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ON A CLASS OF WEIGHTED SOBOLEV SPACES

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Many authors have studied various classes of Sobolev spaces with weights (see for example [2], [3] and the references therein). The aim of the present paper is, roughly speaking, to present some results about the Sobolev theorem and about the inequalities of the type "compacity" for a special class of weighted Sobolev spaces; one can use these results to study spectral properties of a class of degenerated elliptic operators.

Let us mention that the results presented here were obtained together with P. BOLLEY and PHAM THE LAI (Nantes University).

I. The case of the half-line $\mathbb{R}_+$

For an integer $m \in \mathbb{N}$, two real numbers α and $\beta \geqq 0$ and an interval I of $\mathbb{R}_+$, we consider the space

$$V^m_{\alpha,\beta}(I) = \{u \in \mathcal{D}'(I);\ t^\alpha u \in L^2(I),\ t^\beta D^m_t u \in L^2(I)\}$$

equipped with the canonical norm.

<u>*PROPOSITION I.1.*</u> *If* $u \in V^m_{\alpha,\beta}(0, T)$, *where* T *is a strictly positive real number, we have:*

(i) $t^{\beta-j}D^{m-j}_t u \in L^2(0, T)$ *for* $0 \leqq j \leqq \mathrm{Min}(j_0, m)$ *with* $j_0 = [\beta + \frac{1}{2}]_-$;

(ii) $t^{\beta-j_0}D^{m-j}_t u \in L^2(0, T)$ *for* $j_0+1 \leqq j \leqq m$ *if* $j_0+1 \leqq m$;

(iii) $u \in H^{m-\beta}(0, T)$ *if* $\beta-m \neq$ *integer* $+ \frac{1}{2}$.

The notation $[A]_-$ means the greatest integer $< A$.

P r o o f . Let ϕ be an infinitely differentiable function such that $\phi(t) = 1$ if $t \leqq \frac{T}{2}$ and $\phi(t) = 0$ if $t \geqq 3\frac{T}{4}$. Put $v = \phi u$; then $v \in V^m_{\alpha,\beta}(\mathbb{R}_+)$ with bounded support.

Using Hardy's inequality, we obtain (i).

Now for (ii): we have $t^{\beta-j_0}D_t^{m-j_0}v \in L^2(\mathbb{R}_+)$, also $t^{\beta-j_0-1}D_t^{m-j_0}v \in L^2(\mathbb{R}_+)$ and by Hardy's inequality we get $t^{\beta-j_0}D_t^{m-j_0-1}v \in L^2(\mathbb{R}_+)$; repeating the same argument, we obtain (ii).

If $\beta > m$, it results from (i) that $t^{\beta-m}u \in L^2(\mathbb{R}_+)$ and consequently if $\beta-m \neq$ integer $+\frac{1}{2}$, we have ([4]) $u \in H^{m-\beta}(\mathbb{R}_+)$.

If $\beta \leqq m$, then $j_0 \leqq m$ and $-\frac{1}{2} < \beta - j_0 \leqq \frac{1}{2}$. Hence two cases must be distinguished according to $-\frac{1}{2} < \beta - j_0 \leqq 0$ and $0 < \beta - j_0 \leqq \frac{1}{2}$.

First case: $0 < \beta - j_0 \leqq \frac{1}{2}$. We have $t^{\beta-j_0}D_t^{m-j_0-1}v$ and $t^{\beta-j_0}D_t^{m-j_0}v \in L^2(\mathbb{R}_+)$ (notice that $0 < \beta - j_0$ and $\beta \leq m$ implies $j_0 + 1 \leq m$). Then we have $t^{\frac{1}{2}}D_t^{m-j_0-1}v$ and $t^{\frac{1}{2}}D_t^{m-j_0}v \in L^2(\mathbb{R}_+)$, and now we prove that these two conditions imply $D_t^{m-j_0-1}v \in L^2(\mathbb{R}_+)$.

LEMMA I.1. ([1]). *If* $u \in V^1_{\frac{1}{2},\frac{1}{2}}(\mathbb{R}_+)$, *then* $u \in L^2(\mathbb{R}_+)$.

Proof. If $u \in \mathcal{D}(\mathbb{R}_+)$, we can write

$$|u(t)|^2 = 2\,\mathrm{Re}\int_t^{+\infty} u(\sigma)\overline{u'(\sigma)}\,d\sigma$$

and using Fubini's theorem, we obtain

$$\int_0^{+\infty}|u|^2dt \leqq -2\,\mathrm{Re}\int_0^{+\infty}\sigma\, u(\sigma)\overline{u'(\sigma)}\,d\sigma \leqq \int_0^{+\infty}t|u(t)|^2dt + \int_0^{+\infty}t|u'(t)|^2dt.$$

Finally the density of $\mathcal{D}(\mathbb{R}_+)$ in the space $V^1_{\frac{1}{2},\frac{1}{2}}(\mathbb{R}_+)$ proves Lemma I.1.

Now, we prove that $D_t^{m-j_0-1}v \in H^{\varepsilon}(\mathbb{R}_+)$ with $\varepsilon = 1 - (\beta - j_0)$. To this end put $D_t^{m-j_0-1}v = f$ and $D_t^{m-j_0}v = F$ and compute

$$\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^2}{|x-y|^{2\varepsilon+1}}\,dx\,dy = \int_0^{+\infty}\int_0^{+\infty}\frac{|f(x+t)-f(x)|^2}{t^{2\varepsilon+1}}\,dx\,dt\,.$$

However,

$$f(x+t) - f(x) = \int_0^t F(x+\sigma)\,d\sigma\,.$$

Hence

$$\int_0^{+\infty} \frac{|f(x+t)-f(x)|^2}{t^{2\varepsilon+1}}\, dt = \int_0^{+\infty} \frac{1}{t^{2\varepsilon+1}} \left|\int_0^{+\infty} F(x+\sigma)\, d\sigma\right|^2 dt$$

and using Hardy's inequality,

$$\leqq C \int_0^{+\infty} \frac{1}{t^{2\varepsilon-1}} |F(x+t)|^2\, dt \; .$$

(C is a constant.) On the other hand,

$$\int_0^{+\infty} \frac{1}{t^{2\varepsilon-1}} |F(x+t)|^2\, dt = \int_x^{+\infty} \frac{1}{|y-x|^{2\varepsilon-1}} |F(y)|^2\, dy$$

$$= x^{-2(\varepsilon-1)} \int_1^{+\infty} \frac{1}{|\sigma-1|^{2\varepsilon-1}} |F(\sigma x)|^2\, d\sigma$$

and Fubini's theorem yields

$$\int_0^{+\infty} x^{-2(\varepsilon-1)} \left(\int_1^{+\infty} \frac{1}{|\sigma-1|^{2\varepsilon-1}} |F(\sigma x)|^2 d\sigma\right) dx = \int_1^{+\infty} \frac{\sigma^{2(\varepsilon-1)-1}}{|\sigma-1|^{2\varepsilon-1}} d\sigma \int_0^{+\infty} y^{-2(\varepsilon-1)} |F(y)|^2\, dy;$$

thus, $D_t^{m-j_0-1} v \in H^{\varepsilon}(\mathbb{R}_+)$ and $v \in H^{m-\beta}(\mathbb{R}_+)$.

