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# POINCARÉ–SOBOLEV AND ISOPERIMETRIC INEQUALITIES, MAXIMAL FUNCTIONS, AND HALF-SPACE ESTIMATES FOR THE GRADIENT

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At the conference I gave four lectures. The most detailed part of the material below is related to *Lecture 3* which concerns joint work with Prof. Michael Wilson about half-space estimates for the gradient [WWi]. The other three lectures that I gave were about papers that are already published or in press. For the sake of completeness, their surveys are presented here (see the references at the end of this article). In addition, related to Lecture 2, the paper [FLW] concerning Poincaré's inequality for vector fields of Hörmander type now exists in preprint form.

**Lecture 1:** Fractional maximal functions ([W1], [SWZ])

Consider the *fractional maximal function* on $\mathbb{R}^n$,

$$M_\alpha f(x) = \sup_{B:\ x\in B} \frac{1}{|B|^{1-\alpha/n}} \int\limits_B |f(y)|\,dy,$$

where $B$ denotes a ball in $\mathbb{R}^n$, $0<\alpha<n$. The problem is to characterize the weights in the inequality

$$\Big(\int\limits_{\mathbb{R}^n} |M_\alpha f(x)|^q w(x)\,dx\Big)^{1/q} \le c\Big(\int\limits_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}$$

or (in norms)

$$(*)\qquad \|M_\alpha f\|_{L^q_w} \le c\|f\|_{L^p_v},$$

where $1<p\le q<\infty$, $c$ is a positive constant independent of the function $f$, and $v$, $w$ are weight (measurable and a.e. nonnegative) functions.

**History.** (1) D.R. Adams (1973, [A]) proved that a necessary and sufficient condition for the validity of $(*)$, when $v(x) \equiv 1$, $q > p$, $\alpha < n/p$, is

$$\int_B w\,dx \le c|B|^{(\frac{1}{p}-\frac{\alpha}{n})q} \qquad \text{for all balls } B \subset \mathbb{R}^n.$$

(2) B. Muckenhoupt and R.L. Wheeden (1974, [MW]) showed that under the assumptions $1/q = 1/p - \alpha/n$, $\alpha < n/p$, $w = v^{q/p}$, a necessary and sufficient condition for $(*)$ to hold is

$$|B|^{\frac{\alpha}{n}-1}\Big(\int_B w\Big)^{1/q}\Big(\int_B v^{-\frac{1}{p-1}}\Big)^{q/p'} \le c \qquad \text{for all balls } B \subset \mathbb{R}^n,$$

i.e., $v^{q/p} \in A_{1+q/p'}$ $(1/p + 1/p' = 1)$.

(3) E.T. Sawyer (1982, [S]) found a condition, namely,

$$\Big(\int_Q M_\alpha(\chi_Q v^{-\frac{1}{p-1}})^q w\,dx\Big)^{1/q} \le c\Big(\int_Q v^{-\frac{1}{p-1}}\,dx\Big)^{1/p} \quad \text{for all cubes } Q \subset \mathbb{R}^n,$$

which is necessary and sufficient for $(*)$ to be valid provided $p \le q$.

The following characterization holds for $p < q$ (1993, [W1])

**Theorem 1.** *Let $1 < p < q < \infty$ and $0 < \alpha < n$. Then $(*)$ holds if and only if $v$, $w$ satisfy*

$$(**) \qquad \Big(\int_{\mathbb{R}^n} \frac{w(x)\,dx}{(|B|^{1/n} + |x - x_B|^{(n-\alpha)q}}\Big)^{1/q}\Big(\int_B v(x)^{-\frac{1}{p-1}}\,dx\Big)^{1/p'} \le c$$

*for all balls $B \subset \mathbb{R}^n$, where $x_B$ is the center of $B$ and $c$ is a positive constant independent of $B$.*

*Remark.* (1) We can replace balls by cubes.

(2) In case the weight $\sigma = v^{-1/(p-1)}$ satisfies the *reverse doubling condition*, i.e.,

$$(RD) \qquad \exists\, \beta, \theta > 1 \quad \text{such that} \quad \sigma(\theta B) \ge \beta\sigma(B) \quad \text{for all balls } B \subset \mathbb{R}^n$$

(where $\theta B = \{x\colon |x - x_B| < \theta r(B)\}$, $\sigma(B) = \int_B \sigma(x)\,dx$) then the condition is equivalent to the condition

$$(A^{\alpha}_{p,q}) \qquad |B|^{\frac{\alpha}{n}-1} w(B)^{1/q} \sigma(B)^{1/p'} \le C \quad \text{for all balls } B \subset \mathbb{R}^n.$$

This condition is well-known to be necessary and sufficient for the weak-type estimate

$$\sup_{t>0} t\, w(\{x\colon M_\alpha f(x) > t\})^{1/q} \le c\|f\|_{L^p_v},$$

even for $p = q$.

The fractional maximal function was used in [MW] to help control the Riesz fractional integral

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} f(y) \frac{1}{|x-y|^{n-\alpha}}\, dy,$$

but it turns out that $I_\alpha$ arises (or can be made to arise) in the proof of Theorem 1. In fact, the proof shows that the dual weak type behavior of $I_\alpha$ is decisive for $M_\alpha$. The condition on the weights in Theorem 1 is essentially the same as that given in [GaK] in order to characterize the weights for which there is the dual weak-type estimate

$$\sup_{t>0} t\, \sigma(\{x\colon I_\alpha f(x) > t\})^{1/p'} \le c\|f\|_{L^{q'}_{w^{-1/(q-1)}}},$$

where $\sigma = v^{-\frac{1}{p-1}}$, $1 < p < q < \infty$.

By combining these results with a strong type characterization [S] for the boundedness of $I_\alpha$, we get

**Theorem 2.** *For $1 < p < q < \infty$, we have*

$$(**) \qquad I_\alpha\colon L^p_v \to L^q_w$$

*(i.e., in norms, $\|I_\alpha f\|_{L^q_w} \le c\|f\|_{L^p_v}$ with $c > 0$ independent of $f$) if and only if both conditions*

$$M_\alpha\colon L^p_v \to L^q_w$$

*and*

$$M_\alpha\colon L^{q'}_{w^{-1/(q-1)}} \to L^{p'}_{v^{-1/(p-1)}}$$

*hold.*

Of course, the operator $I_\alpha$ is self-adjoint, so $(**)$ is equivalent to $I_\alpha\colon L^{q'}_{w^{-1/(p-1)}} \to L^{p'}_{v^{-1/(p-1)}}$.

**More general context.**

Let us consider a *homogeneous space* $(X, d, \mu)$ where $d$ is a quasi-metric and $\mu$ is a measure in the space $X$ that satisfy

(1) $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(3) there is a constant $K > 0$ such that $d(x,y) \le K[d(x,z) + d(z,y)]$, for all $x, y, z \in X$,

(4) there is a constant $C > 0$ such that $\mu\big(B(x,2r)\big) \le C\mu\big(B(x,r)\big)$, where $B(x,r) = \{y \colon d(x,y) < r\}$ (such a measure $\mu$ is said to satisfy the *doubling condition*, and we write $\mu \in (DC)$).

We also assume that $B(x,R) \setminus B(x,r)$ is not empty for $0 < r < R < \infty$ and that $X$ has a group structure with respect to "+" such that

(5) $d(x+z, y+z) = d(x,y)$,

(6) $\mu(-B+z) = \mu(B)$ where $-B = \{x \colon -x \in B\}$.

