ON THE BEHAVIOUR OF THE GENERALIZED SOLUTION OF THE BIHARMONIC EQUATION IN TWO DIMENSIONS NEAR SINGULAR POINT OF THE BOUNDARY

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The results of this contribution were obtained commonly by O. A. Olejnik, V. A. Kondratjev from Moscow University and me and will be published in [1].

1. Notations and definitions

Let $\Omega \subset R^2$ be a bounded domain, $\Gamma \subset \delta\Omega$. By $H^2(\Omega, \Gamma)$ we denote the closure in the norm $\|u\|^2 = \int_\Omega \sum_{|\alpha| \leq 2} |D^\alpha u|^2 \, dx$ of the set of functions from $C^\infty(\overline{\Omega})$, vanishing in some neighbourhood of Γ. By generalized solution of the problem

(1) $\Delta^2 u = f$ in Ω, $u = \delta u/\delta n = 0$ on Γ

we understand the function $u \in H^2(\Omega, \Gamma)$, satisfying the identity

(1') $$\int_\Omega \sum_{|\alpha|=2} D^\alpha u \cdot D^\alpha v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all} \quad v \in H^2(\Omega, \delta\Omega).$$

2. The main theorem

<u>Theorem.</u> For Ω, Γ defined above let 0 be a point from Γ. Suppose that for some $T > 0$ the assumptions

i) $\delta\Omega \cap \{x \in R^2, \|x - 0\| = t\} \neq \emptyset$ for all $t \in (0, T)$,

ii) $\delta\Omega \cap \delta\Omega_T \subset \Gamma$

hold, where $\Omega_t = \Omega \cap \{x, \|x - 0\| < t\}$.
Then there exist constants C_i depending on T (but not on Ω and u) such that for arbitrary generalized solution u of (1) with $f = 0$ in Ω_T following estimates hold

(3) $$\int_{\Omega_t} E(u) \, dx \leq C_1 \cdot t^{0,304} \int_{\Omega_T} E(u) \, dx \quad \text{for} \quad t \in (0, T),$$

(4) $$|u(x)|^2 \leq C_2 \|x - 0\|^{2,304} \int_{\Omega_T} E(u) \, dx \quad \text{for} \quad x \in \Omega, \|x - 0\| \leq T/2,$$

where $E(u) = u^2_{x_1x_1} + 2\, u^2_{x_1x_2} + u^2_{x_2x_2}$.

If in addition to above assumptions Ω satisfies

iii) for each $x^o \in \Gamma$, $\|x^o - 0\| \leq T/2$ the circle $\|x - x^o\| = t$ intersects Γ for all $t \in (0, \|x^o - 0\|/2\rangle$,

then the estimate

$$(4') \quad |\operatorname{grad} u(x)|^2 \leq C_3 \|x - 0\|^{0,304} \int_{\Omega_T} E(u)\, dx$$

holds too.

3. The sketch of the proof of the theorem

We suppose $0 = (0, 0)$ and introduce polar coordinates r and φ.

a) for $\phi \in C^1(0, T\rangle$, $\phi' \leq 0$, ϕ'' piecewise continuous on $(0, T\rangle$, $\phi(t) = 1$, $\phi'(T) = 0$ let $\tilde{\phi}(r, t) = \phi(r)$ for $t \leq r \leq T$, $\tilde{\phi} = 1$ for $r \geq T$, $\tilde{\phi}$ linear on $(0, t)$ and in $C^1(0, T)$. Substituting in (1') $v = u.(\tilde{\phi}(r, t) - 1)$, we get after some integrations by parts the identity

$$(5) \quad \int_{\Omega_t} \{[\phi(t) + (r - t)\cdot\phi'(t)]\cdot E(u) + r^{-1}|\phi'(t)|\cdot K_2(u)\}\, dx$$

$$+ \int_{\Omega_T \setminus \Omega_t} [\phi(r)\cdot E(u) - \phi''(r)\cdot K_1(u) - r^{-1}\phi'(r)\cdot K_2(u)]\, dx \leq \int_{\Omega_T} E(u)\, dx$$

with $K_1(u) = u_r^2 + (r^{-1}u_\varphi)^2 - u\cdot u_{rr}$, $K_2(u) = u_r^2 + 2(r^{-1}u_\varphi)^2 - r^{-1}u\cdot u_r$.

b) If ϕ moreover satisfies

$$(6) \quad \int_{S_r} [E(u)\phi(r) - \phi''(r)\cdot K_1(u) - r^{-1}\phi'(r)\cdot K_2 u]\, d\varphi \geq 0$$

for all $r \in (0, T)$ and all $u \in C^2$ in a neighbourhood of $\tilde{S}_r$, vanishing with first derivatives in points of $\delta\Omega \cap \delta\Omega_T$ [by S_r we denote the set of φ such that the point $x = (r\cos\varphi, r\sin\varphi)$ belongs to Ω, by $\tilde{S}_r$ the set of all such points x) we get from (5) using $\phi' \leq 0$, $\phi \geq 1$, $\int_{S_r} K_2(u)\, d\varphi \geq 0$ the estimate

$$(7) \quad \int_{\Omega_t} E(u)\, dx \leq (\phi(t))^{-1} \int_{\Omega_T} E(u)\, dx.$$

c) Now we shall construct the function ϕ. Denoting $z = r\cdot u_{rr}$, $w = r^{-1}\cdot u$, $v = u_r$, we see, that ϕ will satisfy (6) if it satisfies

$$(8) \quad \int_0^{2\pi} (\mathcal{E}(w, v, z)\cdot\phi(r) - r^2\phi''\mathcal{K}_1(w, v, z) - r\cdot$$
$$\cdot\phi'\mathcal{K}_2(w, v, z))\, d\varphi \geq 0$$

for all $r \in (0, T)$ and all $w \in \mathring{H}^2(0, 2\pi)$, $v \in \mathring{H}^1(0, 2\pi)$, $z \in L_2(0, 2\pi)$ where $\mathcal{E}(w, v, z) = z^2 + (w'' + v)^2 + 2(v' - w')^2$, $\mathcal{K}_1 = v^2 + (w')^2 - w\cdot z$,