<u>Second case:</u> $-\frac{1}{2} < \beta - j_0 \leqq 0$. The case $\beta - j_0 = 0$ being trivial, we can assume that $-\frac{1}{2} < \beta - j_0 < 0$. Then $\frac{1}{2} < \beta - j_0 + 1 < 1$ and we have $D_t^{m-j_0} v \in L^2(\mathbb{R}_+)$ and $t^{\beta-j_0+1} D_t^{m-j_0+1} v \in L^2(\mathbb{R}_+)$. By the same computation as before we get that $D_t^{m-j_0} v \in H^{\varepsilon}(\mathbb{R}_+)$ with $\varepsilon =$ $= -(\beta - j_0)$ and finally $v \in H^{m-\beta}(\mathbb{R}_+)$. Proposition I.1 is proved.

<u>REMARK I.1.</u> We can improve the result of Proposition I.1 when $\beta - \alpha > m$, in fact we have: if $\beta - \alpha > m$ and if $u \in V^m_{\alpha,\beta}(0, T)$, then $t^{\alpha+(j/m)(\beta-\alpha)} D_t^j u \in L^2(0,T)$ for $j = 0, \dots, m$. The proof is analogous to that of the following proposition.

<u>*PROPOSITION I.2.*</u> *If* $\beta - \alpha < m$ *and if* $u \in V^m_{\alpha,\beta}(T,+\infty)$ *where* T *is a real number* > 0 *, then*

$$t^{\alpha+\frac{j}{m}(\beta-\alpha)} D_t^j u \in L^2(T,+\infty) \qquad \textit{for } j = 0, \dots, m \; .$$

P r o o f . It will be done in two steps.

<u>First step:</u> Reduction to the case $\alpha = 0$.

<u>*LEMMA I.2.*</u> *If* $u \in V^m_{\alpha,\beta}(T, +\infty)$ *, then* $t^{\beta-m+j} D_t^j u \in L^2(T, +\infty)$.

P r o o f : If $\beta \leqq \frac{1}{2}$, we have obviously $u \in H^m(T, +\infty)$ and then $t^{\beta-j}D_t^{m-j}u \in L^2(T,+\infty)$ for $j = 0, \ldots, m$.

If $\beta > \frac{1}{2}$, then, as in Proposition I.1, we get that $t^{\beta-j}D_t^{m-j}u \in L^2(T,+\infty)$ for $0 \leq j \leq \mathrm{Min}(j_0, m)$ with $j_0 = [\beta + \frac{1}{2}]_-$. Finally, since $D_t^{m-j}u \in L^2(T, +\infty)$ for $j = 0, \ldots, m$, we get that $t^{\beta-j}D_t^{m-j}u \in L^2(T, +\infty)$ for $j = j_0 + 1, \ldots, m$ if $j_0 + 1 \leq m$ ($\beta - j$ is negative).

<u>*LEMMA I.3.*</u> *The map* $u \longmapsto t^\alpha u$ *is an isomorphism from* $V^m_{\alpha,\beta}(T,+\infty)$ *onto* $V^m_{0,\beta-\alpha}(T, +\infty)$.

P r o o f . Let u be an element of $V^m_{\alpha,\beta}(T, +\infty)$, we put $v = t^\alpha u$; then $t^{\beta-\alpha}D_t^m v(t) = \sum_{j=0}^m a_j . t^{\beta-j}D_t^{m-j}u(t)$ and by Lemma I.2 it results that $v \in V^m_{0,\beta-\alpha}(T, +\infty)$.

Conversely, let v be an element of $V^m_{0,\beta-\alpha}(T, +\infty)$, we put $u = t^{-\alpha}v$; then $t^\beta D_t^m u(t) = \sum_{j=0}^m a_j . t^{\beta-\alpha-j}D_t^{m-j}v(t)$ and by Lemma I.2 it results that $u \in V^m_{\alpha,\beta}(T, +\infty)$.

<u>Second step:</u> We assume $\alpha = 0$.

We use the change of variable $y = t^{\frac{m-\beta}{m}}$ and of the function $w(y) = y^{\beta/2(m-\beta)}u(t)$.

By induction on p we show that, for $0 \leq p \leq m$, we have

$$D_y^p w(y) = y^{\beta/2(m-\beta)} \sum_{j=0}^p a_{jp} . t^{j-p+p\frac{\beta}{m}} D_t^j u(t) ,$$

where $a_{pp} \neq 0$. By Lemma I.2 we get $D_y^m w \in L^2(Y,+\infty)$ where $Y = T^{\frac{m-\beta}{m}}$ and consequently $w \in H^m(Y, +\infty)$ since $w \in L^2(Y, +\infty)$. Then $D_y^p w \in L^2(Y, +\infty)$ for $p = 0, \ldots, m$ and using the preceding formula, we get by induction on p and since $j - p + p\frac{\beta}{m} < j\frac{\beta}{m}$ for $j < p$ that $t^{j\frac{\beta}{m}} D_t^j u \in L^2(T, +\infty)$ for $j = 0, \ldots, m$. Proposition I.2 is proved.

We now apply these results to a sub-class of Sobolev spaces with weights which will be useful for the following: let $m \in \mathbb{N}$, let $-\sigma$ and δ be two real numbers > 0 such that $\sigma + m \geq 0$ and $\sigma + \delta m \geq 0$. We consider the space

$$W^m_{\sigma,\delta}(\mathbb{R}_+) = \{u \in H^{-\sigma}(\mathbb{R}_+);\ t^{\sigma+\delta k+j}D_t^j u \in L^2(\mathbb{R}_+) \text{ for } \sigma+\delta k+j \geq 0 \text{ and } k+j \leq m\}$$

equipped with the canonical norm.