It follows easily from (5), (6) that $d(0,-x) = d(0,x)$ and $\mu(-B) = \mu(B)$. This is the sort of situation considered in [SW2], and we will use many ideas from there.

Let

$$M_\gamma(f d\sigma) = \sup_{B \colon x \in B} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f(y)|\, d\sigma(y)$$

where $0 < \gamma < 1$ and $d\sigma$ is a Borel measure and the supremum is taken over all balls $B \subset \mathbb{R}$.

Define $B_{xy} = B\big(x, d(x,y)\big)$ and note that $\mu(B_{xy}) \sim \mu(B_{yx})$ since $\mu \in (DC)$.

The main result is

**Theorem 3.** *If $1 < p < q < \infty$ and $0 < \gamma < 1$, then*

$$(***) \qquad \Big(\int_X \big|M_\gamma(f d\sigma)\big|^q d\omega\Big)^{1/q} \le c\Big(\int_X |f|^p d\sigma\Big)^{1/p}$$

*holds for all $f$, with $c$ independent of $f$, if and only if*

$$\Big(\int_X \frac{d\omega(x)}{[\mu(B) + \mu(B_{x_B x})]^{(1-\gamma)q}}\Big)^{1/q} \sigma(B)^{1/p'} \le C \quad \textit{for all balls } B \subset X$$

*with $C$ independent of $B$.*

It would be nice to find a proof which does not require the group structure. If we drop the assumption of a group structure and only keep the homogeneous space structure, we have

**Theorem 4.** *If $1 < p < q < \infty$, $0 < \gamma < 1$ and $\sigma$ satisfies the doubling condition then the inequality $(***)$ holds if and only if*

$$\mu(B)^{\gamma-1}\omega(B)^{1/q}\sigma(B)^{1/p'} \le C \quad \textit{for all balls } B \subset X$$

*with $C$ independent of $B$.*

Also in [SWZ], we prove analogues for fractional maximal functions of "Hörmander type": i.e., define $\widehat{X} = X \times [0,\infty)$, and for $(x,t) \in \widehat{X}$, let

$$M_\alpha(f d\sigma)(x,t) = \sup_{r:\ r \ge t} \frac{1}{\mu\big(B(x,r)\big)^{1-\gamma}} \int_{B(x,r)} |f(y)|\, d\sigma(y).$$

**Theorem 5.** *If $1 < p < q < \infty$ and $0 < \gamma < 1$ ($X$ has a group structure now), then*

$$\Big(\int_{\widehat{X}} M_\gamma(f d\sigma)(x,t)^q d\omega(x,t)\Big)^{1/q} \le c\Big(\int_X |f(x)|^p d\sigma(x)\Big)^{1/p}$$

*holds for all $f$ with $c$ independent of $f$ if and only if*

$$\bigg(\int_{\widehat{X}} \frac{d\omega(x,t)}{\big[\mu(B) + \mu\big(B(x_B, d(x_B,x)+t)\big)\big]^{(1-\gamma)q}}\bigg)^{1/q} \sigma(B)^{1/p'} \le C$$

*for all balls $B \subset \mathbb{R}^n$, with $C$ independent of $B$.*

*Remarks.* (i) If in addition $\sigma \in (DC)$, then a necessary and sufficient condition is

$$\mu(B)^{\gamma-1}\omega(\widehat{B})^{1/q}\sigma(B)^{1/p'} \le c$$

where $\widehat{B} = B \times \big[0, r(B)\big)$, and in this case we do not need the group structure.

(ii) There are results due to Sergio Zani [Z] of this type for subsets $D_u \cup D$ of $X \times (-\infty,\infty)$.

## Lecture 2: Poincaré inequality ([FGuW], [FGaW], [FLW])

Let us consider the homogeneous space $(\mathbb{R}^N, \rho, dz)$ where $\rho$ is a metric and a family $\{X_j\}_j$ of first order differential operators associated with real continuous vector fields (not always smooth).

The exact form of the *Poincaré inequality* we shall investigate is

$$(P) \qquad \Big( \fint_B |f(z) - f_B|^q u(z)\, dz \Big)^{1/q} \le c \cdot r \Big( \fint_B \sum_j |X_j f(z)|^p v(z)\, dz \Big)^{1/p}.$$

with $c$ independent of $B$ and $f$, where $B$ denotes the $\rho$-ball with radius $r = r(B)$ with respect to the metric $\rho$ which is naturally generated from the vector fields and associated integral curves. Here, $1 \le p \le q < \infty$, $u$ and $v$ are nonnegative locally integrable functions, $f_B = \fint_B fu\, dz$, and we denote

$$\fint_B f(z)u(z)dz = \frac{1}{u(B)} \int_B f(z)u(z)dz.$$

Our methods also yield the *Sobolev inequality* with $f_B$ replaced by 0 in the case $f$ is supported in $B$.

**Applications.** (i) For $p > 1$ Harnack's inequality for positive solutions of type second order p.d.e.

(ii) For $p = 1$ the isoperimetric inequality $|E|^{N-1} \le c \cdot \mathcal{H}_{N-1}(\partial E)$.

**History.**

(i) *Standard case:*

$$\rho(z_1, z_2) = |z_1 - z_2| \quad \text{and} \quad X_j = \frac{\partial}{\partial z_j},$$

[FKS] $1 < p < \infty$, $u = v \in A_p$, $q = q(u,p) > p$ or $p = 2$, $u = v = |\phi'|^{1-2/n}$, $\phi$ quasiconformal, $q > 2$;
[CKN] $u$, $v$ powers of $|z|$, all $p$, $q$;
[CW1,2] $u$, $v$, $p$, $q$ related by a dimensional balance condition, $v \in A_p$;
[DS] $u$, $v$ certain powers of a strong-$A_\infty$ weight $w$; a representation formula;

(ii) *Other metrics and vector fields*:

Non-smooth: [FL], [FS];

Smooth (Hörmander vector fields): [J], [L], $\ldots$ .

The first part of this talk concerns enlarging the class of weights in the standard case as well as in a special case of non-smooth vector fields. To motivate things, we first return to the standard case and recall the main result in [CW1]. For $1 < p < q < \infty$, consider the *balance condition*

$$(BC) \qquad \frac{r(B)}{r(B_0)}\Big(\frac{u(B)}{u(B_0)}\Big)^{1/q} \le c\Big(\frac{v(B)}{v(B_0)}\Big)^{1/p}$$

for all balls $B$, $B_0$ such that $B \subset B_0$.

In the most classical case, i.e., $u = v = 1$, this balance amounts to $1/q \ge 1/p - 1/N$.

The following theorem holds.

**Theorem.** *Let $1 < p < q < \infty$ and $u$, $v$ be a pair of weight functions satisfying the balance condition $(BC)$, let $u \in (DC)$ and $v \in A_p$. Then the inequality $(P)$ holds.*

(Recall that $u$ is *doubling*, $u \in (DC)$, if $u(2B) \le u(B)$ for all balls $B$ where $2B$ denotes the ball concentric with $B$ and with radius $2r(B)$; $v \in A_p$ if $⨍_B v\big(⨍_B v^{-1/(p-1)}\big)^{p-1} \le c$ for all balls $B$.)

