EMBEDDINGS OF SOBOLEV SPACES

David E. Edmunds

1. Introduction

As is very well known, Sobolev spaces provide a natural framework for the modern theory of partial differential equations, and this theory is greatly aided by the possibility of embedding one Sobolev space in a variety of other such spaces, in L^p spaces, or even in spaces of continuous functions, the corresponding embedding maps being continuous and often compact. The compact embeddings are very important; they make it possible to reduce elliptic boundary-value problems to questions involving the Fredholm-Riesz-Schauder theory of compact linear operators, and they are at the heart of much of the work on the asymptotic distribution of eigenvalues of elliptic operators.

In this paper we discuss some of these embedding maps, focussing initially on the question as to whether they have any properties better than mere continuity, such as compactness. A convenient criterion for this is provided by the notion of a k-set contraction, which generalises the idea of a compact map; the compact maps are precisely the 0-set contractions. Some results concerning the k-set contractive nature of certain embedding maps are given, both for bounded and unbounded space domains; the applications include the location of the essential spectrum of an elliptic operator in an unbounded domain. If the embedding map is compact it is desirable to classify it in some way, and to do this we use the approximation numbers of the map, in the sense of PIETSCH [18]; these measure the closeness with which the map can be approximated by maps with finite-dimensional range. We obtain estimates for these numbers by means somewhat different from those employed by BIRMAN and SOLOMJAK [3] and other writers on this topic. The results have applications to the theory of the asymptotic distribution of eigenvalues of elliptic operators, for example.

2. Embeddings and ball-k-set contractions

Denote by $x = (x_1, \ldots, x_n)$ points of n-dimensional real Euclidean space $\mathbb{R}^n$ ($n \geq 2$), and let Ω be a non-empty domain in $\mathbb{R}^n$; that is, a connected open set. The closure and boundary of Ω will be represented by $\bar{\Omega}$ and $\delta\Omega$ respectively. The symbol α will always be reserved for a multi-index $(\alpha_1, \ldots, \alpha_n)$, where each α_i is a non-negative integer; we shall write

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \alpha! = \alpha_1! \ldots \alpha_n!, \text{ and } x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} .$$

Partial derivatives will be expressed by means of the symbol

$$D^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i} .$$

Given any positive integer m and any p, $1 \leq p < \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined to be $\{u : D^\alpha u \in L^p(\Omega) \text{ for all } \alpha, |\alpha| \leq m\}$, endowed with the norm

$$||u||_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} ||D^\alpha u||_{p,\Omega}^p \right)^{1/p},$$

where

$$||v||_{p,\Omega} = \left(\int_\Omega |v(x)|^p \, dx \right)^{1/p} .$$

Here the functions involved may be real- or complex-valued, but for definiteness we shall assume that they are complex-valued; the derivatives are taken in the sense of distributions. The closure in $W^{m,p}(\Omega)$ of the set $C_0^\infty(\Omega)$ of all infinitely differentiable complex-valued functions with compact support in Ω is denoted by $\overset{o}{W}{}^{m,p}(\Omega)$.

One of the most celebrated embedding theorems is that due to Rellich; it asserts that if Ω is bounded and has a smooth enough boundary $\partial\Omega$, then $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. In other words, the identity map $I : W^{1,2}(\Omega) \to L^2(\Omega)$ is compact. As for the conditions to be imposed on $\delta\Omega$, it is enough (cf. for example, [16], p. 17) that $\partial\Omega$ should be of class C, by which we mean that given any $p \in \partial\Omega$, there exist an n-neighbourhood $N(p)$ of p, a Cartesian co-ordinate system $y = (y', y_n)$, $y' = (y_1, \ldots, y_{n-1})$, with $y \neq 0$ at $x = p$, and a function f_p such that

$$\partial\Omega \cap N(p) = \{(y', f_p(y')) : y' \in G_p\},$$

where G_p is a convex (n-1)-neighbourhood of 0 and f_p is a real-valued continuous function on the (n-1)-closure $\overline{G}_p$ of G_p. The corresponding natural map $I_0 : \overset{o}{W}{}^{1,2}(\Omega) \to L^2(\Omega)$ is always compact, no matter what kind of boundary the bounded set Ω may have.

If $\partial\Omega$ is not of class C the embedding map I may not be compact, and to illustrate this we refer to an example constructed by FRAENKEL [10] (see also [6], p. 521 for a somewhat similar example in connection with the Poincaré inequality). The example consists of a subset S of $\mathbb{R}^2$ referred to as 'rooms and passages' and made up of an infinite sequence of square boxes of decreasing size joined together by pipes: the origin is a point of accumulation of S, and it is this point which is responsible for the failure of ∂S to be of class C. However, S is not remarkably pathological, for ∂S is a rectifiable Jordan curve and there are $C^{0,\lambda}$ homeomorphisms of neighbourhoods of $\overline{S}$ which map S onto the open unit ball in R^2. Despite the relatively innocuous nature of ∂S, however, the corresponding map I is not compact. Let us try to analyse what goes wrong in cases such as this.

If Ω is bounded and Ω_0 is any open set such that $\overline{\Omega}_0 \subset \Omega$, then there is an open set Ω_1, with $\partial\Omega_1$ analytic, such that $\overline{\Omega}_0 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$, and hence the natural embedding of $W^{1,2}(\Omega)$ in $L^2(\Omega_0)$ is compact, as it may be represented as the composition of the natural maps

$$W^{1,2}(\Omega) \to W^{1,2}(\Omega_1) \to L^2(\Omega_1) \to L^2(\Omega_0) ,$$

in which the outer two maps are continuous and the middle one is compact. This clarifies the role of the boundary in situations in which I is not compact, and suggests that we should look at L^2 integrals over boundary strips. Thus given any $\varepsilon > 0$, put $\Omega(\varepsilon) = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) < \varepsilon\}$ and set

$$\Gamma_\Omega(\varepsilon) = \sup\left\{\int_{\Omega(\varepsilon)} |u(x)|^2 dx/\|u\|^2_{1,2,\Omega} : u \in W^{1,2}(\Omega) \setminus \{0\}\right\} .$$

Evidently $0 < \Gamma_\Omega(\varepsilon) \leq 1$; and $\Gamma_\Omega(\varepsilon)$ is monotonic non-increasing as $\varepsilon \downarrow 0$, so that we may put

$$\Gamma_\Omega(0) = \lim_{\varepsilon \to 0} \Gamma_\Omega(\varepsilon) .$$

My colleague Amick has pursued this line of enquiry and has shown [1] that I is compact if and only if $\Gamma_\Omega(0) = 0$. This result is a special case of his work on I from the standpoint of ball-k-set contractions, and to explain this we make a short digression.