By Propositions I.1 and I.2, this space coincides with the space $V^m_{\sigma+\delta m,\sigma+m}(\mathbb{R}_+)$.

We now give Sobolev's theorem for the spaces $W^m_{\sigma,\delta}(\mathbb{R}_+)$.

<u>*PROPOSITION I.3.*</u> *(i) If* $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ *, then* u *is continuous on* $\mathbb{R}_+$ *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ *and for every* $t > 0$ *, we have*

$$|u(t)| \leqq C.\ t^{-\frac{\sigma+m}{2m}}\ ||u||^{\frac{1}{2m}}_{W^m_{\sigma,\delta}}\ ||u||^{1-\frac{1}{2m}}_{L^2}. \tag{1.1}$$

(ii) Let us assume $-\sigma > \frac{1}{2}$ *. If* $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ *, then* u *is continuous and bounded on* $\mathbb{R}_+$ *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ *and for every* $t > 0$*, we have*

$$|u(t)| \leqq C.\ ||u||^{-\frac{1}{2\sigma}}_{W^m_{\sigma,\delta}}\ ||u||^{1+\frac{1}{2\sigma}}_{L^2}\ ; \tag{1.2}$$

$$|u(t)| \leqq C.\ t^{-(\sigma+\delta m)+\frac{1}{2}(\delta-1)}\ ||u||_{W^m_{\sigma,\delta}}\ . \tag{1.3}$$

P r o o f . (i) First, we apply the usual Sobolev's theorem: if $v \in H^m(\mathbb{R}_+)$ with $m \geqq 1$, then v is continuous on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}_+)$ and for every $t \geqq 0$, we have

$$|v(t)|^2 \leqq C\ \{\int_0^{+\infty} |D^m_t v(\tau)|^2\, d\tau + \int_0^{+\infty} |v(\tau)|^2\, d\tau\ \}.$$

If $w \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, the function v defined by $v(\tau) = w(\tau+t)$ belongs to $H^m(\mathbb{R}_+)$ for every $t > 0$. Since $-\sigma > 0$ and $\sigma+m \geqq 0$, it is $m \geqq 1$ and for every $w \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ and for every $t > 0$, we have

$$|w(t)|^2 \leqq C.\ \{\int_t^{+\infty} |D^m_t w(\tau)|^2\, d\tau + \int_t^{+\infty} |w(\tau)|^2\, d\tau\ \}.$$

Now, let u be an element of $W^m_{\sigma,\delta}(\mathbb{R}_+)$ and let us apply the preceding inequality to the function w defined by $w(\tau) = u(\lambda\tau)$ where λ

is a positive constant. Then there exists a constant $C > 0$ such that, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$ and for every $\lambda > 0$, we have

$$(1.4) \qquad |u(t)|^2 \leqq \frac{C}{\lambda}\{\int_t^{+\infty} |\lambda^m D_t^m u(\tau)|^2 d\tau + \int_t^{+\infty} |u(\tau)|^2\, d\tau\},$$

and since $t \leqq \tau$, we get

$$|u(t)|^2 \leqq \frac{C}{\lambda}\{\int_t^{+\infty} \lambda^{2m}\, t^{-2(\sigma+m)} |\tau^{\sigma+m} D_t^m u(\tau)|^2\, d\tau + \int_t^{+\infty} |u(\tau)|^2 d\tau\}.$$

Choosing $\lambda = t^{\frac{\sigma+m}{m}}$, a fortiori we obtain

$$|u(t)|^2 \leqq C.\, t^{-\frac{\sigma+m}{m}}\{\int_0^{+\infty} |\tau^{\sigma+m}\, D_t^m u(\tau)|^2 d\tau + \int_0^{+\infty} |u(\tau)|^2\, d\tau\}.$$

Now, we apply this inequality to the function v defined by $v(\tau) = u(\lambda\tau)$ where λ is a constant >0 :

$$|u(\lambda t)|^2 \leqq C.t^{-\frac{\sigma+m}{m}} \lambda^{-1}\{\int_0^{+\infty} \lambda^{-2\sigma} |\tau^{\sigma+m} D_t^m u|^2\, d\tau + \int_0^{+\infty} |u(\tau)|^2\, d\tau\}.$$

Putting $\lambda = r^{\frac{1}{2\sigma}}$, we get for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$ and for every $r > 0$ that

$$|u(tr^{\frac{1}{2\sigma}})|^2 \leqq C.\, (tr^{\frac{1}{2\sigma}})^{-\frac{\sigma+m}{m}}\, r^{\frac{1}{2m}-1}\{\int_0^{+\infty} |\tau^{\sigma+m} D_t^m u|^2 d\tau + r\int_0^{+\infty} |u(\tau)|^2 d\tau\}.$$

Finally, there exists a constant $C > 0$ such that for every $t > 0$, for every $r > 0$ and for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have

$$|u(t)|^2 \leqq C.\, t^{-\frac{\sigma+m}{m}}\, r^{\frac{1}{2m}-1}\{||u||^2_{W^m_{\sigma,\delta}} + r||u||^2_{L^2}\}.$$

Taking $r = ||u||^2_{W^m_{\sigma,\delta}} \,.\, ||u||^{-2}_{L^2}$, we obtain the inequality (1.1).

(ii) If $-\sigma > \frac{1}{2}$, Sobolev's theorem implies that if $v \in H^{-\sigma}(\mathbb{R}_+)$, then v is continuous and bounded on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^{-\sigma}(\mathbb{R}_+)$ and for every $t \geqq 0$, we have

$$|v(t)|^2 \leqq C. \; ||v||^2_{H^{-\sigma}(\mathbb{R}_+)} .$$

However, by Proposition I.1, the space $V^m_{0,\sigma+m}(\mathbb{R}_+)$ is continuously imbedded in $H^{-\sigma}(\mathbb{R}_+)$. Hence for every $t \geqq 0$ and for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have

$$|v(t)|^2 \leqq C. \; \{\int_0^{+\infty} |\tau^{\sigma+m} D^m_t u|^2 \, d\tau + \int_0^{+\infty} |u(\tau)|^2 \, d\tau\} .$$

Using the same change of functions as before, we conclude that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$ and for every $r > 0$, we have

$$|u(t)|^2 \leqq C. \; r^{-1-\frac{1}{2\sigma}} \{||u||^2_{W^m_{\sigma,\delta}} + r \, ||u||^2_{L^2} \}.$$

We obtain the inequality (1.2) by taking $r = ||u||^2_{W^m_{\sigma,\delta}} . ||u||^{-2}_{L^2}$.

