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NUMERICAL IMPERFECTIONS NEAR A CRITICAL POINT

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ABSTRACT - The behaviour of a finite dimensional approximation of a nonlinear problem near a critical point is analysed from the point of view of contact equivalence.

0. The aim of this lecture is to present a short survey on the results obtained by the author in some recent papers. Reference should be made to [1]-[4] for a more detailed treatment. We shall deal with the following framework; assume that we are given:

(0.1) two Banach spaces, V,W

(0.2) a C^∞ mapping G from $V\times R^n (n\geq 1)$ into W

(0.3) a linear compact operator T from W to V

and consider the nonlinear problem:

$$(0.4)\quad \begin{cases} \underline{\text{find}}\ (u,\lambda)\in V\times R^n\ \underline{\text{such that}} \\ u+TG(u,\lambda)=0\,. \end{cases}$$

Assume moreover that we are given a sequence T_h of linear compact operators from W into V, such that

$$(0.5)\quad \lim_{h\to 0} \|T_h-T\|_{\mathcal{L}(W,V)}=0;$$

hence we may consider the "approximated problems":

$$(0.6)\quad \begin{cases} \underline{\text{find}}\ (u,\lambda)\in V\times R^n\ \underline{\text{such that}} \\ u+T_hG(u,\lambda)=0\,. \end{cases}$$

Our aim is to study the behaviour of the set of solutions of (0.6) (if any) in a neighbourhood of a critical point (u_o,λ_o) for (0.4).

Remark. In the practical cases (finite element methods, spectral methods and so on) the operators T_h will have a finite dimensional range V_h; hence, on the computer, the solution of (0.6) will be sought in $V_h\times R^n$. However, from the theoretical point of view, it will be easier to look for solutions of (0.6), a priori, in the whole space $V\times R^n$. On the other hand our theory will apply as well to different cases, in which the range of T_h is not finite dimensional: for instance we may assume that W is a compact subspace of V' (=dual space of V), that A is an isomorphism from V onto V' and that $T=A^{-1}$; if now A_h is a sequence of isomorphisms from V onto V' that G-converges to A, we may set $T_h=A_h^{-1}$ and (0.5) will be fullfilled.

1. Let now (u_o, λ_o) be a solution of (0.4) and consider the Fréchet derivative with respect to u of the mapping

$$(1.1) \qquad F(u,\lambda) \equiv u + TG(u,\lambda)$$

at the point (u_o,λ_o):

$$(1.2) \qquad L = D_u F^o = D_u F(u_o,\lambda_o).$$

By definition $L \in \mathcal{L}(V,V)$. If L is an isomorphism, the implicit function theorem will ensure the existence of a unique mapping $\lambda \to u(\lambda)$ through (u_o,λ_o) such that

$$(1.3) \qquad u(\lambda) + TG(u(\lambda),\lambda) = 0$$

identically in a neighbourhood of λ_o. It is easy to see that, in this case, problem (0.6) shows a similar behaviour for h small enough. Setting, as in (1.1)

$$(1.4) \qquad F_h(u,\lambda) \equiv u + T_h G(u,\lambda)$$

one can also prove (see e.g.[2]) the optimal error bound

$$(1.5) \qquad \|u_h(\lambda) - u(\lambda)\|_V \le c\|F_h(u(\lambda),\lambda)\|_V = c\|(T_h - T)G(u(\lambda),\lambda)\|_V$$

uniformly in a neighbourhood of λ_o independent of h. Obviously, in (1.5), $(u_h(\lambda),\lambda)$ is the solution of (0.6).

Let us turn now to a more interesting case; for this, assume that L, defined in (1.2), has a finite dimensional kernel. For the sake of simplicity we assume

$$(1.6) \qquad \dim(\ker(L)) = 1.$$

We say then that F has a simple critical point at (u_o,λ_o). It is proven in [3] that, in such case, the classical Lyapunov-Schmidt decomposition can be carried out on both F and F_h at the same time, giving rise to the reduced problems

$$(1.7) \qquad f(x,\lambda) = 0 \qquad f \in C^\infty(R \times R^n; R)$$

and

$$(1.8) \qquad f_h(x,\lambda) = 0 \qquad f_h \in C^\infty(R \times R^n; R).$$

Moreover f_h converges uniformly to f in a neighbourhood of the origin with all the derivatives, with no loss in the optimality of error bounds. See [3] for precise statements and details. From now on we shall assume that (1.7) and (1.8) are our original problems.