Let X and Y be Banach spaces and let B be a bounded subset of X. The ball-measure of non-compactness of B is defined to be

$$\beta_X(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many open balls of radius } \varepsilon\} .$$

A map $T \in B(X, Y)$ is called a ball-k-set contraction if for all bounded sets $B \subset X$, $\beta_Y(T(B)) \leq k\,\beta_X(B)$; for such a map we put

$$\gamma(T) = \inf\{k \geq 0 : T \text{ is a ball-k-set contraction}\} .$$

Evidently T is compact if and only if $\gamma(T) = 0$; note also that $\gamma(T) \leq \|T\|$ for all $T \in B(X,Y)$, so that new information about T is gained only if it turns out that $\gamma(T) < \|T\|$. There is now a well-developed theory associated with these ideas, with a number of interesting theorems available; we mention in particular the results that if $X = Y$, then $\lim_{n\to\infty} (\gamma(T^n))^{1/n}$ exists and equals the radius of the essential spectrum of T [17], while $id_X - T$ (id_X is the identity mapping of X to X) is a Fredholm map of index 0 if $\gamma(T) < 1$.

Returning to the embedding I of $W^{1,2}(\Omega)$ in $L^2(\Omega)$, a connection between $\Gamma_\Omega(0)$ and $\gamma(I)$ is provided by AMICK [1], who has shown that

$$\gamma(I) = (\Gamma_\Omega(0))^{1/2} \tag{1}$$

for any bounded domain Ω. To sketch the proof of this interesting result, suppose first that $\eta \overset{\text{def}}{=} |\gamma(I)|^2 - \Gamma_\Omega(0) > 0$, and choose $\varepsilon > 0$ so small that $\Gamma_\Omega(\varepsilon) \leq \Gamma_\Omega(0) + \frac{1}{2}\eta$. Let U be an open set such that $\Omega \setminus \overline{\Omega(\varepsilon)} \subset U \subset \overline{U} \subset \Omega$, with ∂U of class C. Then since $W^{1,2}(\Omega)$ is compactly embedded in $L^2(U)$, the open unit ball B in $W^{1,2}(\Omega)$ is totally bounded as a subset of $L^2(U)$, so that there exist functions $f_1, \ldots, f_k \in L^2(U)$ such that given any $u \in B$, $||u - f_i||^2_{2,U} \leq \eta/8$ for some i, $1 \leq i \leq k$. If we extend the f_j to the whole of Ω by setting them equal to 0 in $\Omega \setminus \overline{U}$, and denote the extensions by $\tilde{f}_j$, then since $\gamma(I) = \beta_{L^2(\Omega)}(I(B))$, there exists $v \in B$ such that $||v - \tilde{f}_j||^2_{2,\Omega} \geq |\gamma(I)|^2 - \eta/8$ for $j = 1, 2, \ldots, k$. Hence for a suitable j,

$$\begin{aligned} ||v||^2_{2,\Omega(\varepsilon)} \geq ||v||^2_{2,\Omega\setminus U} &= ||v - \tilde{f}_j||^2_{2,\Omega} - ||v - \tilde{f}_j||^2_{2,U} \\ &\geq |\gamma(I)|^2 - \eta/4 \\ &= \Gamma_\Omega(0) + \tfrac{3}{4}\eta . \end{aligned}$$

But this implies that

$$\Gamma_\Omega(\varepsilon) \geq \Gamma_\Omega(0) + \frac{3}{4}\eta ,$$

which contradicts the inequality $\Gamma_\Omega(\varepsilon) \leq \Gamma_\Omega(0) + \frac{1}{2}\eta$.

Thus $|\gamma(I)|^2 \leq \Gamma_\Omega(0)$. To prove equality, suppose that $\Gamma_\Omega(0) > |\gamma(I)|^2$, in which case there exists $\delta > 0$ such that

$$\zeta \overset{\text{def}}{=} (\Gamma_\Omega(0) - 2\delta)^{\frac{1}{2}} - \gamma(I) - \delta > 0 .$$

There are functions $g_1, \ldots, g_\ell \in L^2(\Omega)$ such that given any $u \in B$, $||u - g_i||_{2,\Omega} \leq \gamma(I) + \delta$ for some i, $1 \leq i \leq \ell$. Take $\varepsilon_1 > 0$ so small that $\Gamma_\Omega(\varepsilon_1) \leq \Gamma_\Omega(0) + \delta$; there exists $u_1 \in B$ such that

$||u_1||^2_{2,\Omega(\varepsilon_1)} \geqq \Gamma_\Omega(0) - \delta$. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be such that $||u_1||^2_{2,\Omega(\varepsilon_2)} \leqq \delta$, so that $||u_1||^2_{2,\Omega(\varepsilon_1)\setminus\Omega(\varepsilon_2)} \geqq \Gamma_\Omega(0) - 2\delta$. Thus for some ℓ_1, $1 \leqq \ell_1 \leqq \ell$,

$$(\Gamma_\Omega(0) - 2\delta)^{\frac{1}{2}} \leqq ||u_1||_{2,\Omega(\varepsilon_1)\setminus\Omega(\varepsilon_2)} \leqq ||g_{\ell_1}||_{2,\Omega(\varepsilon_1)\setminus\Omega(\varepsilon_2)} + \gamma(I) + \delta ,$$

which shows that $\zeta \leqq ||g_{\ell_1}||_{2,\Omega(\varepsilon_1)\setminus\Omega(\varepsilon_2)}$. Let $u_2 \in B$ be such that $||u_2||^2_{2,\Omega(\varepsilon_2)} \geqq \Gamma_\Omega(0) - \delta$; there exist $\varepsilon_3 \in (0,\varepsilon_2)$ and ℓ_2, $1 \leqq \ell_2 \leqq \ell$, such that $\zeta \leqq ||g_{\ell_2}||_{2,\Omega(\varepsilon_2)\setminus\Omega(\varepsilon_3)}$. Proceeding in this way we obtain a strictly decreasing sequence (ε_i) of positive numbers and a sequence (ℓ_i) of integers, with $1 \leqq \ell_i \leqq \ell$ for all i, such that $\zeta \leqq ||g_{\ell_i}||_{2,\Omega(\varepsilon_i)\setminus\Omega(\varepsilon_{i-1})}$ for $i = 2, 3, \ldots$. Infinitely many of the ℓ_i must be equal, to ℓ_0 say; but this implies that $||g_{\ell_0}||_{2,\Omega} = \infty$. This contradiction establishes (1).

