## EQUADIFF 5

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Attractors of systems under bounded perturbation

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The aim of this talk is to give explicit estimates to uniformly attracting sets of non-autonomous systems of differential equations that don't exhibit any special feature (such as, for example, periodicity, almet periodicity etc.) and to estimate at the same time the basin (the region of attractivity).

It will be assumed that the eyetem considered is close to an autonomous or to a periodic one, the latter posseseing an aeymptotically stable equilibrium or an asymptotically etable periodic solution, respectively- "Close" means that the right hand sides are close in the $C^{0}$ topology, that is, their difference ie lese than a positive number $\eta$ which may be small but ie "finite" and is explicitly estimated too.

The results are based on a theorem of Yoshizawa [5] concerning the existence of an attrantor of a perturbed system. The proofa of the results can be found in papers [1,2].

Consider firet the case in which our system is nlose to an autonomous one.

Let $\Omega \in R^{n}$ be an open set oontaining the orisin, $R^{+}=[0, \infty)$,

$$
f \in C^{0}\left[R^{+} \times \Omega, R^{n}\right], f_{x}^{\prime} \in c^{0}\left[R^{+} \times \Omega, R^{n^{2}}\right], g \in c^{2}\left[\Omega, R^{n}\right]
$$

for any compact $Q \subset \Omega$ let $\left|f_{x}\right|$ be bounded over $R^{+} X Q$ where $x=\left(x_{1}, \ldots, x_{n}\right) \in R^{n}$, and coneider the systems

$$
\begin{equation*}
\dot{x}=f(t, x) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{x}=g(x) \tag{2}
\end{equation*}
$$

where dot denotes differentiation with reepect to $t \in R^{+}$. Aesume further that there exists an $\eta>0$ such that

$$
|f(t, x)-g(x)|<\eta, \quad(t, x) \in R^{+} \times \Omega
$$

$g(0)=0$ and the real parte of all the aigenvalues of the matrix $g^{\prime}(0)$ are ne:ative. (ise are not assuming that $f(t, 0) \equiv 0$ ).

As it ie well known there exists (and one can determine) a positive ueifinite quadratic form $w(x)$ and constants $\rho_{1}>0$, $S_{2}>0$ such that

$$
\begin{aligned}
& U_{\rho_{1}}=\left\{x \in R^{n}| | x \mid<\rho_{1}\right\} \subset \Omega \\
& { }^{w_{(2)}}(x) \leq-\rho_{2}:(x), \quad x \in U_{\rho_{1}}
\end{aligned}
$$

Let us den?te the least and the largest eirenvalues of the quadratic form $w(x)$ by $\lambda_{1}$ and $\lambda_{n}$, respectively ( $0<\lambda_{4} \leq \lambda_{n}$ ). Introuuce the ellipenidal eets

$$
\begin{aligned}
& A^{A^{\prime}}=\left\{x \in R^{n} \mid\right. \\
&\left.B(x) \leq 4 \lambda_{n}^{2} \eta^{2} / \rho_{2}^{2} \lambda_{1}\right\} \\
&=\left\{x \in R^{n} \mid\right. \\
&\left.x(x)<\lambda_{1} \rho_{1}^{2}\right\} .
\end{aligned}
$$

Theorem 1. Under the conditions imposed above if

$$
c<\eta<\lambda_{1} \rho_{1} Q_{2} / 2 \lambda_{n}
$$

then $A_{\eta} \subset B C \Omega$, the eet $R^{+} \times A_{\eta}$ is a uniform asymptotically stable inveriant eot of eystem (1) and ita basin contains the set $\mathrm{R}^{+} \times \mathrm{B}$.

Now, replace system (2) by

$$
\begin{equation*}
\dot{x}=g(t, x) \tag{3}
\end{equation*}
$$

where $g, g_{x}^{\prime}, g_{x x}^{n} \in C^{0}\left[R^{+} \times \Omega\right]$ (now $\Omega$ does not have to contain the origin), let $B$ be periodic in $t$ with period rCo and assume that there exists an $\quad \eta>0$ such that

$$
|f(t, x)-g(t, x)|<\eta \quad, \quad(t, x) \in R^{+} \times \Omega
$$

Aseume further that (3) has a periodic eolution $p$ ith period $T$ and all the characteristic multipliers of the variational system

$$
\begin{equation*}
\dot{z}=B_{X}^{\prime}(t, p(t)) z \tag{4}
\end{equation*}
$$

are in modulus less than one.
Let us denote by $\Gamma$ the path of $p, i .9 . \Gamma=\left\{x \in R^{n} \mid x=p(t), t \in R^{+}\right\}$ and let $Q_{1}>0$ be such that

$$
\psi\left(\Gamma, \rho_{1}\right)=\left\{x \in R^{n}\left\{\rho(x, \Gamma) \leq \rho_{1}\right\} \subset \Omega\right.
$$

