Valter Šeda

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ON A NONLINEAR PERTURBATION OF A SELF-ADJOINT BOUNDARY VALUE PROBLEM

Valter Šeda
Bratislava, Czechoslovakia

The aim of this paper is to give a uniqueness and existence results for a nonlinear perturbation of a self-adjoint boundary value problem which generalizes and strengthens some theorems proved by S. Fučík and A. Kufner in [2, pp. 199, 202, 203], M. Švec in [5, p. 174] and completes the results obtained by J. D. Schuur in [3, p. 23] and by J. R. Ward, Jr. in [8, p. 239].

Preliminaries. Let $n \geq 2$ be a natural number, $a < b$ be real numbers. Let the functions $p_j \in C^{(n-j)}([a, b])$ ($j = 1, 2, \ldots, n$) and the function $f : [a, b] \times R \to R$ locally satisfy Carathéodory conditions. All functions throughout the paper are supposed to be real-valued. Let the boundary conditions

$$B_i(x) = \sum_{j=1}^{n} (M_{ij}\, x^{(j-1)}(a) + N_{ij}\, x^{(j-1)}(b)) \qquad (i = 1, 2, \ldots, n)$$

with $M_{ij}, N_{ij} \in R$ ($i, j = 1, 2, \ldots, n$) be linearly independent. Denote $L(x) = x^{(n)} + \sum_{j=1}^{n} p_j(t)\, x^{(n-j)}$ ($x \in C^{(n)}([a, b])$).

If the problem

(1) $\qquad L(x) = \lambda x$

(2) $\qquad B_i(x) = 0 \qquad (i = 1, 2, \ldots, n)$

is self-adjoint ([1, p. 189]), i. e. $\int_a^b L(u)\, v\, dt = \int_a^b u\, L(v) dt$ for all $u, v \in C^{(n)}([a, b])$ satisfying (2), then there exists an orthonormal basis for $L^2([a, b])$ made up of eigenfunctions y_i of (1), (2) and let λ_i ($i = 0, 1, 2, \ldots$) be the corresponding eigenvalues of (1), (2). We have that $|\lambda_i| \to \infty$ as $i \to \infty$. Denote the inner product in $L^2([a, b])$ by (. , .).

Theorem 1 is a consequence of Theorem 2 in [4, p. 439-440]. It shows the role of the eigenvalues λ_i of (1), (2) in the question of existence of a solution to the nonlinear problem (3), (4).

<u>Theorem 1.</u> Let the following assumptions be satisfied :

(H1) The problem (1), (2) is self-adjoint.

(H2) f is continuous in $[a, b] \times R$.

(H3) There exist real numbers $p < q$ such that

$$p \leq \liminf f(t, x)/x \leq \limsup f(t, x)/x \leq q$$

as $|x| \to \infty$ uniformly for $t \in [a, b]$ and such that $\lambda_i \notin [p, q]$ ($i = 0, 1, 2, \dots$).

Then the problem

(3) $L(x) = f(t, x)$

(4) $B_i(x) = C_i$ ($i = 1, 2, \dots, n$)

has at least one solution for each $C_i \in R$ ($i = 1, 2, \dots, n$).

Proof. We can write $f(t, x) = h(t, x).x + g(t, x)$ $(a \leq t \leq b, x \in R)$ where

$$h(t, x) = \begin{cases} p & \text{if } f(t,x)/x \leq p, \\ f(t,x)/x & \text{if } p \leq f(t,x)/x \leq q \\ q & \text{if } q \leq f(t,x)/x \end{cases} \quad \text{for } a \leq t \leq b, \ |x| \geq 1,$$

$h(t, x) = h(t, -1) + \{[h(t, 1) - h(t, -1)]/2\}(x + 1)$ $(a \leq t \leq b, |x| \leq 1)$ and $g(t, x) = f(t, x) - h(t, x).x$ $(a \leq t \leq b, x \in R)$. g and h satisfy the assumptions of Theorem 2 in [4, p. 439-440]. Hence, by this theorem, there exists a solution of the problem (3), (2). A solution u of the problem (3), (4) can be obtained in the way suggested in [8, p. 238]. If c is not in the spectrum of (1), (2), then the problem $L(x) = cx$, (4) has a unique solution w. Let $\tilde{f}(t, x) = f(t, x+w(t)) - cw(t)$ $(a \leq t \leq b, x \in R)$. Then $\tilde{f}$ satisfies (H2), (H3) and hence there exists a solution v of the problem $L(x) = \tilde{f}(t, x)$, (2). Then $u = v + w$ is a solution to (3), (4).

