ATTRACTORS OF SYSTEMS UNDER BOUNDED PERTURBATION

L. Farkas
Budapest, Hungary

The aim of this talk is to give explicit estimates to uniformly attracting sets of non-autonomous systems of differential equations that don't exhibit any special feature (such as, for example, periodicity, almost periodicity etc.) and to estimate at the same time the basin (the region of attractivity).

It will be assumed that the system considered is close to an autonomous or to a periodic one, the latter possessing an asymptotically stable equilibrium or an asymptotically stable periodic solution, respectively. "Close" means that the right hand sides are close in the C^0 topology, that is, their difference is less than a positive number η which may be small but is "finite" and is explicitly estimated too.

The results are based on a theorem of Yoshizawa [5] concerning the existence of an attractor of a perturbed system. The proofs of the results can be found in papers [1,2].

Consider first the case in which our system is close to an autonomous one.

Let $\Omega \subset R^n$ be an open set containing the origin, $R^+ = [0,\infty)$,

$$f \in C^0[R^+ \times \Omega, R^n], \quad f'_x \in C^0[R^+ \times \Omega, R^{n^2}], \quad g \in C^2[\Omega, R^n] ,$$

for any compact $Q \subset \Omega$ let $|f'_x|$ be bounded over $R^+ \times Q$ where $x = (x_1, \dots, x_n) \in R^n$, and consider the systems

$$\dot{x} = f(t,x) \tag{1}$$

and

$$\dot{x} = g(x) \tag{2}$$

where dot denotes differentiation with respect to $t \in R^+$. Assume further that there exists an $\eta > 0$ such that

$$|f(t,x) - g(x)| < \eta \quad , \quad (t,x) \in R^+ \times \Omega ,$$

$g(0) = 0$ and the real parts of all the eigenvalues of the matrix $g'(0)$ are negative. (We are <u>not</u> assuming that $f(t,0) \equiv 0$).

As it is well known there exists (and one can determine) a positive definite quadratic form $w(x)$ and constants $\rho_1 > 0$, $\rho_2 > 0$ such that

$$U_{\rho_1} = \{x \in R^n \mid |x| < \rho_1\} \subset \Omega \qquad \text{and}$$
$$\dot{w}_{(2)}(x) \le -\rho_2 w(x), \qquad x \in U_{\rho_1} .$$

Let us denote the least and the largest eigenvalues of the quadratic form $w(x)$ by λ_1 and λ_n, respectively ($0 < \lambda_1 \le \lambda_n$).

Introduce the ellipsoidal sets

$$A_\eta = \{x \in R^n \mid w(x) \le 4\lambda_n^2 \eta^2 / \rho_2^2 \lambda_1\}$$
$$B = \{x \in R^n \mid x(x) < \lambda_1 \rho_1^2\} .$$

<u>Theorem</u> 1. Under the conditions imposed above if

$$0 < \eta < \lambda_1 \rho_1 \rho_2 / 2\lambda_n$$

then $A_\eta \subset B \subset \Omega$, the set $R^+ \times A_\eta$ is a uniform asymptotically stable invariant set of system (1) and its basin contains the set $R^+ \times B$.

Now, replace system (2) by

$$\dot{x} = g(t,x) \qquad (3)$$

where $g, g'_x, g''_{xx} \in C^0[R^+ \times \Omega]$ (now Ω does not have to contain the origin), let g be periodic in t with period $\tau > 0$ and assume that there exists an $\eta > 0$ such that

$$|f(t,x) - g(t,x)| < \eta \quad , \quad (t,x) \in R^+ \times \Omega \quad .$$

Assume further that (3) has a periodic solution p with period τ and all the characteristic multipliers of the variational system

$$\dot{z} = g'_x(t,p(t))z \qquad (4)$$

are in modulus less than one.

