Jiří Souček

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SINGULAR LINES IN LIQUID CRYSTALS

SOUČEK J., PRAGUE, Czechoslovakia

Let us consider $H^{1,2}_{\varphi}(\Omega, S^2) = \{u : \Omega \subset \mathbb{R}^3 \to S^2 \subset \mathbb{R}^3 \; : \; u \in H^{1,2}(\Omega, S^2), u_{|\partial\Omega} = \varphi\}$ and the energy $E(u) = \frac{1}{2}\int_\Omega |Du(x)|^2\,dx$. Minimizers can be considered as stable configurations of the liquid crystal, where $u(x) \in S^2$ is the direction of the optical axis at x [2].

In [1] it is shown that the relaxed energy is not E but $F(u) = E(u) + 4\pi\cdot$ "length of a minimal connection"– see [1][2]. Alternative description was introduced in [3][4]. The class of Cartesian currents $\mathrm{cart}^{2,1}_{\varphi}(\Omega, S^2)$ is the set of all 3–dimensional rectifiable boundary-less currents in $\Omega \times S^2$ of the form $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$, $\partial T = [\![G_\varphi]\!]$, where $u_T \in H^{1,2}$ and G_{u_T} denotes the graph of u_T. L_T is related to singularities of u_T by the relation $\partial[\![G_{u_T}]\!] = -L_T \times [\![S^2]\!]$ which follows from $\partial T = 0$. The energy of T is given by the parametric extension of E [3][4]

$$\mathcal{D}(T;\Omega) \;=\; \frac{1}{2}\int_\Omega |Du_T(x)|^2\,dx \;+\; 4\pi\,\mathrm{M}_\Omega(L_T).$$

Now we shall describe some results from [4].

To each $u \in H^{1,2}_{\varphi}(\Omega, S^2)$ there is a class $[u] = \{T \in \mathrm{cart}^{2,1}_{\varphi}(\Omega, S^2) : u_T = u\}$ and $F(u) = min\{\mathcal{D}(T) : T \in [u]\}$. $\mathcal{D}(T)$ is the relaxed energy with respect to the weak convergence of currents

$$\mathcal{D}(T) \;=\; \inf\{\liminf_{k\to\infty} E(u_k) : u_k \in C^1, [\![G_{u_k}]\!] \rightharpoonup T\}.$$

Using the monotonicity formula the partial regularity for minimizers of $\mathcal{D}(T)$ in $\mathrm{cart}^{2,1}_{\varphi}(\Omega, S^2)$ can be proved: u_T is smooth outside of the closed set $\Sigma \subset \Omega$ with $\mathcal{H}^{1+\epsilon}(\Sigma) = 0,\ \forall\epsilon > 0$.

L_T is interpreted as the line singularity of T. This is confirmed by the behavior of smooth approximating sequences. Denote by

$$e_T = \frac{1}{2}|Du_T|^2 \cdot \mathcal{H}^3(dx) + 4\pi \|L_T\|$$

the "energy measure" of T. Then we have

Theorem. *Let $u_k \in C^1(\Omega, S^2)$ be the sequence satisfying*

$$[\![G_{u_k}]\!] \rightharpoonup [\![G_{u_T}]\!] + L_T \times [\![S^2]\!], \quad E(u_k) \to \mathcal{D}(T).$$

Then

$$\frac{1}{2}|Du_k|^2 \cdot \mathcal{H}^3 \rightharpoonup e(T)$$

as measures in Ω and for any neighborhood $\mathcal{U}$ of $\operatorname{spt} L_T$ *we have*

$$u_k \to u \text{ in } H^{1,2}(\Omega \setminus \overline{\mathcal{U}}; S^2).$$

This describes the concentration of the gradient $|Du_k|^2$ at the support of L_T. An analog property in the setting of [1] does not hold: $u_k \rightharpoonup u$ in $H^{1,2}(\Omega, S^2)$ and $E(u_k) \to F(u)$ does not imply that $|Du_k|^2 \cdot \mathcal{H}^3$ converge as measures (for the example see [4]).

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