Since $\gamma(I) \leqq ||I|| \leqq 1$ for all bounded domains Ω, it is natural to ask whether $\gamma(I)$ can be equal to 1 , or whether $\gamma(I) < 1$ for all such sets Ω, no matter what $\partial\Omega$ is like. It turns out that the former is the case, for [1] contains an example of 'rooms and passages' type for which $\Gamma_\Omega(0) = 1$, and hence $\gamma(I) = 1$.

One further comment on this circle of ideas should be made. This concerns the Poincaré inequality, which asserts that for a bounded domain Ω with suitable restrictions on $\partial\Omega$,

$$\int_\Omega |u(x)|^2 \, dx \leqq \text{const.} \left\{ \left| \int_\Omega u(x)dx \right|^2 + \int_\Omega |\text{grad } u(x)|^2 dx \right\}$$

for all $u \in W^{1,2}(\Omega)$, the constant being independent of the particular u . It emerges from the work of [1] that the Poincaré inequality holds if and only if $\Gamma_\Omega(0) < 1$, or, of course, if and only $\gamma(I) <$ < 1 . The ubiquitous 'rooms and passages' can also be used to provide an example of a bounded domain Ω for which the Poincaré inequality holds but I is not compact.

So far we have mentioned <u>bounded</u> domains Ω only. If Ω is unbounded the position is much worse as regards compact embeddings, for even if $\partial\Omega$ is very smooth, neither I nor I_0 is compact unless Ω becomes very thin for large values of $|x|$; the compactness of I is a particularly rare phenomenon. However, there is still something that can be salvaged, for EVANS and I have shown [7] that if $\overset{\circ}{W}{}^{1,2}(\Omega)$ is normed by $\sum_{|\alpha|\leq 1} ||D^\alpha u||_{2,\Omega}$, then

$$\gamma(I_0) \leq \limsup_{|x|\to\infty, x\in\Omega} \{ |\Omega \cap B(x, d)| / |B(x, d)| \}^{1/n} ,$$

where d is any number such that $0 < d \leq 1$, $B(x, d)$ is the open ball in $\mathbb{R}^n$ with centre x and radius d, and $|.|$ denotes Lebesgue n-measure. (Actually we used a measure of noncompactness slightly different from that presented here, but the end result is unaffected.) This kind of result can be used (cf. [7]) to locate the essential spectrum of certain self-adjoint realisations of elliptic operators on unbounded domains, such as cylinders or strips, which do not open out at infinity. For the embedding maps I_p of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ $(p > n)$ estimates of $\gamma(I_p)$ are given in [8]. Of course, weighted Sobolev spaces are, in various ways, more suitable for the study of problems involving unbounded domains, but the embeddings which then arise can often be handled by adaptations of the techniques needed for the results described above, and estimates for γ obtained.

3. Embeddings and approximation numbers

Here we return to the case in which Ω is a bounded domain in $\mathbb{R}^n$, and suppose that the boundary $\partial\Omega$ is smooth enough for various of the natural embeddings, such as that of the Rellich theorem, to be compact. For example, it is known that if $1 \leq p < \infty$, $1 \leq q < \infty$ and r is a positive integer such that $\frac{r}{n} > \frac{1}{p} - \frac{1}{q}$, then $W^{r,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ provided that $\partial\Omega$ is minimally smooth in the sense of STEIN ([20], p. 189). The question then arises as to whether it is possible to clasify these compact embedding maps in some useful way; this matter has been extensively discussed in recent times by numerous authors, in particular by BIRMAN and SOLOMJAK [3] and by TRIEBEL [21], [22].

The basic idea is that of the <u>approximation numbers of</u> a mäp, in the terminology of PIETSCH [18]. Let X, Y be Banach spaces and let $T \in B(X, Y)$. For each $r \in \mathbb{N}$ define

$$a_r(T) = \inf \{||T - F||: F \in B(X, Y), \dim F(X) \leq r\} ;$$

$a_r(T)$ is called the r^{th}-approximation number of T , and it is clear that $||T|| = a_0(T) \geq a_1(T) \geq \ldots \geq 0$. By analogy with the familiar sequence spaces, given any real $s > 0$, T is said to be of <u>type</u> ℓ^s if $\sum_{r=0}^{\infty} (a_r(T))^s < \infty$; T is of type c_0 if $\lim_{r\to\infty} a_r(T) = 0$. Every map of type c_0 is compact, as it is the limit of maps with finite-dimensional range, but the converse is not true, even in separable spaces, as is shown by the result of Per Enflo on the existence of a separable Banach space without the approximation property.

One of the most interesting properties of approximation numbers is that if X and Y are Hilbert spaces, T is compact and (λ_r) is the sequence of eigenvalues of the positive square root $(T^*T)^{\frac{1}{2}}$ arranged in descending order of magnitude, then $a_r(T) = a_r(T^*) = \lambda_r$ for all $r \in \mathbb{N}$. In particular, if $X = Y$ and T is positive, self-adjoint and compact, $a_r(T) = \lambda_r$ for all $r \in \mathbb{N}$. It is this classical result which leads to one of the most natural applications of these notions, namely to questions of the distribution of eigenvalues of elliptic boundary-value problems. Consider, as a simple example, the following eigenvalue problem in a bounded domain Ω in $\mathbb{R}^n$:

$$- \Delta u + \lambda u = 0 \quad \text{in} \quad \Omega \ , \ u = 0 \quad \text{on} \quad \partial\Omega \ .$$

Denote by V the closed subspace of $W^{2,2}(\Omega)$ consisting of all those elements u which are zero on $\partial\Omega$. Under mild smoothness conditions on $\partial\Omega$ it is known that the map $A : V \to L^2(\Omega)$ defined by