The balance condition $(BC)$ is necessary if $u$ is doubling but $v \in A_p$ is certainly not necessary as shown for example by the results of [CKN] or by the second result of [FKS]: $v(z)$ can be $|z|^\alpha$ for arbitrarily large $\alpha$. There are also results in [GW] and [CW3] which allow any $u$, $v$ that satisfy the balance condition and for which

$$v/\Pi \in A_p$$

for some $\Pi = \prod_i |z - a_i|$, $\alpha_i > 0$ (e.g., large).

The problem is the method of proof, which deduces the result from weighted estimates for the fractional integral

$$I_1 g(z) = \int\limits_{\mathbb{R}^N} g(\zeta)\frac{1}{|z-\zeta|^{N-1}}\,d\zeta,$$

by using the fact that

$$|f(z) - f_B| \le cI_1(|\nabla f|\chi_B)(z), \quad z \in B$$

($\chi_B$ denotes the characteristic function of the set $B$). The difficulty is that the fractional integral results require at least $v^{-1/(p-1)} \in L^1_{\mathrm{loc}}$, which rules out weights like $v(z) = |z|^\alpha$ for large $\alpha > 0$.

In [FGuW], we by-pass this difficulty by using a different representation. This allows some unification of earlier results. It leads to new weights even in the standard case and also applies to non-smooth vector fields of Grushin type: consider

$$\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m = \{z = (x,y)\colon x \in \mathbb{R}^n, y \in \mathbb{R}^m\},$$

and vector fields

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \quad \lambda(x)\frac{\partial}{\partial y_1}, \dots, \lambda(x)\frac{\partial}{\partial y_m},$$

where $\lambda(x) \geq 0$ and continuous, but not necessarily differentiable. We assume that $\lambda$ satisfies

$$(RH_\infty) \qquad \fint_{|x-x_0|<r} \lambda(x)\,dx \sim \max_{|x-x_0|<r} \lambda(x)$$

and some other conditions to be stated later. A typical example is $\lambda(x) = |x|^\alpha$, $\alpha \geq 0$, and the results will be new even when $\lambda \equiv 1$.

Denote

$$|\nabla_\lambda f(z)|^2 = |\nabla_x f(z)|^2 + \lambda(x)^2 |\nabla_y f(z)|^2.$$

Define a metric $\rho$ by means of sub-unit curves, i.e., absolutely continuous curves $\gamma(t)$ in $\mathbb{R}^N$ so that for all $\zeta = (\xi,\eta) \in \mathbb{R}^N$,

$$(\dot\gamma(t),\zeta)^2 \leq |\xi|^2 + \lambda\big(\gamma(t)\big)^2 |\eta|^2;$$

if $z_1,\ z_2 \in \mathbb{R}^N$, let

$$\rho(z_1,z_2) = \inf\{T\colon \exists \text{ sub-unit curve } \gamma \text{ with } \gamma(0) = z_1 \text{ and } \gamma(T) = z_2\}.$$

This is the sort of situation studied in depth in [F]. In particular, the corresponding metric balls are comparable to rectangles with fairly simple structure, and the balls are doubling with respect to Lebesgue measure.

To state the main result, we must first define the class of *strong–$A_\infty$ weights* in our context. Let $w = w(z) \in A_\infty = \cup_{p>1} A_p$ with respect to Lebesgue measure and $\rho$-balls. For $z_1,\ z_2 \in \mathbb{R}^N$, consider the quasi-metric

$$\delta(z_1,z_2) = \inf_{B\colon\ z_1,z_2\in B} \Big( \int_B w(z)\lambda(z)^{m/(N-1)} dz \Big)^{1/N},$$

where $B$ is a $\rho$-ball and $\lambda(z) = \lambda(x)$ if $z = (x, y)$. If $\gamma\colon [0, T] \to \mathbb{R}^N$ is a sub-unit curve, its $w$-length is defined by

$$\ell(\gamma) = \liminf_{|\sigma| \to 0} \sum_i \delta\big(\gamma(t_{i+1}), \gamma(t_i)\big),$$

where $\sigma = \{t_i\}$ is a partition of $[0, T]$. Define a "distance" $d(z_1, z_2)$ by

$$d(z_1, z_2) = \inf\{\ell(\gamma)\colon \gamma \text{ is a sub-unit curve connecting } z_1 \text{ and } z_2\}.$$

Then $d$ is a pseudo-metric ($d(z_1, z_2) = 0$ may not imply $z_1 = z_2$). If

$$d(z_1, z_2) \sim \delta(z_1, z_2),$$

then $d$ is a metric, and we say $w$ is a strong-$A_\infty$ weight for the metric $\rho$.

When $\lambda \equiv 1$, this notion was introduced in [DS].

**Examples.** (a) $w(z) \equiv 1$,

(b) $w(z) = \rho(z, z_0)^\alpha$, $\alpha \geq 0$,

(c) $w(z) = |\phi'|$, $\phi$ quasiconformal, $\lambda \equiv 1$.

*Technical usefulness of strong–$A_\infty$.*

Say $\lambda \equiv 1$ for simplicity. Then

$$d(z, z^*) \sim \Big( \int_{B(z^*, |z - z^*|)} w \cdot 1^{m/(N-1)} \Big)^{1/N}$$

by definition. It is not hard to see by using the triangle inequality that

$$D_z d(z, z^*) \leq c w(z)^{1/N},$$

whereas

$$D_z \Big( \int_{B(z^*, |z - z^*|)} w \Big)^{1/N}$$

will spread onto the surface of the ball $B(z^*, |z - z^*|)$. An analogous basic fact about quasiconformal maps is that

$$|\phi(z) - \phi(z^*)| \sim \Big( \int_{B(z^*, |z - z^*|)} |\phi'| \Big)^{1/N},$$

and of course the derivatives of the right side spread while those of the left side localize at $z$.

We now state the remaining **assumptions on $\lambda$ and $w$.** We assume

(i) $\lambda(x)^n$ is in strong–$A_\infty$ with respect to standard Euclidean balls in $\mathbb{R}^n$;

(ii) $\lambda = 0$ at most in a finite subset of each ball;

(iii) if $\lambda(x_1) = 0$, then $w(x, y)$ is bounded as $x \to x_1$ uniformly in $y$ for $y$ in any bounded set.

In the case $n = 1$ ($m = N - 1$), we can replace (ii) by the assumption $\lambda > 0$ a.e., and (iii) is not needed at all.

Finally, $A_p(d\mu)$ means $A_p(\mathbb{R}^N, \rho, d\mu)$ for a doubling measure $\mu$.

**Main Theorem.** *Suppose that $1 \le p < q < \infty$ and that $u$, $v$ satisfy the balance condition $(BC)$. If $u$ is doubling and there exists a strong–$A_\infty$ weight $w$ such that $vw^{-(1-1/N)} \in A_p(w^{1-1/N} dz)$ then*

$$\Big( \fint_B |f(z) - f_B|^q u(z)\, dz \Big)^{1/q} \le c \cdot r(B) \Big( \fint_B |\nabla_\lambda f(z)|^p v(z)\, dz \Big)^{1/p}$$

*for all metric balls $B$.*

*Remarks.* (i) Note that $p = 1$ is included. The case $q = p$ is not included but there is a separate result for it.

(ii) The main point is that any high order zeros of $v$ can be tamed by multiplication by $w^{-(1-1/N)}$. In the $A_p$ condition, only integrals of $v$ and $v^{-1/(p-1)} w^\theta$ (for $\theta > 0$) arise, which is very good.