To establish the inequality (1.3), we start from the inequality (1.4) in which we choose $\lambda = (\int_t^{+\infty} |u(\tau)|^2 d\tau)^{\frac{1}{2m}} (\int_t^{+\infty} |D^m_t u(\tau)|^2 d\tau)^{-\frac{1}{2m}}$, which yields

$$|u(t)|^2 \leqq C. \; (\int_t^{+\infty} |D^m_t u|^2 d\tau)^{\frac{1}{2m}} (\int_t^{+\infty} |u(\tau)|^2 d\tau)^{1-\frac{1}{2m}};$$

then we notice that, since $t \leq \tau$, we have

$$\int_t^{+\infty} |D^m_t u|^2 \, d\tau \leqq t^{-2(\sigma+m)} \int_t^{+\infty} \tau^{2(\sigma+m)} |D^m_t u|^2 d\tau \leqq t^{-2(\sigma+m)} ||u||^2_{W^m_{\sigma,\delta}}$$

and

$$\int_t^{+\infty} |u(\tau)|^2 d\tau \leqq t^{-2(\sigma+\delta m)} \int_t^{+\infty} \tau^{2(\sigma+\delta m)} |u(\tau)|^2 d\tau \leqq t^{-2(\sigma+\delta m)} ||u||^2_{W^m_{\sigma,\delta}}$$

which implies the inequality (1.3).

II. The case of the half space $\mathbb{R}^n_+$, $n > 1$.

Let m be an integer, $-\sigma$ and δ two real numbers >0 such that $\sigma+m \geqq 0$ and $\sigma+\delta m \geqq 0$. We consider the space

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+);\ t^{\sigma+\delta|\alpha|+j} D^j_t D^\alpha_x u \in L^2(\mathbb{R}^n_+) \quad \text{for} \quad \sigma+\delta|\alpha|+j \geqq 0 \quad \text{and} \quad |\alpha|+j \leqq m\}$$

equipped with the canonical norm.

The space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ (cf. for example [2]) and we have also

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) = \{u \in \mathcal{D}'(\mathbb{R}^n_+);\ t^{\mathrm{Max}(0,\sigma+\delta|\alpha|+j)} D^j_t D^\alpha_x u \in L^2(\mathbb{R}^n_+) \text{ for } |\alpha|+j \leqq m\}.$$

<u>*PROPOSITION II.1.*</u> *(i) If* $m > n/2$ *and if* $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ *, then* u *is continuous on* $\mathbb{R}^n_+$ *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ *and for every* $(t,x) \in \mathbb{R}^n_+$*, we have*

$$(2.1) \qquad |u(t,x)| \leqq C.\ t^{-\frac{\sigma+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} \|u\|^{\frac{n}{2m}}_{W^m_{\sigma,\delta}} \|u\|^{1-\frac{n}{2m}}_{L^2}$$

(ii) If $\mathrm{Min}\,(-\sigma, -\sigma/\delta) > n/2$ *and if* $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ *, then* u *is continuous and bounded on* $\mathbb{R}^n_+$ *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ *and for every* $(t,x) \in \mathbb{R}^n_+$ *, we have*

$$(2.2) \qquad |u(t,x)| \leqq C.\ \|u\|^{-\frac{1+\delta(n-1)}{2\sigma}}_{W^m_{\sigma,\delta}} \|u\|^{1+\frac{1+\delta(n-1)}{2\sigma}}_{L^2}$$

P r o o f . The proof is analogous to those in Chapter I.

(i) First we apply the usual Sobolev's theorem: if $v \in H^m(\mathbb{R}^n_+)$ with $m > n/2$ then v is continuous on $\overline{\mathbb{R}^n_+}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}^n_+)$ and for every $(t,x) \in \mathbb{R}^n_+$, we have

$$|u(t,x)|^2 \leqq C.\ \{\sum_{j+|\alpha|=m} \int_{\mathbb{R}^n_+} |D^j_t D^\alpha_x v(\tau,y)|^2 d\tau\, dy + \int_{\mathbb{R}^n_+} |v(\tau,y)|^2 d\tau\, dy\}.$$

If $w \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, the function v defined by $v(\tau,y) = w(\tau+t,\ y)$

belongs to $H^m(\mathbb{R}^n_+)$ for every $t > 0$. Hence for every $w \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ and for every $(t,x) \in \mathbb{R}^n_+$, we have

$$|w(t,x)|^2 \leqq C.\{\sum_{|\alpha|+j=m}\int_t^{+\infty}\int_{\mathbb{R}^{n-1}} |D_t^j D_x^\alpha w(\tau,y)|^2 d\tau dy + \int_t^{+\infty}\int_{\mathbb{R}^{n-1}} |w(\tau,y)|^2 d\tau dy\}.$$

Let now u be an element of $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ and let us apply the preceding inequality to the function w defined by $w(\tau, y) = u(\lambda\tau, \mu y)$ where λ and μ are two constants. Hence there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t, x) \in \mathbb{R}^n_+$ and for every λ and $\mu > 0$, we have

$$|u(t,x)|^2 \leqq \frac{C}{\lambda.\mu^{n-1}} \{\sum_{|\alpha|+j=m}\int_t^{+\infty}\int_{\mathbb{R}^{n-1}} \lambda^{2j}\mu^{2(m-j)}|D_t^j D_x^\alpha u(\tau, y)|^2 d\tau\, dy + \int_t^{+\infty}\int_{\mathbb{R}^{n-1}} |u(\tau, y)|^2 d\tau\, dy\},$$

and since $t \leqq \tau$, this yields

$$|u(t,x)|^2 \leqq$$
$$\leqq \frac{C}{\lambda.\mu^{n-1}}\{\sum_{|\alpha|+j=m}\int_t^{+\infty}\int_{\mathbb{R}^{n-1}} \lambda^{2j}\mu^{2(m-j)}t^{-2(\sigma+\delta(m-j)+j)}|\tau^{\sigma+\delta|\alpha|+j}D_t^j D_x^\alpha u|^2 d\tau dy + \int_t^{+\infty}\int_{\mathbb{R}^{n+1}} |u(\tau,y)|^2 d\tau dy\}.$$