$Au = -\Delta u$ $(u \in V)$ is an isomorphism. Denote by V_A the space V endowed with the inner product $(u, v)_A = (Au, Av)_{L^2(\Omega)}$, and let J_A be the canonical embedding of V_A in $L^2(\Omega)$. Since A is an isomorphism, $(.,.)_A$ induces a norm on V_A equivalent to the original norm on V, and A is an isometry between V_A and $L^2(\Omega)$. If $\partial\Omega$ is smooth enough, J_A is compact and the compact self-adjoint map $J_A A^* = J_A A^{-1} : L^2(\Omega) \to L^2(\Omega)$ has a countable sequence (μ_r) of eigenvalues, each of finite multiplicity and such that $|\mu_r| \to 0$ as $r \to \infty$, the corresponding eigenfunctions u_r forming a complete orthonormal set in $L^2(\Omega)$. If we put $T = J_A A^*$, then given any $u \in L^2(\Omega)$ there is a unique $v \in V$ such that $\Delta v = u$, so that

$(Tu, u)_{L^2(\Omega)} = -\int_\Omega v(x)\, \Delta v(x)\, dx \geq 0$. It follows that T is positive and each $\mu_r > 0$. Thus $T = (T^*T)^{1/2}$ (the positive square root) has eigenvalues μ_r, and as $T^*T = J_A A^* A J_A^* = J_A J_A^*$ we have $\mu_r = a_r(J_A)$. The eigenvalues μ_r satisfy $J_A A^{-1} u_r = \mu_r u_r$, that is, $-\Delta u_r = \mu_r^{-1} u_r$; hence the μ_r^{-1} are eigenvalues of the original eigenvalue problem. Further, all the eigenvalues of the original problem are included in the μ_r^{-1}, and we conclude that the r^{th} eigenvalue of this problem, λ_r say, is given by $\lambda_r = 1/a_r(J_A)$. Knowledge of the approximation numbers thus gives the λ_r, and this can be exploited when the distribution of the λ_r for large r is of importance, as in the celebrated problem of the behaviour of

$$N(t) \overset{\text{def}}{=} \sum_{\lambda_r \leq t} 1$$

for large t. For it turns out to be relatively easy to estimate $a_r(J_A)$, and that $a_r(J_A) \approx r^{2/n}$ for large r, so that $N(t) \approx t^{n/2}$ as $t \to \infty$. These crude estimates can be appreciably refined so as to give the leading term of the asymptotic development of $N(t)$ and good estimates for the remainder; BIRMAN and SOLOMJAK [4] and

and MÉTIVIER [15] have impressive results in this direction, valid for the eigenvalues of arbitrary uniformly elliptic operators. A particular virtue of this kind of approach is that it eliminates the need to have the very precise estimates related to Green's functions upon which most other methods of estimation of $N(t)$ rely.

Another natural use which can be made of estimates of approximation numbers is to determine the rapidity of convergence of approximative processes. This kind of question is taken up in AUBIN's book [2] and by SCHOCK [19].

We now give one of the fundamental results concerning the approximation numbers of embedding maps.

THEOREM 1. Let Ω *be a bounded domain in* $\mathbb{R}^n$, *let* $1 \leq p < \infty$, $1 \leq q < \infty$, *and let* r *be a positive integer such that* $\frac{r}{n} > \frac{1}{p} - \frac{1}{q}$. *Then the* s^{th} *approximation number* $a_s(I_0)$ *of the embedding map* I_0: $\overset{o}{W}{}^{r,p}(\Omega) \to L^q(\Omega)$ *is* $O(s^{-h})$ *as* $s \to \infty$, *where* $h = \frac{r}{n} - \max(0, \frac{1}{p} - \frac{1}{q})$; *thus* I_0 *is of type* ℓ^t *if* $ht > 1$. *When* $p = q$, $a_s(I_0) \geq \text{const. } s^{-r/n}$ *for all large* s. *All these results hold also for the embedding* $I : W^{r,p}(\Omega) \to L^q(\Omega)$ *provided that* $\partial\Omega$ *is minimally smooth.*

Results of this kind seem to have been obtained first by BIRMAN and SOLOMJAK [3], who used a method involving piecewise polynomial approximation. Since their work there have been numerous developments and embellishments of the theory, and various alternative methods of procedure have emerged, including the Fourier approximation method of [9]. Here we shall present a short proof of the theorem which uses piecewise polynomial approximations but is different from that given in [3]. The key step is given by the following lemma, the proof of which is due to D. J. Harris:

LEMMA 1. Let $Q = \{x \in \mathbb{R}^n : a_i < x_i < b_i$ *for* $i = 1, 2, \ldots, n\}$, *and suppose that* $1 \leq p \leq q < \infty$, $\frac{r}{n} > \frac{1}{p} - \frac{1}{q}$, *where* r *is a positive integer. For all* $u \in W^{r,p}(Q)$ *and all* $x \in \mathbb{R}^n$ *put*

$$(P_{r,Q}u)(x) = \frac{\chi_Q(x)}{|Q|} \sum_{|\alpha| \leq r-1} \int_{\mathbb{R}^n} \chi_Q(y) \frac{(x-y)^\alpha}{\alpha!} D^\alpha u(y)\, dy ,$$

where χ_Q *is the characteristic function of* Q. *Let* Q *be subdivided into* 2^{nN} *congruent boxes* Q_j, *and set*

$$(P_N u)(x) = \sum_{j=1}^{2^{nN}} \chi_{Q_j}(x)(P_{r,Q_j}u)(x) , \qquad x \in \mathbb{R}^n.$$

Then for all $u \in W^{r,p}(Q)$ *with* $||u||_{r,p,Q} = 1$,

$$||u - P_N u||_{q,Q} \leq C(2^{-nN}|Q|)^h ,$$

where C *depends only on* n, r, p *and* q .