<u>Remark</u>. The setting of (0.6) in $V \times R^n$ instead of $V_h \times R^n$ could seem unimportant at first sight; however it is <u>crucial</u> in order to carry

out the L-S decomposition <u>for the two problems at the same time.</u>

2. Our setting, from now on, will be the following. We are given a mapping

(2.1) $f \in C^{\infty}(R\times R^n; R)$

and a sequence of mappings

(2.2) $f_h \in C^{\infty}(R\times R^n; R)$

converging to f with all the derivatives in a neighbourhood of the origin. We assume that the origin is a simple critical point for f, in the sense that

(2.3) $f(0,0)=f_x(0,0)=0$

and we look for the solutions of

(2.4) $f(x,\lambda)=0$

and

(2.5) $f_h(x,\lambda)=0$

in a neighbourhood of the origin.

We recall first the two basic concepts of "codimension" and of "contact equivalence" (see [6]) in the case n=1 (i.e. $\lambda\in R$).

<u>Definition</u> 2.1. <u>Let</u>

$G=\{\text{germs of } C^{\infty}(R^2; R)\}$, $G_{(\lambda)}=\{\text{germs of } C^{\infty}(R; R)\}$, ..

<u>let</u> $f\in G$ <u>and set</u>

$\tilde{T}f=\{g_o f+g_1 f_x \mid g_i\in G\}$

$Tf=\tilde{T}f \oplus \{g(\lambda)f_\lambda \mid g\in G_{(\lambda)}\}$;

<u>if</u> $G/\tilde{T}f$ <u>has finite dimension we define</u>

codim f = dim (G/Tf);

<u>otherwise we say that f has infinite codimension.</u>

<u>Definition</u> 2.2. <u>Let</u> f,g <u>be two germs in</u> G. <u>We say that</u> $f \overset{c.e.}{\simeq} g$ (f <u>is contact equivalent to</u> g) <u>if there exists</u> $\tau(x,\lambda)\in G$, $X(x,\lambda)\in G$ <u>and</u> $\Lambda(\lambda)\in G_{(\lambda)}$ <u>such that</u>

$\tau(0,0)\neq 0$, $X(0,0)=0$, $\Lambda(0)=0$, $\Lambda_\lambda(0)>0$, $X_x(0,0)>0$

<u>and</u>

$g(x,\lambda)=\tau(x,\lambda)f(X(x,\lambda), \Lambda(\lambda))$.

The following theorems are proved in [1].

Theorem 2.1. If f has codimension 0 then there exists a neighbourhood U of the origin, and an $h_o>0$ such that for all $h<h_o$ there exists a unique point (x_o^h,λ_o^h) in U such that

$$f_h(x+x_o^h,\lambda+\lambda_o^h)\overset{c.e.}{\simeq}f(x,\lambda).$$

Remark. Here and in the following, when speaking of the codimension of a function, we mean the codimension of the corresponding germ.

Theorem 2.2. Assume that f has codimension one, and let $g(x,\lambda,\mu)$ be a one-parameter universal unfolding of f, that is a C^∞ mapping $R^3\to R$ such that:

$$g(x,\lambda,0)\equiv f(x,\lambda),$$

$$G\equiv\{a+cb \mid a\in Tf,\ b=g_\mu(x,\lambda,0),\ c\in R\}.$$

Let $g_h(x,\lambda,\mu)$ be a sequence of C^∞ functions converging to g, with all the derivatives, in a neighbourhood of the origin. Then there exists a neighbourhood of the origin U and an $h_o>0$ such that for all $h<h_o$ there exists a unique point $(x_o^h,\lambda_o^h,\mu_o^h)$ in U such that.

$$g_h(x+x_o^h,\ \lambda+\lambda_o^h,\ \mu_o^h)\ \overset{c.e.}{\simeq}f(x,\lambda),$$

$g_h(x+x_o,\lambda+\lambda_o^h,\mu+\mu_o^h)$ is a universal unfolding of $g_h(x+x_o^h,\lambda+\lambda_o^h,\mu_o^h)$.