Let us denote by Γ the path of p , i.e. $\Gamma = \{x \in R^n \mid x = p(t), t \in R^+\}$ and let $\varrho_1 > 0$ be such that

$$U(\Gamma, \varrho_1) = \{x \in R^n \mid \varrho(x,\Gamma) \le \varrho_1\} \subset \Omega$$

(here $\varrho(x,\Gamma)$ is the Euclidean distance of x from the set Γ).
Set

$$h(t,z) = g(t,p(t)+z) - g(t,p(t)) - g'_x(t,p(t))z \quad ,$$

more exactly let h be the Lagrangian remainder of order two of g. From our assumptions follows that a $\varrho_2 > 0$ exists (and can be determined by estimating the second partial derivatives of g) such that

$$|h(t,z)| < \varrho_2 |z|^2 \ , \quad t \in R^+, |z| < \varrho_1 \quad .$$

By Floquet's theory the periodic linear system (4) is reducible, that is, there exists a continuously differentiable, regular, τ-periodic matrix function S(t) such that the transformation $z = S(t)y$ carries (4) into a linear system with constant coefficients

$$\dot{y} = By \tag{5}$$

which is asymptotically stable. Let us denote by $\lambda_s(t)$ the least eigenvalue of the periodic positive definite matrix $S^{-1T}(t)S^{-1}(t)$
Clearly, $\lambda = \min_{t \in R^+} \lambda_s(t) > 0$. One can determine a positive definite quadratic form w(y) and a constant $\varrho_3 > 0$ such that

$$\dot{w}_{(5)}(y) \le -\varrho_3 |y|^2 \quad , \quad y \in R^n \quad .$$

Define $v(t,z) = w(S^{-1}(t)z)$ which is a positive definite quadratic form in z whose coefficients are periodic functions of t. Let us denote the least and largest eigenvalues of v(t,z) by $\lambda_{1v}(t)$ and $\lambda_{nv}(t)$ respectively. Clearly,

$$0 < \lambda_1 = \min_{t \in R^+} \lambda_{1v}(t) \le \lambda_n = \max_{t \in R^+} \lambda_{nv}(t) \quad .$$

Finally, let us introduce the notations

$$\alpha_1 = \lambda_1 \left(\min\left(\rho_1, \rho_3\lambda / 4\lambda_n\rho_2\right)\right)^2, \quad \alpha_2 = \lambda_n \left(4\lambda_n\eta / \rho_3\lambda\right)^2$$

and the sets

$$A_\eta = \left\{(t,x) \in R^+ \times R^n \,\middle|\, v(t,x-p(t)) \le \alpha_2\right\}$$
$$B = \left\{(t,x) \in R^+ \times R^n \,\middle|\, v(t,x-p(t)) < \alpha_1\right\}.$$

<u>Theorem</u> 2. If the conditions imposed upon systems (1), (3) and (4) hold and

$$0 < \eta < \left(\lambda_1/\lambda_n\right)^{1/2} \min\left(\frac{\rho_1\rho_3\lambda}{2\lambda_n}, \frac{\rho_3^2\lambda^2}{8\lambda_n^2\rho_2}\right)$$

then $A_\eta \subset B \subset R^+ \times \Omega$, the set A_η is a uniform asymptotically stable invariant set of system (1) and its basin contains the set B.

The first Theorem can be applied to the perturbed van der Pol equation

$$\ddot{u} + m(1-u^2)\dot{u} + u = F(t,u,\dot{u}), \quad m > 0 \qquad (6)$$

and the second to the perturbed Duffing's equation

$$\ddot{u} = -k^2u + m(-b\dot{u} + cu) + a\sin t + F(t,u,\dot{u})$$

(see [3]).

<u>References</u>

[1] Farkas, M.: Attractors of systems close to autonomous ones (to appear).

[2] Farkas, M.: Attractors of systems close to periodic ones, Nonlinear Analysis, 5.(1981) 845–851.

[3] Farkas, M.: The attractor of Duffing's equation under bounded perturbation (to appear in Annali di Matematica).

[4] Малкин, И.Г.: Некоторые задачи теории нелинейных колебаний, Гостехиздат, 1956, Москва.

[5] Yoshizawa, T.: Stability theory by Liapunov's second method, The Math. Soc. Japan, 1966, Tokyo.