In many boundary value problems the set of all eigenvalues is bounded from below (from above). Then the following uniqueness theorem is true.

Theorem 2. Suppose that besides the assumption (H1) the following assumptions be fulfilled:

(H4) All eigenvalues of (1), (2) are greater or equal to the eigenvalue λ_0 (are smaller or equal to the eigenvalue λ_0).

(H5) The function $f(t, x) - \lambda_0 x$ is decreasing (increasing) in $x \in R$ for each $t \in [a, b]$.

(H6) f is locally majorized by L^2 functions.

Then the problem (3), (4) has at most one solution for each $C_i \in R$ ($i = 1, 2, \dots, n$).

Proof. Without loss of generality we may assume that $\lambda_0 = 0$, since if we let $L_1(x) = L(x) - \lambda_0 x$, $g(t, x) = f(t, x) - \lambda_0 x$, then (3) is equivalent to $L_1(x) = g(t, x)$ and all of our hypotheses are satisfied by the pair L_1, g with $\lambda_0 = 0$. Further only the case $\lambda_i \geq 0$ and hence $f(t, .)$ is decreasing will be considered. Let there exist two solutions x_1, x_2 of (3), (4). Then $y = x_2 - x_1$ satisfies (2) as well as the equation $L(y) = h(t, y)$ where $h(t, y) = f[t,$

$x_1(t) + y] - f[t, x_1(t)]$. h enjoys properties (H5) and (H6) of the function f. Moreover $h(t, y)y < 0$ ($a \leq t \leq b$, $y \neq 0$). Hence

(5) $$(L(y), y) = \int_a^b L(y)(t)y(t)dt = \int_a^b h[t, y(t)]y(t)dt < 0.$$

On the other hand, since $\{y_i\}$ forms an orthonormal basis in $L^2([a, b])$, $y = \sum_{i=0}^{\infty} (y, y_i)y_i$ and as $L(y) \in L^2([a, b])$, $L(y) = \sum_{i=0}^{\infty} (L(y), y_i)y_i = \sum_{i=0}^{\infty} (y, L(y_i))y_i = \sum_{i=0}^{\infty} \lambda_i(y, y_i)y_i$. Thus

(6) $$(L(y), y) = \sum_{i=0}^{\infty} \lambda_i(y, y_i)^2 \geq 0$$

which contradicts (5).

Remarks. 1. Hypotheses (H4), (H5) can be replaced by the following ones:

(H4') All eigenvalues of (1), (2) are positive (negative).

(H5') The function f(t, .) is nonincreasing (nondecreasing) in $x \in R$ for each $t \in [a, b]$.

2. Consider the differential operator M which is defined on $D(M) = \{x \in C^{(n)}([a, b]) : x \text{ satisfies } (2)\}$ by $M(x) = L(x)$. The problem (1), (2) is self-adjoint iff M is symmetric. By (6) M is positively definite in D(M) iff all eigenvalues of (1), (2) are positive.

Under the assumption (H4) the existence of a solution to (3), (4) will be proved. The proof can be based either on Hammerstein's theorem ([6, p. 266]) or on Vajnberg's theorem ([7, p. 275]) or on Ward's results [8]. J. R. Ward, Jr. has used his own results to derive an existence theorem ([8, p. 239]) which is very similar to our next theorem. From the mentioned results Hammerstein's theorem is the most elementary and its proof contains constructive elements which can be used in calculating approximative solutions to the problem (3), (4).