Choosing $\lambda = t^{\frac{\sigma+m}{m}}$ and $\mu = t^{\frac{\sigma+\delta m}{m}}$, a fortiori we get

$$|u(t,x)|^2 \leqq C.t^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)}\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+} |\tau^{\sigma+\delta|\alpha|+j}D_t^j D_x^\alpha u|^2 d\tau\, dy + \int_{\mathbb{R}^n_+} |u(\tau, y)|^2 d\tau\, dy\}.$$

We now apply this inequality to the function v defined by $v(\tau, y) = u(\lambda\tau, \mu x)$ where λ and μ are some constants:

$$|u(\lambda t, \mu x)|^2 \leqq$$

$$\leq C.\ \frac{t^{-\frac{\sigma+m}{m}-\frac{n-1}{m}(\sigma+\delta m)}}{\lambda.\mu^{n-1}}\Big\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+}\lambda^{-2(\sigma+\delta(m-j))}\mu^{2(m-j)}|\tau^{\sigma+\delta|\alpha|+j}D_t^jD_x^\alpha u|^2\, d\tau dy +$$

$$+\int_{\mathbb{R}^n_+}|u|^2\, d\tau\, dy\Big\}\ .$$

Putting $\lambda = r^{\frac{1}{2\sigma}}$ and $\mu = \lambda^\delta$, we deduce that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t, x) \in \mathbb{R}^n_+$ and for every $r > 0$, we have

$$|u(tr^{\frac{1}{2\sigma}},\ xr^{\frac{\delta}{2\sigma}})|^2 \leq$$

$$\leq C.(tr^{\frac{1}{2\sigma}})^{-\frac{\sigma+m}{m}-\frac{n-1}{m}(\sigma+\delta m)}\ r^{\frac{n}{2m-1}}\ \Big\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+}|\tau^{\sigma+\delta|\alpha|+j}D_t^jD_x^\alpha u|^2 d\tau dy$$

$$+ r\int_{\mathbb{R}^n_+}|u|^2 d\tau\ dy\ \Big\}.$$

Finally, there exists a constant $C > 0$ such that for every $(t, x) \in \mathbb{R}^n_+$, for every $r > 0$ and for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, we have

$$|u(t,x)|^2 \leq C.\ t^{-\frac{\sigma+m}{m}-\frac{n-1}{m}(\sigma+\delta m)}\ r^{\frac{n}{2m-1}}\ \{||u||^2_{W^m_{\sigma,\delta}} + r||u||^2_{L^2}\}\ .$$

This yields the inequality (2.1) by choosing $r = ||u||^2_{W^m_{\sigma,\delta}}\ .\ ||u||^{-2}_{L^2}$.

(ii) We begin by proving

LEMMA II.1. *We have the algebraical and topological imbedding*

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) \subset H^{\mathrm{Min}(-\sigma,-\sigma/\delta)}(\mathbb{R}^n_+)\ .$$

P r o o f . By Chapter I, we know that $V^m_{\sigma+\delta m,\sigma+m}(\mathbb{R}_+) \subset H^{-\sigma}(\mathbb{R}_+)$. Hence there exists a constant $C > 0$ such that for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have

$$\int_{-\infty}^{+\infty}(1+\tau^2)^{-\sigma}|F(Pv)|^2 d\tau \leq C.\{\int_0^{+\infty}|t^{\sigma+m}D_t^m v|^2 dt + \int_0^{+\infty}|t^{\sigma+\delta m}v|^2 dt\}\ ,$$

where F stands for the Fourier transform in the variable t and P for a linear and continuous extension operator from $H^{-\sigma}(\mathbb{R})$ (for example, P can be taken as the Babitch extension). If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, then the function $u(t) = v(t\Lambda^{-1/\delta})$, where Λ is a positive constant, belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; for every $\Lambda > 0$ we have

$$\int_{-\infty}^{+\infty} (\Lambda^{2/\delta}+\tau^2)^{-\sigma}|F(Pv)|^2 d\tau \leqq C \{\int_0^{+\infty} |t^{\sigma+m}D^m_t v|^2 dt + \Lambda^{2m}\int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt\} .$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}^n_+})$ and for every $\xi \in \mathbb{R}^{n-1}\setminus\{0\}$, let us consider the function $v(t) = \hat{u}(t, \xi)$, where $\hat{}$ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; then $F(Pv)(\tau) = \mathcal{F}Pu(\tau, \xi)$, where $\mathcal{F}$ means the Fourier transform in the variable (t, x) in $\mathbb{R}^n$. From the preceding inequality we deduce, taking $\Lambda = |\xi|$ and integrating in ξ over $\mathbb{R}^{n-1}$, that there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(\overline{\mathbb{R}^n_+})$, we have

$$||Pu||_{H^{-\sigma^*}(\mathbb{R}_n)} \leqq C.||u||_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)}$$

with $\sigma^* = \mathrm{Min}\ (-\sigma, -\sigma/\delta)$, and thus

$$||u||_{H^{-\sigma^*}(\mathbb{R}^n_+)} \leqq C.||u||_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)}$$

The space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ being dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, the proof of Lemma II.1 is complete.

Now, if $\mathrm{Min}\ (-\sigma, -\sigma/\delta) > n/2$ and if $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, then u is continuous and bounded on $\overline{\mathbb{R}^n_+}$ and there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ and for every $(t, x) \in \mathbb{R}^n_+$, we have

$$|u(t,x)|^2 \leqq C.\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+} \tau^{2(\sigma+\delta(m-j)+j)}|D^j_t D^\alpha_x u(\tau,y)|^2\, d\tau\ dy + \int_{\mathbb{R}^n_+} |u(\tau,y)|^2\, d\tau\ dy\}.$$

The change of variable of (i) yields

$$|u(t, x)|^2 \leqq$$
$$\leqq \frac{C}{\lambda.\mu^{n-1}}\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+} \lambda^{-2(\sigma+\delta(m-j))}\mu^{2(m-j)}|\tau^{2(\sigma+\delta(m-j)+j)}D^j_t D^\alpha_x u(\tau,y)|^2 d\tau dy +$$

$$+ \int_{\mathbb{R}^n_+} |u(\tau,y)|^2 d\tau\ dy\} ;$$

we choose $\lambda = r^{\frac{1}{2\sigma}}$ and $\mu = \lambda^{\delta}$, which gives

$$|u(t,x)|^2 \leq C.\ r^{-\frac{2\sigma+1+\delta(n-1)}{2\sigma}} \{||u||^2_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)} + r\ ||u||^2_{L^2(\mathbb{R}^n_+)}\} ,$$

and taking $r = ||u||^2_{W^m_{\sigma,\delta}} . ||u||^{-2}_{L^2}$, we obtain the inequality (2.2).