## **Lecture 3:** Half-space estimates for the gradient ([WWi])

We consider the following

**Problem.** Given a function $k(x)$ on $\mathbb{R}^n$ with some smoothness, some decay at $\infty$, and $\int_{\mathbb{R}^n} k(x)\, dx = 1$, let $f(x, y) = f * k_y(x)$ where $k_y(x) = y^{-n} k(x/y)$, $y > 0$. We study the problem of determining weights $w(x, y)$ on $\mathbb{R}^{n+1}_+$ and $v(x)$ on $\mathbb{R}^n$ so that

$$\Big( \iint_{\mathbb{R}^{n+1}_+} |\nabla f(x, y)|^q w(x, y)\, dx\, dy \Big)^{1/q} \le c \Big( \int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \Big)^{1/p},$$

where $\nabla = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y} \rangle$.

We study the case when $1 < p \le q < \infty$. We also usually assume $q \ge 2$. Complete proofs will appear in [WWi].

There is an analogous problem with the Hardy $H^p_v$ norm on the right. If $v \in A_p$, these are the same problem. For $v \equiv 1$, there are complete results even for $p > 0$ due to Luecking [L], Shirokov [Sh], and Verbitsky [V]. These results can be generalized if $v \in A_\infty$ , and so one obtains $L^p_v$ results if $v \in A_p$, $1 < p < \infty$, since then $L^p_v$ and $H^p_v$ coincide.

We want to study the $L^p_v$ problem without assuming $v \in A_p$ or $A_\infty$.

The results depend on the rate of decay of $k$. If, e.g., $k(x) = c_n(1 + |x|^2)^{-(n+1)/2}$, then $k_y(x)$ is the Poisson kernel and $f(x, y)$ is the Poisson integral. Then

$$\frac{\partial}{\partial x_i} k_y(x) = -c_n(n+1) \frac{x_i y}{(y^2 + |x|^2)^{\frac{n+3}{2}}},$$
$$\frac{\partial}{\partial y} k_y(x) = c_n \frac{|x|^2 - ny^2}{(y^2 + |x|^2)^{\frac{n+3}{2}}} .$$

For fixed $y$, these have orders $|x|^{-n-2}$, $|x|^{-n-1}$ respectively as $|x| \to \infty$. The results are different, and in this note we will only discuss the case of the $x$-derivatives.

**Reformulation.** For any $k$, by the chain rule, we have

$$\frac{\partial}{\partial x_i} k_y(x) = y^{-1} \phi_y(x) \qquad \Big(\phi(x) = \frac{\partial}{\partial x_i} k(x)\Big),$$
$$\frac{\partial}{\partial y} k_y(x) = y^{-1} \phi_y(x) \qquad \Big(\phi(x) = -nk(x) - \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} k(x)\Big).$$

In either case, $\int_{\mathbb{R}^n} \phi(x)\,dx = 0$ (note that $\int\limits_{\mathbb{R}^n} \big[-nk(x) - \sum\limits_1^n x_i \frac{\partial}{\partial x_i} k(x)\big]\,dx = -n \int\limits_{\mathbb{R}^n} k + \sum\limits_1^n \int\limits_{\mathbb{R}^n} k = 0$). So our problem becomes

$$\left( \iint\limits_{\mathbb{R}^{n+1}_+} |y^{-1}\phi_y * f(x)|^q w(x,y)\,dx\,dy \right)^{1/q} \le c \left( \int\limits_{\mathbb{R}^n} |f(x)|^p v(x)\,dx \right)^{1/p}$$

for $\int\limits_{\mathbb{R}^n} \phi(x)\,dx = 0$, $\phi$ smooth with some decay.

Of course, by taking absolute values and replacing $\phi$ by $|\phi|$, we can obtain results as corollaries of known Carleson type results. Roughly speaking, such results (see [SW1], [SWZ], [GGK]) require the following if $q > p$:

$$\Big(\iint\limits_{\mathbb{R}^{n+1}_+} \Big\{\frac{y^{M-n-1}}{(|x-x_0|+y+l)^M}\Big\}^q w(x,y)\,dx\,dy\Big)^{\frac{1}{q}} \Big(\int\limits_{\mathbb{R}^n} \frac{\sigma(x)\,dx}{(1+\frac{|x-x_0|}{l})^{Mp'}}\Big)^{\frac{1}{p'}} \le c$$

for all $x_0 \in \mathbb{R}^n$, $l > 0$, where $|\phi(x)| \le (1+|x|)^-M$ and $\sigma(x) = v(x)^{1-p'}$, $1/p + 1/p' = 1$. There are also sufficient conditions ([SW1], [Z]) for $p = q$.

Our conditions only involve integrating $w(x,y)$ over sets $T(Q)$ of the form $T(Q) = \{(x,y)\colon x \in Q, l(Q)/2 < y < l(Q)\}$, $l(Q) = |Q|^{1/n}$, where $Q$ is a cube in $\mathbb{R}^n$. Thus $T(Q)$ is only the top half of the usual Carleson box.

For example, if $v \equiv 1$ ($\sigma \equiv 1$) and $f(x,y)$ is the Poisson integral, the condition in [SW1] for $p = q$ is

$$\iint\limits_{\widehat{Q}} \frac{w(x,y)}{y^p}\,dx\,dy \le c|Q|,$$

($\widehat{Q} = \{(x,y)\colon x \in Q, 0 < y < l(Q)\}$), while it will be enough that

$$\iint\limits_{T(Q)} \frac{w(x,y)}{l(Q)^p}\,dx\,dy \le c|Q|.$$

Clearly, $w(x,y) = y^{p-1}$ satisfies the second condition but not the first.

Necessary conditions are difficult to find because of cancellation. However, for the Poisson case, by using the full $\nabla$, we can show that the following is necessary ($d\mu(x,y) = w(x,y)\,dx\,dy$):

$$|\widehat{Q}|^{-1}\mu\big(T(Q)\big)^{1/q}\Big(\int\limits_{\mathbb{R}^n} \frac{\sigma(x)\,dx}{\big(1+\frac{|x-x_Q|}{l(Q)}\big)^{(n+1)p'}}\Big)^{1/p'} \le c,$$

where $x_Q$ denotes the center of $Q$.

The sufficiency results require strengthening the third factor, i.e., replacing the integral involving $\sigma$ by something larger. They are obtained by noting a connection of the original problem with Littlewood–Paley theory. In fact, the original problem, i.e.,

$$\Big(\iint\limits_{\mathbb{R}^{n+1}_+} |y^{-1}\phi_y * f(x)|^q d\mu(x,y)\Big)^{1/q} \le c\Big(\int\limits_{\mathbb{R}^n} |f|^p v\,dx\Big)^{1/p},$$

is equivalent by duality to one for the operator

$$Tg(x) = \iint\limits_{\mathbb{R}^{n+1}_+} g(t,y)y^{-1}\phi_y(t-x)\,d\mu(t,y),$$

namely, is equivalent to

$$\Big(\int\limits_{\mathbb{R}^n} |Tg(x)|^{p'}\sigma(x)\,dx\Big)^{1/p'} \le c\Big(\iint\limits_{\mathbb{R}^{n+1}_+} |g(x,y)|^{q'}\,d\mu(x,y)\Big)^{1/q'}.$$