$\mathcal{K}_2 = v^2 + 2(w')^2 - v.w$. (We use that $S_r \subset (0, 2\pi)$ and the equality $E(u) = u_{rr}^2 + r^{-2}((r^{-1}.u_{\varphi\varphi} + u_r)^2 + 2(u_{r\varphi} - r^{-1}.u_{\varphi})^2)$.)
For fixed w, v, z with $\mathcal{E}(w, v, z) = 1$ (8) will be satisfied by the function

$$(9) \quad (\beta(r/T)^{-\alpha} + \alpha(r/T)^{\beta})/(\alpha+\beta)$$

where

$$(9') \quad \alpha = \frac{-(k_1 - k_2) + \sqrt{(k_1 - k_2)^2 + 4\,k_1}}{2\,k_1}, \qquad \beta = \frac{(k_1 - k_2) + \sqrt{(k_1 - k_2)^2 + 4\,k_1}}{2\,k_1}$$

which solves the equation

$$(10) \quad r^2\phi''(r)k_1 + r\phi'(r)k_2 = \phi, \; \phi(T) = 1, \phi'(T) = 0$$

with $k_i = \int_0^{2\pi} \mathcal{K}_i(w, v, z)\, d\varphi$. (We may assume $k_1 > 0$.)
The function (9) satisfies $\phi' \leq 0, \phi'' \geq 0, \phi''(r) + r^{-1}\phi'(r) \geq 0$ for $0 < r \leq T$. Using this it is easily seen that if ϕ satisfies (10) then it satisfies

$$(10') \quad r^2\phi''(r)\tilde{k}_1 + r\phi'(r)\tilde{k}_2 \leq \phi(r)$$

with $\tilde{k}_1$, $\tilde{k}_2$ satisfying one of the following pairs of conditions

$$\tilde{k}_1 = k_1, \quad \tilde{k}_1 \leq k_1, \quad \tilde{k}_1 \leq k_1,$$
$$\tilde{k}_2 \geq k_2, \quad \tilde{k}_2 = k_2, \quad \tilde{k}_1 - \tilde{k}_2 = k_1 - k_2.$$

Now we prove the estimates

$$\max k_1(w, v, z) = \gamma_1 \leq 3{,}2679,$$
$$\max (k_1(w, v, z) - k_2(w, v)) \leq 2{,}29395$$

where the max are taken on the set $\{w,v,z, \int_0^{2\pi} \mathcal{E}(w,v,z)\, d\varphi = 1\}$.
So the function (9) corresponding to $k_1 = 3{,}2679$ and $k_2 = 3{,}2679 - 2{,}29395$ satisfies (8) for all w, v, z. Calculating the corresponding α from (9'), we obtain $\geq 0{,}304$. So the estimate (3) is proved.
(4) follows from (3) e.g. by imbedding theorem.

<u>Remark 1</u>. From results of [2], [3] it follows that for $\Omega = \{x \in R^2, x = (x_1, x_2), \|x\| \neq x_1\}$ the estimate $|u(x)|^2 \leq C|x|^3$ holds for arbitrary generalized solution considered in theorem proved above. It is an open problem if such estimate is valid for general Ω as in our theorem.

Remark 2. One may derive other sufficient conditions for ϕ to satisfy (6) e.g.

(I) $\phi(T) = 1, \phi'(T) = 0, \phi' \leq 0, \phi'' \geq 0, \phi''(r) + r^{-1}\phi'(r) \geq 0,$
$0 < r \leq T,\ 2(\ell(r)/\pi)^2(\phi''(r) + r^{-1}\phi'(r)) + (\ell(r)/2\pi)^2\phi''(r)$
$+ 4\ r(\ell(r)/2r\pi)^3 |\phi'| \leq \phi$

or

(II) instead of inequality from (I) one assume the inequality

$$(2 + 2\sqrt{2})(\ell(r)/2\pi)^2\phi''(r) \leq \phi\ (r),\quad 0 < r \leq T,$$

where $\ell(r)$ is the length of largest arc in $\tilde{S}_r$.

These conditions yield in the case of general domain not so good estimates as in our theorem, but in special cases of Ω may give e.g. exponential decay of $\int_{\Omega_t} E(u)\,dx$ as $t \to 0$, since they exploit the form of Ω.

4. Further results

a) From above results one may derive uniqueness theorems for Dirichlet problem for $\Delta^2 u = f$ in unbounded domains.
b) Analogously one may study the decay of energy or of the solution of (1) in unbounded domains as $|x| \to \infty$. One can prove e.g. that for the domain $\Omega^\omega = \{x = (x_1, x_2) = (r\cos\varphi, r\sin\varphi), \varphi \in (0,\omega)\}$ the solution with finite energy tends to zero as $|x| \to \infty$ for ω sufficiently small. Since for $\omega = 2\pi$ this is not true, it is an open problem to find the largest ω for which this is true.

References

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[3] Mazja, V. G.: DAN SSSR 235 (1977), 1263 - 1266.