(here $\rho(x, \Gamma)$ is the Euclidean distance of $x$ from the set $\Gamma$ ). Set

$$
h(t, z)=g(t, p(t)+z)-g(t, p(t))-g_{x}^{\prime}(t, p(t)) z,
$$

more exactly let $h$ be the Lagrangian remainder of order two of $g$. From our assumptions follows that a $\rho_{2}>0$ exists (and can be determined by estimating the second partial derivatives of g) such that

$$
|h(t, z)|<\rho_{2}|z|^{2}, \quad t \in R^{+},|z|<\rho_{1} .
$$

By Floquet'e theory the periodic linear fy stem (4) is reducible, that ie, there exists a continuously differentiable, regular, $Z$-periodic matrix function $S(t)$ such that the transformation $z=S(t) y$ carries (4) into a linear system with constant coefficients

$$
\begin{equation*}
\dot{y}=B y \tag{5}
\end{equation*}
$$

which is asymptotically stable. Let us denote by $\lambda_{s}(t)$ the least eigenvalue of the periodic positive definite matrix $S^{-1 T}(t) S^{-1}(t)$ clearly, $\lambda=\min _{t \in \mathbb{R}^{+}} \lambda_{b}(t)>0$. One can determine a positive definite quadratic form $w(y)$ and a constant $\rho_{3}>0$ such that

$$
\dot{w}_{(5)}(y) \leq-\rho_{3}|y|^{2}, y \in R^{n}
$$

Define $v(t, z)=w\left(S^{-1}(t) z\right)$ which is a positive definite quadratic form in $z$ those coefficients are periodic functions of t. Let us denote the least and largest eigenvalues of $v(t, z)$ by $\lambda_{0}(t)$ and $\lambda_{\text {se o }}(t)$ respectively. clearly,

$$
0<\lambda_{1}=\min _{t \in R^{+}} \lambda_{10}(t) \leq \lambda_{n}=\max _{t \in R^{+}} \lambda_{n \delta}(t)
$$

Finally, let us introduce the notations

$$
x_{1}=\lambda_{1}\left(\min \left(\rho_{1}, \rho_{3} \lambda / 4 \lambda_{n} \rho_{2}\right)\right)^{2}, \alpha_{2}=\lambda_{n}\left(4 \lambda_{n} 4 / \rho_{3} \lambda\right)^{2}
$$

and the sets

$$
\begin{aligned}
& A_{\eta}=\left\{(t, x) \in R^{+} \times R^{n} \mid v(t, x-p(t)) \leq \alpha_{2}\right\} \\
& B=\left\{(t, x) \in R^{+} \times R^{n} \mid v(t, x-p(t))<\alpha_{1}\right\} .
\end{aligned}
$$

Theorem 2. If the conditions imposed upon systems (1),
(3) and (4) hold and

$$
0<\eta<\left(\lambda_{1} / \lambda_{n}\right)^{1 / 2} \min \left(\frac{\rho_{1} \rho_{3} \lambda}{2 \lambda_{n}}, \frac{\rho_{3}^{2} \lambda^{2}}{8 \lambda_{n}^{2} \rho_{2}}\right)
$$

then $A_{q} \subset B C R^{+} \times \Omega$, the set $A_{q}$ is a uniform asymptotically stable invariant set of system (1) and its basin contains the set B.

The first Theorem can be applied to the perturbed van der Pol equation

$$
\begin{equation*}
\ddot{u}+m\left(1-u^{2}\right) \dot{u}+\dot{u}=F(t, u, \dot{u}) \quad, m>0 \tag{6}
\end{equation*}
$$

and the second to the perturbed Duffing's equation

$$
\ddot{u}=-k^{2} u+m(-b \dot{u}+c u)+a s \operatorname{int}+F(t, \dot{u}, \dot{u})
$$

(see [31).

## References

[1] Parkas, Lis: Attractors of systems close to autonomous ones (to appear).
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