P r o o f . In view of the density of $C^r(\overline{Q})$ in $W^{r,p}(Q)$ it is enough to prove the lemma when $u \in C^r(\overline{Q})$. For such a u we have, by Taylor's formula, when $x \in \mathbb{R}^n$,

$$u(x) - (P_{r,Q}u)(x) =$$
$$= \frac{\chi_Q(x)}{|Q|} \sum_{|\alpha|=r} \frac{r}{\alpha!} \int_{\mathbb{R}^n} \chi_Q(y)dy \int_0^1 (1-\tau)^{r-1}(x-y)^\alpha D^\alpha u(\tau x+y-\tau y)d\tau$$
$$= \sum_{|\alpha|=r} \frac{r}{\alpha!} F_\alpha(x) , \text{ say.}$$

Then

$$|F_\alpha(x)| \leq \frac{\chi_Q(x)}{|Q|} \int_{\mathbb{R}^n} \chi_Q(x - \tfrac{z}{\tau})dz \int_0^1 |z|^\alpha \tau^{-n-1} |D^\alpha u(x - z)|d\tau .$$

Let Q_0 be the box centred at 0 and obtained by translation of Q. A simple calculation shows that

$$\chi_Q(x)\ \chi_Q(x - \tfrac{z}{\tau}) \leq \chi_Q(x - z)\chi_{2Q_0}(z/\tau) ,$$

and hence

$$|F_\alpha(x)| \leq \frac{1}{|Q|}\int_{\mathbb{R}^n} \left\{\int_0^1 |z|^\alpha \tau^{-n-1}\chi_{2Q_0}(z/\tau)\,d\tau\right\}\chi_Q(x-z)\,|D^\alpha u(x-z)|\,dz$$

$$= |Q|^{-1}\,(g_\alpha * (\chi_Q|D^\alpha u|))(x), \text{ say.}$$

Now put $Q_0 = kU$, where k is a positive number so chosen that $|U| = 1$, and set $z = k\rho\xi$, $\xi \in \partial U$. Thus

$$g_\alpha(z) = \int_{\rho/2}^1 |z^\alpha|\tau^{-n-1}\chi_{2Q_0}(z/\tau)\,d\tau$$

$$\leq \frac{1}{n}\left\{\left(\frac{2}{\rho}\right)^n - 1\right\}\,(k\rho)^{|\alpha|}\,|\xi^\alpha|\,\chi_{2Q_0}(z)$$

$$\leq \frac{1}{n}\left(\frac{2}{\rho}\right)^n\,(k\rho)^r\,|\xi^\alpha|\chi_{2Q_0}(z) \quad ,$$

from which we see that

$$||g_\alpha||^m_{m,\mathbb{R}^n} \leq \left(\frac{2^n}{n}\right)^m k^{rm+n}\int_{\partial U}|\xi^\alpha|^m\,d\xi\int_0^2 \rho^{(r-n)m+n-1}\,d\rho$$

$$= 2^{rm+n}\,n^{-m}|Q|^{1+mr/n}\{(r-n)m+n\}^{-1}\int_{\partial U}|\xi^\alpha|^m\,d\xi \,,$$

provided that $(r - n)m + n > 0$. We take m so that $\frac{1}{m} = 1 + \frac{1}{q} - \frac{1}{p}$, which ensures that this condition is satisfied since $\frac{1}{q} - \frac{1}{p} > -\frac{r}{n}$. Now use Young's theorem on convolutions and Holder's inequality for sums; we have

$$||u - P_{r,Q}u||_{q,Q} \leq \sum_{|\alpha|=r}\frac{r}{\alpha!}||F_\alpha||_{q,Q}$$

$$\leq \sum_{|\alpha|=r}\frac{r}{\alpha!}\,|Q|^{-1}||g_\alpha||_{m,\mathbb{R}^n}||\chi_Q D^\alpha u||_{p,\mathbb{R}^n}$$

$$\leq C|Q|^h\,||u||_{r,p,Q} \,,$$

where C is a constant depending only on n, r, p and q. This inequality, with Q_j in place of Q, gives

$$||u - P_N u||_{q,Q} = \left\{\sum_j ||\chi_{Q_j}(u - P_{r,Q_j}u)||^q_{q,Q_j}\right\}^{1/q}$$

$$\leq C(|Q|2^{-nN})^h\left(\sum_j ||u||^q_{r,p,Q_j}\right)^{1/q}$$

$$\leq C(|Q|2^{-nN})^h \left(\sum_j ||u||^p_{r,p,Q_j}\right)^{1/q}$$

$$= C(|Q|2^{-nN})^h ,$$

which establishes the lemma.

COROLLARY. Under the same conditions as the lemma save that the condition $1 \leq p \leq q < \infty$ *is replaced by* $1 \leq q \leq p < \infty$,

$$||u - P_N u||_{q,Q} \leq C|Q|^{\frac{r}{n}+\frac{1}{q}-\frac{1}{p}} 2^{-hnN}$$

for all $u \in W^{r,p}(Q)$ *with* $||u||_{r,p,Q} = 1$.

P r o o f . By Hölder's inequality and the lemma,

$$||u - P_N u||_{q,Q} \leq ||u - P_N u||_{p,q} |Q|^{\frac{1}{q}-\frac{1}{p}}$$

$$\leq C(2^{-nN}|Q|)^{r/n} |Q|^{\frac{1}{q}-\frac{1}{p}} ,$$

as required.

The lemma and the corollary give the O-estimates of the theorem almost immediately in the special case when $\Omega = Q$. For the map $P_N : u \to \sum_{j=1}^{2^{nN}} \chi_{Q_j} P_{r,Q_j} u$ is linear and of rank at most $2^{nN} \sum_{|\alpha|<r} 1 = 2^{nN} M$, say. Thus as $||I - P_N||_{q,Q} \leq 2^{-nNh}|Q|^{\frac{r}{n}-\frac{1}{p}+\frac{1}{q}} C$, we see that

$$a_s(I) \leq C|Q|^{\frac{r}{n}+\frac{1}{q}-\frac{1}{p}} M^h s^{-h}$$

when s is of the form $2^{nN} m$. However, given any positive integer s , there exists $N \in \mathbb{N}$ such that $2^{nN} M \leq s \leq 2^{n(N+1)} M$, and $a_{2^{n(N+1)}M}(I) \leq a_s(I) \leq a_{2^{nN}M}(I)$, from which it easily follows that $a_s(I) = O(s^{-h})$ as $s \to \infty$.