Hammerstein's theorem. Let the following assumptions hold:

1. The function $G \in C([a, b] \times [a, b])$ and it is symmetric, i.e. $G(t, s) = G(s, t)$ in $[a, b]\times[a, b]$.
2. All eigenvalues μ_i of the function G, i.e. the numbers for which there exists a nontrivial solution z_i (the corresponding eigenfunction of G) of the equation $z_i(t) = \mu_i \int_a^b G(t, s)z_i(s)ds$ can be written in a form of a nondecreasing sequence $\mu_0 \leq \mu_1 \leq \ldots \leq \mu_i \leq \ldots$ tending to ∞ (of a nonincreasing sequence $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_i \geq \ldots$ tending to $-\infty$).
3. $w \in C([a, b])$.
4. $f \in C([a, b] \times R)$ and there exist such $\varepsilon > 0$, $C \in R$ that the function F given by the relation

(7) $F(t, x) = \int_0^x f[t, w(t) + u]du \quad ((t, x) \in [a, b] \times R)$

fulfils

(8) $F(t, x) \leq (\frac{\mu_0}{2} - \varepsilon) x^2 + C$

($F(t, x) \geq (\frac{\mu_0}{2} + \varepsilon) x^2 + C$)

in $[a, b] \times R$.

Then there exists a solution $x \in C([a, b])$ of the integral equation

(9) $x(t) = w(t) + \int_a^b G(t, s) f[s, x(s)] ds.$

Remarks. 1. The lemma has been proved under the assumptions that all $\mu_i > 0$ and $w(t) \equiv 0$. If the latter assumption remains valid, then the proof is true also in the case when finitely many eigenvalues $\mu_0, \dots, \mu_p$ are negative. The second part of the theorem follows from the first one by considering (9) (with $w(t) \equiv 0$) in the form $x(t) = \int_a^b [-G(t, s)][-f(s, x(s))]ds$. The assumption $w(t) \equiv 0$ can be removed by transforming (9) to the equivalent equation $y(t) = \int_a^b G(t, s) f[s, w(s) + y(s)] ds$ using the transformation $x(t) - w(t) = y(t)$.

2. Hammerstein's theorem is generalized in a certain sense by Vajnberg's theorem . Still the assumption on continuity of Nemyckij operator in the latter puts restriction on the growth of f.

Theorem 3. Let the assumptions (H1), (H2), (H4) and

(H7) There exists a function $\alpha \in C([a, b])$, $\alpha(t) \geq 0$ in $[a, b]$, $\alpha \not\equiv 0$ such that

(10) $\limsup_{|x| \to \infty} f(t, x)/x \leq \lambda_0 - \alpha(t) \quad (\liminf_{|x| \to \infty} f(t, x)/x \geq \lambda_0 + \alpha(t))$

uniformly for $t \in [a, b]$ be valid.

Then the problem (3), (4) has a solution for each $C_i \in R$ ($i = 1, 2, \dots, n$).

Proof. Only the case that all $\lambda_i \geq \lambda_0$ will be considered. Similarly as in the proof of Theorem 2 we may assume that $\lambda_0 = 0$. Now we consider the equation (3) in the form $L_2(x) = h(t, x)$ where $L_2(x) = L(x) + \alpha(t)x$, $h(t, x) = f(t, x) + \alpha(t)x$. Then (10) implies that

(11) $\limsup_{|x| \to \infty} h(t, x)/x \leq 0$

uniformly for $t \in [a, b]$. On the other hand, by Lemma 1 in [8, p. 237] which is also valid under assumption that all $p_j \in C^{(n-j)}([a, b])$ all eigenvalues $\tilde{\lambda}_i$ ($i = 0, 1, 2, \dots$) of the problem $L_2(x) = \lambda x$, (2), are positive. Thus by (11) there exists an ε, $0 < \varepsilon < \tilde{\lambda}_0$ and an $M > 0$ such that

$$(12) \qquad h(t, x)/x \leq \tilde{\lambda}_0 - \varepsilon \qquad (a \leq t \leq b, |x| \geq M).$$

The problem (3), (4) is equivalent to the integral equation

$$(13) \qquad x(t) = w(t) + \int_a^b G(t, s)h[s, x(s)]ds$$

where w is the solution to (4),

$$(14) \qquad L_2(x) = 0$$

and G is the Green function of (14), (2). The function F_1 given by (7) is of the form

$$(15) \qquad F_1(t, x) = \int_0^x h[t, w(t) + u]du = \int_{w(t)}^{w(t)+x} h(t, v)dv.$$

We shall show that there is such a C that

$$(16) \qquad F_1(t, x) \leq (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + C \qquad ((t, x) \in [a, b] \times R)$$

which will give, on basis of the Hammerstein theorem, the existence of a solution to (13) and thus to the problem (3), (4). Denote $M_1 = \max_{a \leq t \leq b} |w(t)|$, $M_2 = \max (M, M_1)$. First there exist $C_1 < 0$, $C_2 > 0$ such that

$$(17) \qquad C_1 \leq h(t, x) \leq C_2 \qquad (a \leq t \leq b, |x| \leq M_2).$$

We shall consider the following cases.