PROPOSITION II.2. Let ℓ be an integer, $0 \leq \ell < -\sigma - \frac{1}{2}$; then the map $u \longmapsto \gamma_\ell u = D^\ell_t u(t=0): \mathcal{D}(\overline{\mathbb{R}^n_+}) \to \mathcal{D}(\mathbb{R}^{n-1})$ can be extended to a linear and continuous map from $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ into $H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(\mathbb{R}^{n-1})$.

P r o o f . By Chapter I there exists a constant $C > 0$ such that for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have

$$|D^\ell_t v(0)|^2 \leq C.\{\int_0^{+\infty} |t^{\sigma+m} D^m_t v|^2 dt + \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt\} .$$

If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, then the function $u(t) = v(t\ \Lambda^{-\frac{1}{\sigma}})$, where Λ is a positive constant, belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; hence there exists a constant $C > 0$ such that for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ and for every $\Lambda > 0$, we have

$$\Lambda^{-\frac{2(\sigma+\ell)+1}{2\delta}} |D^\ell_t v(o)|^2 \leq C.\ \{\int_0^{+\infty} |t^{\sigma+m} D^m_t v|^2 dt + \Lambda^{2m}\int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt\} .$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}^n_+})$, and for every $\xi \in \mathbb{R}^{n-1}\setminus\{0\}$ let us consider the function $v(t) = \hat{u}(t, \xi)$, where ^ is the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; as in Lemma II.1, we deduce

$$||\gamma_\ell u||_{H^{-\frac{2(\sigma+\ell)+1}{2\delta}}} \leq C.\ ||u||_{W^m_{\sigma,\delta}} .$$

It will be very useful in the sequel to have an inequality of the "compacity" type for the spaces $W^m_{\sigma,\delta}$:

PROPOSITION II.3. Let m *be an integer* $\geqq 1$ *and put* $\delta_1 = \mathrm{Min}\,(1, \delta)$. *There exists a constant* $C > 0$ *such that for every* $\varepsilon > 0$ *and for every* $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ *with* $\operatorname{supp} u \subset \{|t| \leq 1\}$, *we have*

$$(2.3) \qquad ||u||_{W^{m-1}_{\sigma+\delta_1,\delta}} \leq C.\,\{\varepsilon\,||u||_{W^m_{\sigma,\delta}} + \varepsilon^{-(m-1)}||u||_{L^2}\}\ .$$

P r o o f : We begin by establishing a lemma:

LEMMA II.2. The map $u \longmapsto \{||t^{\sigma+m}D^m_t u||^2_{L^2} + \sum_{|\alpha|=m} ||t^{\sigma+\delta m}D^\alpha_x u||^2_{L^2} + ||u||^2_{L^2}\}^{\frac{1}{2}}$ *is an equivalent norm for the space* $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$.

P r o o f . Let k and j be integers such that $\sigma+\delta k+j \geq 0$ and $k+j \leq m$. It follows from Chapter I that if $v(t) \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, then $t^{\sigma+\delta k+j}D^j_t v \in L^2(\mathbb{R}_+)$ and

$$\int_0^{+\infty} |t^{\sigma+\delta k+j}\ D^j_t v|^2 dt \leq C.\{\int_0^{+\infty} |t^{\sigma+m}D^m_t v|^2 dt + \int_0^{+\infty} |t^{\sigma+\delta m}v|^2 dt\}$$

where C is a constant >0 which does not depend on v .

If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, then the function $u(t) = v(t\,\Lambda^{-1/\sigma})$, where Λ is a positive constant, belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; hence there exists a constant $C > 0$ such that for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$ and for every Λ , we have

$$(2.4) \quad \Lambda^{2k}\int_0^{+\infty} |t^{\sigma+\delta k+j}D^j_t v|^2 dt \leq C.\{\int_0^{+\infty} |t^{\sigma+m}D^m_t v|^2 dt + \Lambda^{2m}\int_0^{+\infty} |t^{\sigma+\delta m}v|^2 dt\}.$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}^n_+})$ and for every $\xi \in \mathbb{R}^{n-1}\setminus\{0\}$, let us consider the function $v(t) = \hat{u}(t, \xi)$, where ^ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$. From the preceding inequality we deduce, taking $\Lambda = |\xi|$ and integrating in ξ over $\mathbb{R}^{n-1}$, that there exists a constant $C > 0$ such that for every $u \in \mathcal{D}(\overline{\mathbb{R}^n_+})$, we have

$$||u||^2_{W^m_{\sigma,\delta}} \leq C.\ \{||t^{\sigma+m}D^m_t u||^2_{L^2} + \sum_{|\alpha|=m} ||t^{\sigma+\delta m}D^\alpha_x u||^2_{L^2} + ||u||^2_{L^2}\}\ .$$

The space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ being dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, Lemma II.2 is a consequence of this inequality and the Banach theorem.

P r o o f of Proposition II.3. The inequality (2.4) with $j = m-1$, $k = 1$ and $\Lambda^{-1} = \varepsilon > 0$ implies

$$\int_0^{+\infty} |t^{\sigma+\delta+m-1} D_t^{m-1} v|^2 \, dt \leqq$$
$$\leqq C.\{ \varepsilon^2 \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 \, dt + \varepsilon^{-2(m-1)} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 \, dt\} .$$

We apply this inequality to the function $v(t) = \hat{u}(t, \xi)$ for $u \in \overline{\mathcal{D}}(\mathbb{R}^n_+)$ and $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$ and integrate in ξ over $\mathbb{R}^{n-1}$. This yields

$$(2.5) \qquad \begin{aligned} &||t^{\sigma+\delta+m-1} D_t^{m-1} u||^2_{L^2(\mathbb{R}^n_+)} \leqq \\ &\leqq C. \{\varepsilon^2 ||t^{\sigma+m} D_t^m u||^2_{L^2(\mathbb{R}^n_+)} + \varepsilon^{-2(m-1)} ||u||^2_{L^2(\mathbb{R}^n_+)}\} \end{aligned}$$

provided $\operatorname{supp} u \subset \{|t| \leqq 1\}$.