Remark. In both cases (see [1]) an estimate could be provided for the speed of convergence of the discrete critical point (x_o^h,λ_o^h) to the origin. An estimate for $|\mu_o^h|$ can also be found.

Remark. In the case of codimension one, in general, $f_h(x,\lambda)$ does not have itself a critical point. Theorem 2.2 shows that, from one hand, a small perturbation of f_h allows the recovery of the same type of criticality of f; from the other hand it shows that, for h small enough, the behaviour of f_h is similar to any universal unfolding of f for a suitable value of the perturbation parameter; finally it shows that, in some sense, the addition of a suitable perturbation parameter produces a g_h that matches perfectly the behaviour of g.

Remark. In [1] a guess is done that the result of theorem 2.2 should hold in a more general case: roughly speaking, for a problem of codimension k, the addition of k perturbation parameters should be necessary and sufficient in order to recover the whole bifurcation diagram in the discrete case; however this has not yet been proved at my knowledge.

3. I will recall now some results obtained in [1] on a particular case of codimension 2. For this assume now that

(3.1) $f(x,\lambda)=x^3-\lambda x$

and that the following two parameter universal unfolding is given

(3.2) $g(x,\lambda,\mu,\alpha)=x^3-\lambda x+\mu+\alpha x^2$.

Assume furthermore that g_h is sequence of C^∞ mappings from R^4 to R that converges to g in a neighbourhood of the origin with all the derivatives. The following result is proved in [1].

Theorem 3.1. There exists a neighbourhood U of the origin and an $h_o>0$ such that for all $h<h_o$ there exists a unique point $(x_o^h,\lambda_o^h,\mu_o^h,\alpha_o^h)$ in U such that:

$$g_h(x+x_o^h,\lambda+\lambda_o^h,\mu_o^h,\alpha_o^h)\overset{c.e.}{\simeq} f(x,\lambda),$$

$$g_h(x+x_o^h,\lambda+\lambda_o^h,\mu+\mu_o^h,\alpha+\alpha_o^h) \text{ is a u.u. of } g_h(x+x_o^h,\lambda+\lambda_o^h,\mu_o^h,\alpha_o^h)$$

$$g_h(x+x_o^h,\lambda_o^h,\mu+\mu_o^h,\alpha_o^h)\overset{c.e.}{\simeq} x^3+\mu.$$

Remark. The case

(3.3) $x^3-\lambda x+\mu=0$

is often present, in the applications, as a true two-parameter problem (see e.g. [5]). Although there is no definition, yet, of codimension in the case (n=2) of a two-parameter problem, theorem 3.1 suggests, somehow, that (3.3) behaves as a problem of codimension 1, at least from our point of view.

REFERENCES

[1] F. BREZZI, H. FUJII: Numerical imperfections and perturbations in the approximation of nonlinear problems. (MAFELAP IV, J. Whiteman ed. (Brunel, April 81) - (To appear).

[2] F. BREZZI, J. RAPPAZ, P.-A. RAVIART: Finite dimensional approximations of non linear problems: P.I. Branches of non singular solutions - Numer. Mat. - 36 (1980) 1-25.

[3] F. BREZZI, J. RAPPAZ, P.-A. RAVIART: Finite dimensional approximations of non linear problems: P. II Limit Points - Numer. Math. - 37 (1981) 1-28.

[4] F. BREZZI, J. RAPPAZ; P.-A. RAVIART: Finite dimensional approximations of non_linear problems: P. III Simple Bifurcation Points - Num. Math. 38(1981) 1-30.

[5] P.G. CIARLET, P. RABIER: Les équations de von Kármán. Springer (Berlin) 1980 - Lecture Notes in Mathematics n. 826.

[6] M. GOLUBITSKY, D. SCHAFFER: A theory for imperfect bifurcation via singularity theory. Comm. Pure Appl. Math. 32 (1979) 21-98.