The formula for $Tg$ is reminiscent of the *Calderón reproducing formula*:

$$f(x) = \iint\limits_{\mathbb{R}^{n+1}_+} f*\psi_y(t)\psi_y(x-t)\frac{dt\,dy}{y},$$

for $\psi \in C_0^\infty$, radial, $\int \psi = 0$ and

$$\int\limits_0^\infty |\widehat{\psi}(y\xi)|^2\frac{dy}{y} = 1 \quad \text{for all } \xi \ne 0.$$

Generally speaking, norms of $f$ can be estimated in terms of norms of square functions. This is how Littlewood–Paley theory enters. We now introduce the sort of square function that we will use. A classical square function associated with $f$ is the Lusin area integral

$$\begin{aligned} S(f)^2(x) &= \iint\limits_{|t-x|<y} |\nabla f(t,y)|^2\frac{dtdy}{y^{n-1}} \\ &\sim \iint\limits_{|t-x|<y} |f*\psi_y(t)\cdot y^{-1}|^2\frac{dtdy}{y^{n-1}} \\ &\sim \sum_{k=-\infty}^{\infty}\Big(\iint\limits_{\substack{|t-x|<2^k\\ y\sim 2^k}} |f*\psi_y(t)|^2dt\,dy\Big)\frac{1}{(2^k)^{n+1}} \\ &\sim \sum_{Q\ni x}\frac{1}{|T(Q)|}\iint\limits_{T(Q)} |f*\psi_y(t)|^2dt\,dy, \end{aligned}$$

where $\{Q\}$ is a dyadic grid of cubes $Q$.

Also, since $\mathbb{R}^{n+1}_+ = \bigcup_Q T(Q)$ (disjoint),

$$f(x) = \sum_Q \iint\limits_{T(Q)} f * \psi_y(t)\psi_y(x-t)\frac{dt\,dy}{y} = \sum_Q \Lambda_Q b_Q(x)$$

where

$$b_Q(x) = \frac{1}{\Lambda_Q}\iint\limits_{T(Q)} f * \psi_y(t)\psi_y(x-t)\frac{dt\,dy}{y}$$

and

$$\Lambda_Q = |Q|^{\frac{1}{2}}\Big(\frac{1}{|T(Q)|}\iint\limits_{T(Q)} |f*\psi_y(t)|^2 dtdy\Big)^{1/2} \sim \Big(\iint\limits_{T(Q)} |f*\psi_y(t)|^2\frac{dt\,dy}{y}\Big)^{1/2}.$$

The reason for this choice of $\Lambda_Q$ is to obtain desirable size estimates for $b_Q$ and $\nabla b_Q$. Note that by using the first formula for $\Lambda_Q$, the square function is essentially

$$S(f)(x) = \Big(\sum_{Q\ni x}\frac{\Lambda_Q^2}{|Q|}\Big)^{1/2}.$$

The properties of $b_Q$ are atom-like:

(1) $\int b_Q(x)\,dx = 0$ since $\int \psi\,dx = 0$,
(2) $\operatorname{supp} b_Q \subset 3Q$ since $\operatorname{supp}\psi \subset \{|x| \le 1\}$,
(3) $\|b_Q\|_{L^2} \le C$ and $\|\nabla b_Q\|_{L^2} \le c/l(Q)$ since $\|b_Q\|_{L^\infty} \le c/(|Q|^{1/2})$ and $\|\nabla b_Q\|_{L^\infty} \le c/(l(Q)|Q|^{1/2})$ due to the choice of $\Lambda_Q$: e.g.,

$$\begin{aligned}|\nabla b_Q(x)| &\le \frac{1}{\Lambda_Q}\iint\limits_{T(Q)} |f*\psi_y(t)|\frac{c}{y^{n+1}}\frac{dt\,dy}{y}\\ &\le \frac{c}{\Lambda_Q}\Big(\iint\limits_{T(Q)} |f*\psi_y(t)|^2\frac{dt\,dy}{y}\Big)^{1/2}\Big(\iint\limits_{T(Q)}\frac{dt\,dy}{y^{2n+3}}\Big)^{1/2}\\ &\le c\cdot l(Q)^{-1}|Q|^{-\frac{1}{2}} \quad \text{by definition of } \Lambda_Q.\end{aligned}$$

We say $f$ is in "standard form" if

$$f(x) = \sum \Lambda_Q b_Q(x)$$

for $b_Q$ satisfying (1), (2), (3).

In our case, write

$$Tg(x) = \sum_Q \iint_{T(Q)} g(t,y)y^{-1}\phi_y(t-x)\,d\mu(t,y) = \sum_Q \Lambda_Q b_Q(x),$$

where now

$$b_Q(x) = \frac{1}{\Lambda_Q}\iint_{T(Q)} g(t,y)y^{-1}\phi_y(t-x)\,d\mu(t,y)$$

and

$$\Lambda_Q = |Q|^{-\frac{1}{2}} l(Q)^{-1}\mu\big(T(Q)\big)^{1/q}\Big(\iint_{T(Q)} |g(t,y)|^{q'}\,d\mu(t,y)\Big)^{1/q'}.$$

This choice of $\Lambda_Q$ guarantees by Hölder's inequality that

$$\|b_Q\|_{L^\infty} \le \frac{c}{|Q|^{1/2}}, \quad \|\nabla b_Q\|_{L^\infty} \le \frac{c}{l(Q)|Q|^{1/2}}.$$

Also, $\int b_Q\,dx = 0$ since $\int \phi\,dx = 0$. But the support of $b_Q$ is not generally compact.

If we assume that $\phi$ is supported in $\{|x|<1\}$, then $\operatorname{supp}(b_Q) \subset 3Q$, and it is natural to hope that we can estimate norms of $Tg$ by norms of $\widetilde{S}(Tg)$ defined by

$$\widetilde{S}(Tg)(x) = \Big(\sum \frac{\Lambda_Q^2}{|Q|}\chi_{\widetilde{Q}}(x)\Big)^{1/2},$$

where $\widetilde{Q} = 3Q$ and $\Lambda_Q$ is as above.

For weighted norms there are sharp results of this type due to [CWiWo] and [Wi]. These help us prove the following result. We use the notation

$$\sigma(Q,\eta) = \int_Q \sigma(x)\log^\eta\Big(e + \frac{\sigma(x)}{\sigma_Q}\Big)\,dx \quad \text{for } \eta > 0$$

$$\text{and} \quad \sigma_Q = \frac{1}{|Q|}\int_Q \sigma(x)\,dx \quad (Q \text{ may not be dyadic}).$$

**Theorem 1.** *Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int \phi = 0$ and* $\operatorname{supp}\phi \subseteq \{|x| < 1\}$. *Let $1 < p \le q < \infty$ and $q \ge 2$. If $\eta > p'/2$ and*

$$|\widehat{Q}|^{-1}\mu\big(T(Q)\big)^{1/q}\sigma(\widetilde{Q},\eta)^{1/p'} \le c,$$

*then*

$$\Big(\iint\limits_{\mathbb{R}^{n+1}_+} |y^{-1}\phi_y * f(x)|^q d\mu(x,y)\Big)^{1/q} \le C\Big(\int\limits_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}.$$

To obtain results when $\phi$ does not have compact support, we make the following definition.

**Definition.** Let $\sigma$, $\tau$ be weights and $1 < r < \infty$. We say $(\sigma,\tau)$ is an $r$-pair if $\sigma(x) \le \tau(x)$ a.e. and if

$$\|h\|_{L^r_\sigma} \le \|\widetilde{S}(h)\|_{L^r_\tau}$$

for all $h$ in standard form.