Besides, we know that there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and for every $v(x) \in H^m(\mathbb{R}^{n-1})$, we have

$$(2.6) \qquad \sum_{|\alpha|=m-1} \int_{\mathbb{R}^{n-1}} |D_x^\alpha v|^2 dx \leqq C.\{\varepsilon^2 \sum_{|\alpha|=m} \int_{\mathbb{R}^{n-1}} |D_x^\alpha v|^2 dx + \varepsilon^{-2(m-1)} \int_{\mathbb{R}^{n-1}} |v|^2 dx\}.$$

We use this inequality to the function $v(x) = u(t,x)$, $t > 0$, where $u \in \overline{\mathcal{D}}(\mathbb{R}^n_+)$; we multiply by $t^{\sigma+\delta m}$ and integrate in $t > 0$ over $\mathbb{R}_+$, thus obtaining

$$(2.7) \qquad \sum_{|\alpha|=m-1} ||t^{\sigma+\delta m} D_x^\alpha u||^2_{L^2(\mathbb{R}^n_+)} \leqq C. \{\varepsilon^2 \sum_{|\alpha|=m} ||t^{\sigma+\delta m} D_x^\alpha u||^2_{L^2(\mathbb{R}^n_+)} + \varepsilon^{-2(m-1)} ||u||^2_{L^2(\mathbb{R}^n_+)}\}$$

provided $\operatorname{supp} u \subset \{|t| \leqq 1\}$.

The inequality (2.3) for $\delta \leqq 1$ is a consequence of (2.5) and (2.7). For $\delta \geqq 1$, we replace the inequality (2.5) by the inequality

$$(2.8)\quad ||t^{\sigma+m}D_t^{m-1}u||^2_{L^2(\mathbb{R}^n_+)} \leqq C.\ \{\varepsilon^2||t^{\sigma+m}D_t^m u||^2_{L^2(\mathbb{R}^n_+)} + \varepsilon^{-2(m-1)}||u||^2_{L^2(\mathbb{R}^n_+)}\}$$

if supp $u \subset \{|t| \leqq 1\}$. This inequality is easy to prove in the same way as (2.5).

After that we multiply (2.7) by $t^{2(\sigma+1+\delta(m-1))}$ and choose $\varepsilon = \eta t^{\delta-1}$, $\eta > 0$. We complete the proof as before.

III. The case of a bounded open set Ω of $\mathbb{R}^n$, $n > 1$

Let Ω be a bounded open set of $\mathbb{R}^n$ with a boundary Γ. We assume that Ω is a compact C^∞-manifold. We introduce a C^∞-function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$(3.1)\qquad \begin{cases} \Omega = \{x \in \mathbb{R}^n;\ \phi(x) > 0\}, \\ \Gamma = \{x \in \mathbb{R}^n;\ \phi(x) = 0\}, \\ \text{grad}\ \phi(x) \neq 0 \quad \text{for} \quad x \in \Gamma, \end{cases}$$

where grad $\phi(x) = (\frac{\partial\phi}{\partial x_1}(x), \ldots, \frac{\partial\phi}{\partial x_n}(x))$ is the gradient vector associated with ϕ. Let $(X_i)_{0\leqq i\leqq q}$ be vector fields with C^∞-coefficients on $\mathbb{R}^n$ such that

(3.2) X_0 is transversal to Γ on Γ, i. e. $(X_0\phi)(x) \neq 0$ for $x \in \Gamma$;

(3.3) X_i is tangent to Γ on Γ for $i = 1, \ldots, q$, i. e. $(X_i\phi)(x) = 0$ for $x \in \Gamma$;

(3.4) for every $x \in \overline{\Omega}$, the rank of the system $(X_i(x))_{0\leqq i\leqq q}$ is equal to n.

Let m be an integer, $-\sigma$ and δ two real numbers >0 such that $\sigma+m \geqq 0$ and $\sigma+\delta m \geqq 0$. We consider the space

$$W^m_{\sigma,\delta}(\Omega) = \{u \in L^2(\Omega); \phi^{\text{Max}(0,\sigma+<\delta,\alpha>)}X^\alpha u \in L^2(\Omega) \quad \text{for } |\alpha| \leqq m\}$$

equipped with the canonical norm. (We have used the notation $X^\alpha =$

$= X_0^{\alpha_0} \ldots X_q^{\alpha_q}$ for $\alpha = (\alpha_0, \ldots, \alpha_q) \in \mathbb{N}^{q+1}$ and $<\delta,\alpha> =$
$= \delta \sum_{i=1}^{q} \alpha_i + \alpha_0.$)

<u>*PROPOSITION III.1.*</u> *Under the above assumptions, we have*

(i) $W^m_{\sigma,\delta}(\Omega) \subset H^m_{loc}(\Omega)$;

(ii) $\Phi\, u \in W^m_{\sigma,\delta}(\Omega)$ *for every* $\Phi \in C^\infty(\overline{\Omega})$ *and for every* $u \in W^m_{\sigma,\delta}(\Omega)$.

P r o o f . (i) Under the assumption (3.4), for every $x_0 \in \Omega$ there exists a neighbourhood $V(x_0)$ of x_0 in Ω in which we can write

$$\frac{\partial}{\partial x_k} = \sum_{i=0}^{q} \beta_i^k(x)\, X_i$$

for $k = 1, \ldots, n$ with some convenient functions β_i^k which are C^∞ in $V(x_0)$, and we can easily verify (i).

(ii) Let Φ be a C^∞-function on $\overline{\Omega}$ and $u \in W^m_{\sigma,\delta}(\Omega)$. Then $\Phi\, u \in$ $\in L^2(\Omega)$ and for $|\alpha| \leqq m$, we have

$$X^\alpha(\Phi u) = \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} (X^\beta \Phi)\, (X^{\alpha-\beta} u) .$$

It results that $\phi^{\mathrm{Max}(0,\sigma+<\delta,\alpha>)} X^\alpha(\Phi u) \in L^2(\Omega)$, that is to say $\Phi\, u \in$ $\in W^m_{\sigma,\delta}(\Omega)$.