In the more general case in which we merely assume that Ω is bounded and has a minimally smooth boundary we know ([20], p. 181) that there is an extension map $E : W^{r,p}(\Omega) \to W^{r,p}(\mathbb{R}^n)$ such that for all $u \in W^{r,p}(\Omega)$,

$$||u||_{r,p,\Omega} \leqq ||Eu||_{r,p,\mathbb{R}^n} \leqq c\, ||u||_{r,p,\Omega} ,$$

where c is a constant independent of u and p ; it depends, however, on r . Let Q be a box such that $\bar{\Omega} \subset Q$. Since $W^{r,p}(Q)$ coincides with the set of restrictions to Q of functions in $W^{r,p}(\mathbb{R}^n)$, given any $u \in W^{r,p}(\Omega)$ we define $\tilde{u} = Eu|_Q \in W^{r,p}(Q)$; clearly $\tilde{u}(x) = u(x)$ for all $x \in \Omega$, and

$$||\tilde{u}||_{r,p,Q} \leqq ||Eu||_{r,p,\mathbb{R}^n} \leqq c||u||_{r,p,\Omega} .$$

Hence

$$\begin{aligned} ||u - \sum_{j=1}^{2^{nN}} \chi_{Q_j \cap \Omega} P_{r,Q_j} \tilde{u}||_{q,\Omega} &\leqq ||\tilde{u} - \sum_{j=1}^{2^{nN}} \chi_{Q_j} P_{r,Q_j} \tilde{u}||_{q,Q} \\ &\leqq C\, c|Q|^{\frac{r}{n} + \frac{1}{q} - \frac{1}{p}} 2^{-nNh} ||u||_{r,p,\Omega} \end{aligned} \tag{2}$$

from which we conclude that, as before, $a_s(I) = O(s^{-h})$ as $s \to \infty$. For I_0 the results follow as for I , but without any conditions on $\partial\Omega$ since the extensions needed may be made without them.

R e m a r k . An examination of the arguments used above shows that the constant C may be taken to be of the form

$$C(r, n, p, q) = K(r, n)\, 2^{n\ell}\, \ell^{\ell}\, h^{-\ell} ,$$

where $\ell = 1 - \max\left(0, \frac{1}{p} - \frac{1}{q}\right)$ and $h = \frac{r}{n} - \max\left(0, \frac{1}{p} - \frac{1}{q}\right)$.

To complete the proof of the theorem it is enough to obtain the lower bounds for the approximation numbers of the map $I_0 : W^{r,p}(\Omega) \to L^p(\Omega)$ since $a_s(I) \geqq a_s(I_0)$ for all $s \in \mathbb{N}$. The argument given to establish this owes much to my colleague V. B. Moscatelli, and begins with two simple lemmas.

Lemma 2. Let X *be an* r*-dimensional Banach space* $(r \geqq 1)$, *and let* $id : X \to X$ *be the identity map. Then* $a_{r-1}(id) = 1$.

P r o o f . Let $F \in B(X)$ be such that $\dim F(X) \leqq r - 1$. By

Riesz's lemma, given any $\theta \in (0, 1)$, there exists $x_\theta \in X$ such that $||x_\theta|| = 1$ and $||x_\theta - F(x)|| > \theta$ for all $x \in X$. Hence

$$||id - F|| = \sup \{||x - F(x)|| : x \in X_r, ||x|| = 1\} > \theta \cdot$$

Thus $a_{r-1}(id) \geqq 1$. But $a_{r-1}(id) \leqq a_0(id) = 1$. The lemma follows.

<u>*Lemma 3.*</u> *Let X and Y be Banach spaces such that $X \subset Y$ algebraically and topologically, and let $i : X \to Y$ be the natural embedding map. Let X_r be an r-dimensional subspace of X ($r \geqq 1$) and suppose there is a positive number c such that for all $x \in X_r$, $||x||_X \leqq c||x||_Y$. Let P_r be a projection of Y onto $i(X_r)$. Then*

$$a_{r-1}(i) \geqq c^{-1}||P_r||^{-1} .$$

P r o o f . Let $id_r : X_r \to X_r$ be the identity map, let $i_r : X_r \to X$ be the natural map and let $j_r : i(X_r) \to X_r$ be the identity map. Then $id_r = j_r \circ P_r \circ i \circ i_r$, so that by lemma 2,

$$1 = a_{r-1}(id_r) \leqq ||j_r|| \, ||P_r|| \, ||i_r|| \, a_{r-1}(i) \leqq c||P_r||a_{r-1}(i) .$$

<u>Completion of the proof of Theorem 1.</u>

Take Q to be the unit cube $(0, 1)^n$, let s be a positive integer, let j be the integer such that $(j - 1)^n \leqq s < j^n$, let K be the set of all multi-indices $k = (k_1, \ldots, k_n)$ such that $0 \leqq k_i \leqq j - 1$ for $i = 1, \ldots, n$, and for each $k \in K$ put $Q_k =$ $= \{x : k_i/j < x_i < (k_i + 1)/j$ for $i = 1, \ldots, n\}$. Thus $Q \setminus \bigcup_{k\in K} Q_k$ has zero measure. Let $\phi \in C_0^\infty(Q)$ be such that $||\phi||_{2,Q} = 1$; then the functions ϕ_k $(k \in K)$ defined by $\phi_k(x) = j^{n/2}\phi(jx - k)$ are in $C_0^\infty(Q_k)$ and satisfy

$$(\phi_k, \phi_{k'})_{L^2(Q)} = \delta_{kk'}, \quad ||\phi_k||_{q,Q} = j^{n(\frac{1}{2} - \frac{1}{q})}||\phi||_{q,Q} \quad (q \geqq 1) .$$

Let V be the linear span of the ϕ_k $(k \in K)$ and define $\tilde{P}_j : L^2(Q) \to V$ by

$$\tilde{P}_j u = \sum_{k\in K} (u,\ \phi_k)_{L^2(Q)}\ \phi_k\ ;$$

then P_j is a projection and

$$||P_j u||^p_{p,Q} = j^{n(\frac{p}{2}-1)} ||\phi||^p_{p,Q} \sum_{k\in K} |(u,\ \phi_k)_{L^2(Q)}|^p\ .$$