**Examples.**

(1) If $\sigma \in A_\infty$, then $(\sigma, C\sigma)$ is an $r$-pair for $C = C_{\sigma,r}$.

(2) If $k > r/2$ then $(\sigma, CM^k\sigma)$ is an $r$-pair for $C = C_{k,n}$ and $M^k$ is $k$-fold iteration of Hardy–Littlewood maximal function ([Wi]).

We will state a better result below. First,

**Theorem 2.** *Let $1 < p \le q < \infty$ and $q \ge 2$. Let $\phi \in C^\infty$, $\int \phi = 0$ and $|\phi(x)| \le (1+|x|)^{-M}$, $|\nabla\phi(x)| \le (1+|x|)^{-M-1}$ for some integer $M \ge n+2$. If $(\sigma,\tau)$ is a $p'$-pair and if*

$$|\widehat{Q}|^{-1}\mu\big(T(Q)\big)^{1/q}\Big(\int\limits_{\mathbb{R}^n} \tau(x) s_Q(x)^{p'/q'}\,dx\Big)^{1/p'} \le c,$$

*where*

$$s_Q(x) = \frac{\log\big(e + \frac{|x-x_Q|}{l(Q)}\big)}{\big(1 + \frac{|x-x_Q|}{l(Q)}\big)^M},$$

*then*

$$\Big(\iint\limits_{\mathbb{R}^{n+1}_+} |y^{-1}\phi_y * f(x)|^q d\mu(x,y)\Big)^{1/q} \le C\Big(\int\limits_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}.$$

Note that the restriction $M \geq n+2$ allows the $x$-derivatives of the Poisson kernel but not the $y$-derivative.

The next result gives some information about $r$-pairs.

Given $\sigma(x)$ and $\alpha \geq 0$, let

$$W_\alpha \sigma(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \sigma(t) \log^\alpha \Big(e + \frac{\sigma(t)}{\sigma_Q}\Big)\, dt.$$

Note $W_0\sigma = M\sigma$, the Hardy–Littlewood maximal function. It is a consequence of [St] that $W_k\sigma \leq c_{k,n} M^{k+1}\sigma$ for $k = 1, 2, \ldots$, where $M^{k+1}$ is the $(k+1)$-fold iteration of $M$.

**Theorem 3.** *Let $r \geq 2$. If $\alpha + 1 > r/2$ then $(\sigma, c_{\alpha,n} W_\alpha \sigma)$ is an $r$-pair.*

In particular, if $k$ is an integer greater than $r/2$, then $(\sigma, cW_{k-1}\sigma)$ is an $r$-pair, and so $(\sigma, cM^k\sigma)$ is an $r$-pair. This last statement was proved in [Wi] and is also true for $1 < r < 2$ with $k = 1$. In fact, Wilson showed that $(\sigma, \tau)$ is an $r$-pair if there exists an $\eta > r/2$ so that

$$\int_Q \sigma(x) \log^\eta \Big(e + \frac{\sigma(x)}{\sigma_Q}\Big)\, dx \leq c \int_Q \tau(x)\, dx$$

for all $Q$. Thus, it is enough to show that

$$\int_Q \sigma(x) \log^{\alpha+1} \Big(e + \frac{\sigma(x)}{\sigma_Q}\Big)\, dx \leq c \int_Q W_\alpha \sigma(x)\, dx.$$

This was also proved in [GI] if $\alpha$ is an integer ($\alpha = 1$). The proofs are different.

If we combine the last two theorems and remember that $r = p'$, we get the estimate

$$\|y^{-1}\phi_y * f(x)\|_{L^q_{d\mu}(\mathbb{R}^{n+1}_+)} \leq c\|f\|_{L^p_v(\mathbb{R}^n)}$$

for $\phi$ with $\int \phi = 0$, $|\phi(x)| \leq c(1+|x|)^{-M}$, $|\nabla\phi(x)| \leq c(1+|x|)^{-M-1}$, $M \geq n+2$ if

for $2 < p \leq q < \infty$:

$$|\widehat{Q}|^{-1} \mu(T(Q))^{1/q} \Big(\int_{\mathbb{R}^n} M\sigma(x) s_Q(x)^{p'/q'}\, dx\Big)^{1/p'} \leq c,$$

for $1 < p \le 2 \le q < \infty$:

$$|\widehat{Q}|^{-1}\mu\big(T(Q)\big)^{1/q}\Big(\int_{\mathbb{R}^n} W_\alpha \sigma(x) s_Q(x)^{p'/q'}\,dx\Big)^{1/p'} \le c$$

for some $\alpha > p'/2 - 1$.

*Sketch of the proof of Theorem 2.* Assume $|\phi(x)| \le c(1+|x|)^{-M}$ and $|\nabla\phi(x)| \le c(1+|x|)^{-M-1}$ for an integer $M \ge n+2$. This includes the $x$-derivatives of the Poisson kernel. Also assume that we know Theorem 1, i.e., the compactly supported case. The proof involves a reduction to the compactly supported case by using the following (modified) theorem of [U]. Write $M = (n+2)+m$ for $m = 0, 1, \ldots$.

**Lemma 1.** *Let $m = 0, 1, \ldots$ and*

(i) $|\phi(x)| + |\nabla\phi(x)| \le c(1+|x|)^{-n-2-m}$,
(ii) $\int_{\mathbb{R}^n} \phi(x)P(x)\,dx = 0$ *for every polynomial $P$ with degree* $\le m+1$.

*Then there are functions $\{v_i(x)\}_{i=0}^{\infty}$ so that*

$$\phi(x) = \sum_{i=0}^{\infty} 2^{-i(m+2)}(v_i)_{2^i}(x) = \sum_{i=0}^{\infty} 2^{-i(n+2+m)} v_i\Big(\frac{x}{2^i}\Big)$$

*with*

(a) $\operatorname{supp}(v_i) \subset \{|x| \le 1\}$,
(b) $\operatorname{supp}(v_i) \subset \{c \le |x| \le 1\}$, $0 < c < 1$, *if* $i \ne 0$,
(c) $\|v_i\|_\infty \le C$,
(d) $\int_{\mathbb{R}^n} v_i(x)P(x)\,dx = 0$ *if* $\deg(P) \le m+1$.

In [U], the stronger hypothesis that 0 does not belong to $\operatorname{supp}(\widehat{\phi})$ is made but not required in the proof. Also, (b) is not stated there but follows from the construction.

Our $\phi$ has enough decay but not enough moments vanish. This is easy to fix. Pick $\phi_1 \in C_0^\infty$ with support in $\{|x| \le 1\}$ so that $\int \phi_1 P = \int \phi P$ if $\deg(P) \le m+1$. Write

$$\phi = \phi_1 + (\phi - \phi_1).$$

Then

- $\phi_1$ satisfies $\int \phi_1 = \int \phi = 0$ and has compact support so can be treated by Theorem 1.
- $\phi - \phi_1$ has lots of vanishing moments and the same decay as $\phi$.