<u>REMARK III.1.</u> It is easy to prove that the space $W^m_{\sigma,\delta}(\Omega)$ does not depend on the choice of the vector fields $(X_i)_{0\leqq i\leqq q}$ satisfying the conditions (3.2), (3.3), (3.4).

<u>*PROPOSITION III.2.*</u> (i) *If* $m > n/2$ *and if* $u \in W^m_{\sigma,\delta}(\Omega)$ *, then* u *is continuous on* Ω *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\Omega)$ *and for every* $x \in \Omega$ *, we have*

$$(3.5) \qquad |u(x)| \leqq C.\ \phi(x)^{-\frac{\sigma+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} \,||u||^{n/2m}_{W^m_{\sigma,\delta}}\, ||u||^{1-n/2m}_{L^2} ;$$

(ii) *if* $\mathrm{Min}(-\sigma, -\sigma/\delta) > n/2$ *and if* $u \in W^m_{\sigma,\delta}(\Omega)$ *, then* u *is continuous and bounded on* Ω *and there exists a constant* $C > 0$ *such that for every* $u \in W^m_{\sigma,\delta}(\Omega)$ *and for every* $x \in \Omega$ *, we have*

$$(3.6) \qquad |u(x)| \leq C. \; ||u||_{W^m_{\sigma,\delta}}^{-\frac{1+\delta(n-1)}{2\sigma}} \; ||u||_{L^2}^{1+\frac{1+\delta(n-1)}{2\sigma}}$$

P r o o f . (i) The inequality (3.5) can be obtained by means of Proposition III.1 and of a partition of unity for functions $u \in W^m_{\sigma,\delta}(\Omega)$ with supports in a neighbourhood of the boundary Γ of Ω.

Let x_0 be a point of Γ ; from the properties (3.1) we see that there exists a neighbourhood $V(x_0)$ of x_0 in R^n and a diffeomorphism $\Theta = (\theta_1, \dots, \theta_n)$ with $\theta_n = \phi$ from $V(x_0)$ onto the unit ball of $\mathbb{R}^n$ such that

$$(3.7) \qquad \begin{cases} \Theta(V \cap \Omega) = B_+ = \{y \in \mathbb{R}^n \; ; \; |y| \leq 1 \; , \; y_n > 0\} \; ; \\ \Theta(V \cap \Gamma) = B_0 = \{y \in \mathbb{R}^n \; ; \; |y| \leq 1 \; , \; y_n = 0\} \; ; \\ X_0(\theta_k) = 0 \quad \text{in} \quad V \quad \text{for} \quad k = 1, \dots, n-1 \; . \end{cases}$$

Under these conditions, if $u \in W^m_{\sigma,\delta}(\Omega)$ with supp $u \subset V$ and if $v = u \circ \Theta^{-1}$, then $v \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ with supp $v \subset \overline{B_+}$. In fact, to this end it suffices to notice that the diffeomorphism Θ transforms the vector fields $(X_i)_{0\leq i\leq q}$ into the vector fields $(I_i)_{0\leq i\leq q}$ with

$$(3.8) \qquad I_0 = \alpha\frac{\partial}{\partial y_n}, \; \alpha(y) \neq 0 \quad \text{for} \quad y \in B = \{y \in \mathbb{R}^n ; \; |y| \leq 1\} \; ;$$

$$(3.9) \qquad I_i = I_i^t + \left[(X_i\phi) \circ \Theta^{-1}\right]\frac{\partial}{\partial y_n} \quad \text{for} \quad i = 1, \dots, q \; ,$$

where I_i^t means a homogeneous differential operator of order 1, with C^∞-coefficients in the variables $y_1, \dots, y_{n-1}$;

(3.10) for every $y \in B = \{y \in \mathbb{R}^n \; ; \; |y| \leq 1\}$, the rank of the system $(I_i)_{0\leq i\leq q}$ is equal to n .

Hence, the inequality (3.5) follows from the inequality (2.1) in Proposition II.1.

(ii) In the same way, the inequality (3.6) at the boundary follows from the inequality (2.2) in Proposition II.1.

In the interior, it follows from the fact that if $u \in W^m_{\sigma,\delta}(\Omega)$, then $u \in H^m_{loc}(\Omega)$ and then it belongs to $H^{m'}_{loc}(\Omega)$ as well where $m' = -\frac{\sigma n}{1+\delta(n-1)}$; in fact, since $\sigma+m \geqq 0$ and $\sigma+\delta m \geqq 0$, we have $m' \leqq m$. Then the inequality (3.6) in the interior is a consequence of the classical inequality

$$|u(x)| \leqq C.\ ||u||^{n/2m'}_{H^{m'}}\ ||u||^{1-n/2m'}_{L^2}\ .$$

<u>*PROPOSITION III.3.*</u> *Let* ℓ *be an integer,* $0 \leqq \ell < -\sigma - \frac{1}{2}$; *then the map* $u \to \gamma_\ell u = \frac{\partial^\ell u}{\partial n^\ell}\Big|_\Gamma : \mathcal{D}(\bar{\Omega}) \to \mathcal{D}(\Gamma)$ *can be extended to a linear and continuous map from* $W^m_{\sigma,\delta}(\Omega)$ *into* $H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(\Gamma)$.

($\frac{\partial}{\partial n}$ means the derivative along the unit normal vector to Γ which points into the interior of Ω.) This proposition follows from Proposition II.2.

<u>*PROPOSITION III.4.*</u> *Let* m *be an integer* $\geqq 1$ *and* $\delta_1 =$ $= \mathrm{Min}\ (1, \delta)$. *There exists a constant* $C > 0$ *such that for every* $\varepsilon > 0$ *and for every* $u \in W^m_{\sigma,\delta}(\Omega)$, *we have*

$$(3.11) \qquad ||u||_{W^{m-1}_{\sigma+\delta_1,\delta}} \leqq C.\{\ \varepsilon\ ||u||_{W^m_{\sigma,\delta}} + \varepsilon^{-(m-1)}||u||_{L^2}\ \}.$$

P r o o f . As before, we see that the inequality (3.11) at the boundary follows from the inequality (2.3) and, in the interior, from the classical inequality for the usual Sobolev spaces:

$$||u||_{H^{m-1}} \leqq C.\ \{\ \varepsilon||u||_{H^m} + \varepsilon^{-(m-1)}||u||_{L^2}\}.$$

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