ <u>and</u> $\bar{y}_0, \bar{y}_1$ <u>and let</u> $\bar{y}$ <u>denote the corresponding</u> β^1<u>-solution on</u> $[0,\infty) \times \mathbb{R}$. <u>Define</u>

$$\zeta(t) = \sup_{x\in\mathbb{R}}|y_x(t,x) - \bar{y}_x(t,x)| + \sup_{x\in\mathbb{R}}|y_t(t,x) - \bar{y}_t(t,x)|;$$

<u>then there exists a constant</u> $M_2 = (\sigma, M_0) > 0$ <u>such that</u>

$$(3.4)\quad \zeta(t) \le e^{2M_2M_1t}\left(\zeta(0) + \int_0^t e^{-2M_2M_1\tau}\sup_{x\in\mathbb{R}}|g(\tau,x) - \bar{g}(\tau,x)|d\tau\right) \qquad (0 \le t < \infty),$$

<u>where</u> M_0, M_1 <u>are bounds in</u> (3.2), (3.3) <u>respectively; moreover</u> (using $y(t,x) = y_0(x) + \int_0^t y_t(\tau,x)d\tau$)

$$\sup_{x\in\mathbb{R}}|y(t,x) - \bar{y}(t,x)| \le \sup_{x\in\mathbb{R}}|y_0(x) - \bar{y}_0(x)|$$
$$+ \frac{e^{2M_1M_2t}}{M_1M_2}\Big\{\sup_{x\in\mathbb{R}}|y_0'(x) - \bar{y}_0'(x)| + \sup_{x\in\mathbb{R}}|y_1(x) - \bar{y}_1(x)|$$
$$+ \int_0^t e^{-2M_1M_2\tau}\sup_{x\in\mathbb{R}}|g(\tau,x) - \bar{g}(\tau,x)|d\tau\Big\} \qquad (0 \le t < \infty).$$

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