Thus, by replacing $\phi$ by $\phi - \phi_1$, we may assume that $\phi$ satisfies the hypothesis of Lemma 1 and

$$|\nabla\phi(x)| \le c(1+|x|)^{-n-m-3}\big( = c(1+|x|)^{-M-1}\big).$$

Recall that $Tg(x) = \iint_{\mathbb{R}^{n+1}_+} g(t,\eta)\eta^{-1}\phi_\eta(t-x)\,d\mu(t,\eta)$ and we want to show that

$$\Big(\int_{\mathbb{R}^n} |Tg|^{p'}\sigma\,dx\Big)^{1/p'} \le c\Big(\iint_{\mathbb{R}^{n+1}_+} |g|^{q'}\,d\mu(x,y)\Big)^{1/q'}.$$

Since $(\sigma,\tau)$ is a $p'$-pair, the left side is less than or equal to

$$\Big(\int_{\mathbb{R}^n} |\widetilde{S}(Tg)|^{p'}\tau\,dx\Big)^{1/p'},$$

where

$$\widetilde{S}(Tg)(x) = \Big(\sum_{\widetilde{Q}\ni x} \frac{\Lambda(Q)^2}{|Q|}\Big)^{1/2},$$

$$\Lambda(Q) = \Big(\iint_{T(Q)} |Tg*\psi_y(x)|^2\frac{dx\,dy}{y}\Big)^{1/2},$$

and $\psi$ is as in the Calderón reproducing formula. By definition of $T$,

$$|Tg*\psi_y(x)| \le \iint_{\mathbb{R}^{n+1}_+} |g(t,\eta)|\eta^{-1}|\psi_y*\phi_\eta(t-x)|\,d\mu(t,\eta).$$

We must estimate $\psi_y*\phi_\eta$.

**Lemma 2.** *For $\phi$, $\psi$ as above*

$$|(\psi_y*\phi_\eta)(x)| \le c\begin{cases} \dfrac{y\eta^{m+2}}{\eta+|x|^{n+m+3}} & \text{if } \eta\ge y \text{ or if } \eta\le y \text{ and } |x|\ge 5y\\[2ex] \dfrac{\eta^{m+2}\log(e+\frac{y}{\eta})}{y^{n+m+2}} & \text{if } \eta\le y \text{ and } |x|<5y. \end{cases}$$

This uses [U] Lemma 1 and involves some computation.

Now write

$$
\begin{aligned}
|Tg * \psi_y(x)| &\le \iint\limits_{\mathbb{R}^{n+1}_+} |g(t,\eta)|\eta^{-1}(\psi_y * \phi_\eta)(t-x)|\,d\mu(t,\eta) \\
&= \sum_Q \iint\limits_{T(Q)} \cdots .
\end{aligned}
$$

Let $Q'$ be a fixed dyadic cube and let $(x,y) \in T(Q')$. The sum is

$$
\sum_{Q:\ Q\subset\widetilde{Q'}} + \sum_{Q:\ Q\not\subset\widetilde{Q'}} .
$$

Think of $y$ as $l(Q')$, and $\eta$ as $l(Q)$. Also, $x \in Q'$ and $t \in Q$, so if $Q \subset \widetilde{Q'}$ then both $\eta \le y$ and $|x-t| < 5y$. This corresponds to the second estimate in Lemma 2. Similarly, if $Q \not\subset \widetilde{Q'}$ then either $Q$ is too big or $Q$ is not too big but centered far away from $Q'$. This corresponds to the two possibilities in the first estimate.

We get

$$
\begin{aligned}
&\sup_{(x,y)\in T(Q')} |Tg * \psi_y(x)| \\
\le & c \sum_{Q:\ Q\subset\widetilde{Q'}} \Big[\iint\limits_{T(Q)} |g|\,d\mu\Big]\Big\{\frac{l(Q)^{m+1}}{l(Q')^{n+m+2}} \log\Big(e + \frac{l(Q')}{l(Q)}\Big)\Big\} \\
&+ c \sum_{Q:\ Q\not\subset\widetilde{Q'}} \Big[\iint\limits_{T(Q)} |g|\,d\mu\Big]\Big\{\frac{l(Q')l(Q)^{m+1}}{\big(l(Q) + |x_Q - x_{Q'}|\big)^{n+m+3}}\Big\} \\
= & I + II.
\end{aligned}
$$

We will consider only $I$, which is easier than $II$. Since

$$
\Lambda(Q') = \Big(\iint\limits_{T(Q')} |Tg * \psi_y(x)|^2 \frac{dx\,dy}{y}\Big)^{1/2},
$$

the contribution of $I$ to $\Lambda(Q')$ is just

$$
I \cdot |Q'|^{1/2} = |Q'|^{1/2} \sum a(Q',Q)G(Q),
$$

where

$$G(Q) = \iint\limits_{T(Q)} |g| \, d\mu$$

and

$$a(Q', Q) = \frac{l(Q)^{m+1}}{l(Q')^{n+m+2}} \log\Big(e + \frac{l(Q')}{l(Q)}\Big) \quad \text{if } Q \subset \widetilde{Q'}$$

(and $a(Q', Q) = 0$ otherwise).

The corresponding part of

$$\widetilde{S}(Tg)(x) = \Big(\sum_{\widetilde{Q'} \ni x} \frac{\Lambda(Q')^2}{|Q'|}\Big)^{1/2}$$

is

$$\Big[\sum_{\widetilde{Q'} \ni x} \Big(\sum_{Q} a(Q', Q) G(Q)\Big)^2\Big]^{1/2}.$$

Since we want to show that

$$\|\widetilde{S}(Tg)\|_{L^{p'}_\tau} \le c\Big(\iint\limits_{R^{n+1}_+} |g|^{q'} \, d\mu\Big)^{1/q'}$$

and since (by Hölder's inequality)

$$G(Q)^{q'} \mu\big(T(Q)\big)^{1-q'} \le \iint\limits_{T(Q)} |g|^{q'} \, d\mu,$$

it is enough to show that

$$\begin{aligned}
&\Big(\int\limits_{\mathbb{R}^n} \Big(\sum_{\widetilde{Q'} \ni x} \Big[\sum_{Q} a(Q', Q) G(Q)\Big]^2\Big)^{p'/2} \tau(x) \, dx\Big)^{1/p'} \\
\le & c\Big(\sum_{Q} G(Q)^{q'} \mu\big(T(Q)\big)^{1-q'}\Big)^{1/q'}.
\end{aligned}$$

The argument now depends on whether $p \le 2$ or $p > 2$. Consider $1 < p \le 2$ (only). Use duality to estimate the left side: for $h(x) \in L^{(p'/2)'}_\tau$ with norm less than or equal to 1, consider the inequality

$$\Big(\sum_{Q'} \Big[\sum_{Q} a(Q', Q) G(Q)\Big]^2 \int\limits_{\widetilde{Q'}} h\tau \, dx\Big)^{1/2} \le c\Big(\sum_{Q} G(Q)^{q'} \mu\big(T(Q)\big)^{1-q'}\Big)^{1/q'}.$$

This is a sequence space problem.

Let

$$\begin{aligned} Y(Q) &= G(Q)/|\widehat{Q}|, \\ A(Q',Q) &= a(Q',Q)|\widehat{Q}|, \\ \nu(Q') &= \int_{\widetilde{Q'}} h\tau\, dx, \\ \rho(Q) &= |\widehat{Q}|^{q'} \mu\big(T(Q)\big)^{1-q'}. \end{aligned}$$

**Question.**

$$\Big(\sum_{Q'}\Big[\sum_{Q} A(Q',Q)Y(Q)\Big]^2 \nu(Q')\Big)^{1/2} \le c\Big(\sum_{Q} Y(Q)^{q'}\rho(Q)\Big)^{1/q'} ?$$

I.e., does the kernel $A(Q',Q)$ map

$$l^{q'}\big(\rho(Q)\big) \to l^2\big(\nu(Q)\big)\,?$$

Recall that $q' \le 2$. By the Riesz–Thorin theorem, it is enough to show that

$$l^\infty \to l^\infty$$

and

$$l^1\big(\rho(Q)\big) \to l^{2/q'}\big(\nu(Q)\big)$$

since $1/2 = (1-t)/\infty + t/(2/q')$ means $1/q' = (1-t)/\infty + t/1$.