1. If $0 \leq x \leq M_2 - w(t)$, $a \leq t \leq b$, then by (17) we get that (15) gives

$$F_1(t, x) \leq C_2 x \leq (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + \max_{\substack{a \leq t \leq b \\ 0 \leq x \leq M_2 - w(t)}} \{ C_2(M_2 - w(t)) + (\frac{\varepsilon}{4} - \frac{\tilde{\lambda}_0}{2}) x^2 \} = (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + A.$$

2. $x > M_2 - w(t)$, $a \leq t \leq b$. Then (12) and (17) imply that

$$F_1(t, x) \leq C_2 (M_2 - w(t)) + \frac{\tilde{\lambda}_0 - \varepsilon}{2} [(w(t) + x)^2 - M_2^2] \leq (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + \max_{\substack{a \leq t \leq b \\ M_2 - w(t) \leq x < \infty}} \{ C_2(M_2 - w(t)) + \frac{\tilde{\lambda}_0 - \varepsilon}{2} (w^2(t) - M_2^2) + x^2(- \frac{\varepsilon}{4} + (\tilde{\lambda}_0 - \varepsilon). w(t)/x) \} = (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + B.$$

3. $0 \geq x \geq - M_2 - w(t)$, $a \leq t \leq b$. Now we have

$$F_1(t, x) \leq C_1 x \leq C_1 (- M_2 - w(t)) \leq (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + \max_{\substack{a \leq t \leq b \\ 0 \leq x \geq -M_2 - w(t)}} \{ C_1(-M_2 - w(t)) + (\frac{\varepsilon}{4} - \frac{\tilde{\lambda}_0}{2}) x^2 \} = (\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}) x^2 + D.$$

4. When $x < - M_2 - w(t)$, $a \leq t \leq b$, from (17) and (12) it follows

that

$$F_1(t, x) \leq C_1(-M_2 - w(t)) + \frac{\tilde{\lambda}_0 - \varepsilon}{2}\left[(w(t) + x)^2 - M_2^2\right]$$

$$\leq \left(\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}\right) x^2 + \max_{\substack{a \leq t \leq b \\ -\infty < x \leq -M_2 - w(t)}} \left\{ C_1(-M_2 - w(t)) + \frac{\tilde{\lambda}_0 - \varepsilon}{2}(w^2(t) - M_2^2) + x^2\left(-\frac{\varepsilon}{4} + \frac{(\tilde{\lambda}_0 - \varepsilon) w(t)}{x}\right)\right\}$$

$$= \left(\frac{\tilde{\lambda}_0}{2} - \frac{\varepsilon}{4}\right) x^2 + E.$$

In all cases the inequality (16) is satisfied with C = max (A, B, D, E).

REFERENCES

[1] E. A. Coddington, N. Levinson , Theory of Ordinary Differential Equations, Mc Graw-Hill Book Co., Inc. New York-Toronto-London 1955.

[2] S. Fučík, A. Kufner , Nelineární diferenciální rovnice, SNTL, Praha 1978.

[3] J. D. Schuur , Perturbation at Resonance for a Fourth Order Ordinary Differential Equation, J. Math. Anal. Appl. 65 (1978), 20-25.

[4] V. Šeda , A Class of Differential Equations Similar to Linear Equations, Math. Slovaca 30 (1980), 433-441.

[5] M. Švec, Existence of Periodic Solutions of Differential Equations of Second Order, G. E. O. Giacaglia, Periodic Orbits, Stability and Resonance, Dordrecht (1970), 168-175.

[6] F. G. Tricomi , Integral Equations, Interscience Publ., Inc. New York 1957 (Russian Translation, Izdat. Inostran. Lit., Moskva 1960).

[7] M. M. Vajnberg , Variacionnyje metody issledovanija nelinejnych operatorov, Gos. Izdat. Tech.-Teoret. Lit., Moskva 1956.

[8] J. R. Ward, Jr., Existence Theorems for Nonlinear Boundary Value Problems at Resonance, J. Differential Equations 35 (1980), 232-247.