Since

$$|(u,\phi_k)_{L^2(Q)}| \leqq j^{n(\frac{1}{p}-\frac{1}{2})} ||u||_{p,Q_k}\ ||\phi||_{p',Q}$$

it follows easily that

$$||\tilde{P}_j|| \leqq ||\phi||_{p,Q}\ ||\phi||_{p',Q} \qquad (p' = p/(p-1)).$$

Given any $u \in V$, $u = \tilde{P}_j u \in \overset{o}{W}{}^{r,p}(Q)$ and

$$||u||_{r,p,Q} = \left\{ \sum_{|\alpha|\leqq r} \sum_{k\in K} |(u,\phi_k)_{L^2(Q)}|^p\ ||D^\alpha\phi_k||^p_{p,Q_k} \right\}^{1/p}$$

$$= j^{n(\frac{1}{2}-\frac{1}{p})} \left(\sum_{|\alpha|\leqq r} j^{|\alpha|p} ||D^\alpha\phi||^p_{p,Q} \right)^{1/p} \left(\sum_{k\in K} |(u,\phi_k)_{L^2(Q_k)}|^p \right)^{1/p}$$

$$= \left(\sum_{|\alpha|\leqq r} j^{|\alpha|p} ||D^\alpha\phi||^p_{p,Q} \right)^{1/p} ||u||_{p,Q}\ ||\phi||^{-1}_{p,Q} \tag{3}$$

$$\leqq j^r\ ||\phi||_{r,p,Q}\ ||u||_{p,Q}\ ||\phi||^{-1}_{p,Q}$$

$$\leqq (s^{1/n}+1)^r ||\phi||_{r,p,Q}\ ||u||_{p,Q}\ ||\phi||^{-1}_{p,Q}\ .$$

Let X_{s+1} be the linear span of any set of $(s+1)$ functions ϕ_k; this is possible because $\dim V = j^n \geqq s+1$. Define a projection P_{s+1} of $L^2(Q)$ onto X_{s+1} (viewed as a subspace of $L^p(Q)$) by

$$P_{s+1}(u) = \sum_{\phi_k\in X_{s+1}} (u,\ \phi_k)_{L^2(Q)}\ \phi_k\ .$$

Then $||P_{s+1}|| \leqq ||\tilde{P}_j|| \leqq ||\phi||_{p,Q}\ ||\phi||_{p',Q}$, so that by (3) and lemma 3,

$$a_s(I_0) \geqq \left(||\phi||_{p,Q}\ ||\phi||_{p',Q} \right)^{-1} (s^{1/n}+1)^{-r}\ ,$$

which is the required lower bound for $a_s(I_0)$, at least in the case where $\Omega = (0,1)^n$ and hence also for any cube in $\mathbb{R}^n$. To finish the proof, let Ω be any bounded domain in $\mathbb{R}^n$, and let Q be an open cube contained in Ω. Then the natural embedding map I_Q : $\overset{o}{W}{}^{r,p}(Q) \to L^p(Q)$ satisfies $I_Q = R \circ I_0 \circ E$, where $R : L^p(\Omega) \to$ $\to L^p(Q)$ is the restriction map and $E : \overset{o}{W}{}^{r,p}(Q) \to \overset{o}{W}{}^{r,p}(\Omega)$ is the extension map which extends elements of $\overset{o}{W}{}^{r,p}(Q)$ by zero in $\Omega \setminus Q$. Hence

$$a_s(I_Q) \leqq ||R|| \; ||E|| \; a_s(I_0) = a_s(I_0) \; ,$$

which gives $a_s(I_0) \geqq \text{const.}\; s^{-r/n}$.

4. Other developments

The techniques used in § 3 can be adapted to analyse, from the same point of view, the embedding maps which turn up when we consider unbounded domains Ω, spaces of fractional order, traces on lower dimensional manifolds and non-isotropic spaces. For accounts of these topics we refer to [4], [5], [11],[12], [13], [14], [21] and [22]. Rather than go into such matters in any detail here, however, we prefer to conclude with a brief discussion of Sobolev spaces and Orlicz spaces.

Let us recall that an Orlicz function is a non-negative convex function ϕ on $[0,\infty)$ with $\phi(0) = 0$, $\lim_{t\to 0} \phi(t)/t = 0$ and $\lim_{t\to\infty} \phi(t)/t = \infty$. The Orlicz space $L^\phi(\Omega)$ is the Banach space of all (equivalence classes of) measurable functions u on Ω such that for some $\lambda > 0$,

$$\int_\Omega \phi(|u(x)|/\lambda)dx < \infty ,$$

with norm

$$||u||_{\phi,\Omega} = \inf \{\lambda > 0 : \int_\Omega \phi(|u(x)|/\lambda)dx \leqq 1\} .$$

Note that $L^\infty(\Omega) \subset L^\phi(\Omega) \subset L^1(\Omega)$, the inclusions being proper, in general. The importance of these spaces in the theory of partial differential equations with strong non-linearities is now well-known, and a good deal of their interest stems from a remarkable result of TRUDINGER [23] which closes a gap in the Sobolev embedding theory by means of Orlicz spaces. The gap referred to arises in connection with the space $W^{r,p}(\Omega)$ when $rp = n$, and as a consequence of Trudinger's work it is now known that if $1 \leq r < n$ and $1 < \nu < n/(n-r)$ then $W^{r,n/r}(\Omega)$ is compactly embedded in $L^{\phi_\nu}(\Omega)$, where $\phi_\nu(t) = \exp(t^\nu) - 1$. Here we shall estimate the approximation numbers of the embedding of $W^{r,p}(\Omega)$ in $L^{\phi_\nu}(\Omega)$, when this embedding exists, under the conditions $1 \leq n/r \leq p < \infty$, $\nu < 1$ and Ω bounded with minimally smooth boundary.

Let $u \in W^{r,p}(\Omega)$ and set $U = u - \sum_{j=1}^{2^{nN}} \chi_{Q_j \cap \Omega} P_{r,Q_j} \tilde{u}$, with the notation used in (2). Then if $\lambda > 0$ we have from (2),

$$\int_\Omega \phi_\nu(|U(x)|/\lambda)dx = \int_\Omega \sum_{k=1}^{\infty} \frac{1}{k!} (|U(x)|/\lambda)^{k\nu} dx$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left(||U||_{k\nu,\Omega} \lambda^{-1} \right)^{k\nu}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ C(r,n,p,k\nu) c 2^{-nNh(k\nu)} |Q|^{\frac{r}{n}+\frac{1}{k\nu}-\frac{1}{p}} |u|_{r,p,\Omega} \lambda^{-1} \right\}^{k\nu}$$

where $h(k\nu) = \frac{r}{n} - \max\left(0, \frac{1}{p} - \frac{1}{k\nu}\right)$. It follows that if $p > n/r$,

$$\int_\Omega \phi_\nu(|U(x)|/\lambda)dx \leq 1$$

provided that

$$||u||_{r,p,\Omega} (|Q| 2^{-nN})^{\frac{r}{n} - \frac{1}{p}} \leq \lambda A ,$$

where A is a certain positive constant. This implies that for the embedding $J : W^{r,p}(\Omega) \to L^{\phi_\nu}(\Omega)$ we have

$$a_s(J) = O\left(s^{-\frac{r}{n}+\frac{1}{p}}\right) \text{ as } s \to \infty ,$$

a result first derived in [9].

If $p = n/r$ we are in the interesting case treated by Trudinger, and our estimates above become

$$\int_\Omega \phi_\nu(|U(x)|/\lambda)\,dx \leq$$
$$\leq \text{const.}|Q|2^{-nN}\sum_{k=1}^{\infty}\frac{1}{k!}\left(cK\|u\|_{r,p,\Omega}2^{n(1-\frac{1}{p})}\lambda^{-1}\right)^{k\nu}\{1+k\nu(1-\tfrac{1}{p})\}^{1+k\nu(1-\frac{1}{p})}.$$

We thus have to deal with the series

$$S(z) = \sum_{k=1}^{\infty}\frac{1}{k!}\,(1+bk)^{1+bk}\,z^k$$

where $b = \nu(1-\frac{1}{p})$ and $z = \left(cK\|u\|_{r,p,\Omega}\,2^{n(1-\frac{1}{p})}\,\lambda^{-1}\right)^{\nu}$. This series converges for all z if $b < 1$, and can be majorised successively by constant multiples of $\sum_{k=0}^{\infty}\frac{e^{bk}\,z^k}{(k!)^{1-b}}$ and $\exp(z^{1/(1-b)})$.

It follows easily that

$$a_s(J) = O\left((\log s)^{1-\frac{r}{n}-\frac{1}{\nu}}\right) \text{ as } s \to \infty,$$

so that J is of type c_0. This also was obtained in [9], by different methods. Whether this result can be improved I do not know. Our conclusions are summarised as follows:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with minimally smooth boundary, let r and p be positive numbers (r an integer) such that $1 \leq n/r < p < \infty$, and define ϕ_ν by $\phi_\nu(t) = \exp(t^\nu) - 1$ for $t \geq 0$, where $\nu > 1$. Then the approximation numbers of the embedding map $J : W^{r,p}(\Omega) \to L^{\phi_\nu}(\Omega)$ satisfy*

$$a_s(J) = O\left(s^{-\frac{r}{n}+\frac{1}{p}}\right) \textit{ as } s \to \infty.$$

If $p = n/r > 1$ *and* $1 < \nu < n/(n-r)$,

$$a_s(J) = O\left((\log s)^{1-\frac{r}{n}-\frac{1}{\nu}}\right) \textit{ as } s \to \infty.$$

References

[1] C. J. Amick: Some remarks on Rellich's theorem and the Poincaré inequality. J. London Math. Soc. (to appear)

[2] J.-P. Aubin: Approximation of elliptic boundary-value problems. New York, Wiley, 1972.

[3] M. S. Birman and M. Z. Solomjak: Piecewise polynomial approximations of functions of class W_p^α. Mat. Sb. 115 (1967), 331-355.

[4] M. S. Birman and M. Z. Solomjak:Spectral asymptotics of nonsmooth elliptic operators I. Trans. Moscow Math. Soc. 27 (1972), 1-52; II. ibid. 28 (1973), 1-32.

[5] V. V. Borzov: On some applications of piecewise polynomial approximations of functions of anisotropic classes W_p^r. Soviet Math. Dokl. 12 (1971), 804-807.

[6] R. Courant and D. Hilbert: Methoden der mathematischen Physik II. Berlin, Springer, 1937.

[7] D. E. Edmunds and W. D. Evans: Elliptic and degenerate-elliptic operators in unbounded domains. Ann. Scuola Norm. Sup. Pisa 27 (1973), 591-640.

[8] D. E. Edmunds and W. D. Evans: Orlicz and Sobolev spaces on unbounded domains. Proc. Roy. Soc. London A 342 (1975), 373-400.

[9] D. E. Edmunds and V. B. Moscatelli: Fourier approximation and embeddings of Sobolev spaces. Dissertationes Math. 145 (1977),1-50.

[10] L. E. Fraenkel: On regularity of the boundary in the theory of Sobolev spaces. (to appear)

[11] H. König: Operator properties of Sobolev imbeddings over unbounded domains. J. Functional Anal. 24 (1977), 32-51.

[12] J. S. Martins: Embeddings of Sobolev spaces of fractional order. Proc. Roy. Soc. Edinburgh 79 A (1977), 1-24.

[13] J. S. Martins: Embeddings of Sobolev spaces on unbounded domains. Ann. Math. Pura Appl. (to appear)

[14] J. S. Martins: Embeddings of nonisotropic spaces. Quart. J. Math. Oxford. (to appear)

[15] G. Métivier: Valeurs propres de problèmes aux limites elliptiques irréguliers. Bull. Soc. Math. France, Mémoire 51-52 (1977), 125--219.

[16] J. Nečas: Les méthodes directes en théorie des équations elliptiques. Paris - Prague, Masson - Academia, 1967.

[17] R. D. Nussbaum: The radius of the essential spectrum. Duke J. Math. 37 (1970), 473-478.

[18] A. Pietsch: Nuclear and locally convex spaces. Berlin, Springer, 1972.

[19] E. Schock: Approximation numbers of bounded operators. J. Math. Anal. Appl. 51 (1975), 440-448.

[20] E. M. Stein: Singular integrals and differentiability properties of functions. Princeton, University Press, 1970.

[21] H. Triebel: Über die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev - Besov - Räumen. Invent. Math. 4 (1967), 275-293.

[22] H. Triebel: Über die Approximationszahlen der Einbettungsoperatoren $J(B^r_{p,q}(\Omega) \to B^{r'}_{p',q'}(S))$. Arch. Math. 19 (1968), 305-312.

[23] N. S. Trudinger: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. 17 (1967), 473-483.