$\underline{l^\infty \to l^\infty}$: We must show that

$$\sum_{Q} A(Q',Q) \le c \quad \text{for every } Q',$$

or, by definition of $A$, that

$$\sum_{Q:\ Q\subset\widetilde{Q'}} \frac{l(Q)^{m+1}}{l(Q')^{n+m+2}} \log\Big(e + \frac{l(Q')}{l(Q)}\Big) l(Q)^{n+1} \le c.$$

The sum is

$$
\begin{aligned}
&\sum_{Q:\ Q\subset\widetilde{Q'}} \Big(\frac{l(Q)}{l(Q')}\Big)^{n+m+2} \log\Big(e+\frac{l(Q)'}{l(Q)}\Big)\\
\le c &\sum_{Q:\ Q\subset\widetilde{Q'}} \Big(\frac{|Q|}{|Q'|}\Big)^{1+\epsilon} \qquad \text{for some } \epsilon>0 \text{ since } n+m+2>n\\
\le c &\qquad \text{by a standard argument.}
\end{aligned}
$$

$\underline{l^1 \to l^{2/q'}}$: By Minkowski's inequality,

$$
\begin{aligned}
&\Big(\sum_{Q'}\Big[\sum_{Q} A(Q',Q)Y(Q)\Big]^{2/q'}\nu(Q')\Big)^{q'/2}\\
&\le \sum_{Q}\Big[\sum_{Q'} A(Q',Q)^{2/q'}\nu(Q')\Big]^{q'/2} Y(Q),
\end{aligned}
$$

so it is enough to show that

$$
\Big[\sum_{Q'} A(Q',Q)^{2/q'}\nu(Q')\Big]^{q'/2} \le c\rho(Q) \qquad \text{for every } Q.
$$

We will use the fact that for $\alpha$, $\beta>0$,

$$
(*) \qquad \sum_{k\ge j} k^{\alpha}e^{-k\beta} \le c_{\alpha,\beta}\, j^{\alpha}e^{-j\beta}, \quad j\ge 1.
$$

Write

$$
\begin{aligned}
&\sum_{Q'} A(Q',Q)^{2/q'}\nu(Q)\\
&=\sum_{Q':\ Q\subset\widetilde{Q'}} \Big(\frac{l(Q)}{l(Q')}\Big)^{(n+m+2)\frac{2}{q'}} \log^{2/q'}\Big(e+\frac{l(Q')}{l(Q)}\Big)\int_{\widetilde{Q'}} h\tau\,dx\\
&=\int_{\mathbb{R}^n}\Big\{\sum_{Q':\ Q\subset\widetilde{Q'}} \Big(\frac{l(Q)}{l(Q')}\Big)^{(n+m+2)\frac{2}{q'}} \log^{2/q'}\Big(e+\frac{l(Q')}{l(Q)}\Big)\chi_{\widetilde{Q'}}(x)\Big\}h(x)\tau(x)\,dx.
\end{aligned}
$$

In the inner sum, the cubes $Q'$ satisfy $|x-x_Q|\le c\,l(Q')$ and $l(Q)\le c\,l(Q')$, i.e., $l(Q')\ge c[|x-x_Q|+l(Q)]$.

By $(*)$, we get at most

$$
\begin{aligned}
&c\int\limits_{\mathbb{R}^n} \Big(\frac{l(Q)}{|x-x_Q|+l(Q)}\Big)^{(n+m+2)\frac{2}{q'}} \log^{2/q'}\Big(e+\frac{|x-x_Q|}{l(Q)}\Big)h(x)\tau(x)\,dx\\
=&c\int\limits_{\mathbb{R}^n} s_Q(x)^{2/q'}h(x)\tau(x)\,dx \quad \text{ by definition of } s_Q.
\end{aligned}
$$

Now use Hölder's inequality with exponents $p'/2$ and $(p'/2)'$, and recall that $\|h\|_{L_\tau^{(p'/2)'}} \le 1$. We get at most

$$
c\Big(\int\limits_{\mathbb{R}^n} s_Q(x)^{p'/q'}\tau(x)\,dx\Big)^{2/p'} \le c\Big(\frac{|\widehat{Q}|}{\mu\big(T(Q)\big)^{1/q}}\Big)^2 \quad \text{ by hypothesis.}
$$

Raise this to the power $q'/2$ and note that the result equals $c\rho(Q)$ as desired.

## Lecture 4: Operators related to starlike sets ([CWaW], [W2])

We shall study weighted norm inequalities for certain generalizations of the Riesz fractional integral operators and associated maximal operators. We employ the idea that the geometry associated with the operators should be reflected in the conditions imposed on the weights.

**Model case.** One such operator is the following: on $\mathbb{R}^n$, $n>1$, define

$$
I_{\alpha,\beta}f(x)=f*k_{\alpha,\beta}(x),
$$

where

$$
k_{\alpha,\beta}(x)=\frac{1}{|x|^{n-1-\alpha}|x_n|^{1-\beta}},
$$

for $x=(x_1,\dots,x_{n-1},x_n)$. Here $-\beta<\alpha<n-1$ and $0<\beta<1$.

Putting $S=\{x\colon k_{\alpha,\beta}(x)>1\}$ we can easily see that $S$ depends only on $\gamma=(n-1-\alpha)/(1-\beta)$; we can write $S=S_\gamma$, $0<\gamma<\infty$.

Rewrite

$$
k_{\alpha,\beta}(x)=\frac{1}{|x|^{n-1-\alpha}|x_n|^{1-\beta}}=\frac{(|x_n|/|x|)^{\beta-1}}{|x|^{n-(\alpha+\beta)}}=\frac{\Omega(x)}{|x|^{n-\mu}},
$$

where $\mu=\alpha+\beta$, $0<\mu<n$ and $\Omega(x)=(|x_n|/|x|)^{\beta-1}$. The function $\Omega(x)$ is nonnegative, homogeneous of degree 0 and unbounded.

In general, let

$$I_{\Omega,\mu} f = f * \frac{\Omega(x)}{|x|^{n-\mu}}$$

and consider the set

$$S = \{x : \frac{\Omega(x)}{|x|^{n-\mu}} > 1\} = \{x \colon |x| < \Omega(x)^{1/(n-\mu)}\}.$$

It is starlike about the origin. In polar coordinates, i.e. $x = r\theta$ with $r = |x|$ and $\theta = x/|x| \in \Sigma := \{x : |x| = 1\}$, let

$$\rho(\theta) = \Omega(\theta)^{1/(n-\mu)}.$$

Then

$$S = \{x = r\theta \colon 0 \le r \le \rho(\theta), \theta \in \Sigma\}.$$

Note that

$$|S| = \int_S dx = \int_\Sigma \int_0^{\rho(\theta)} r^{n-1} dr\, d\theta = \frac{1}{n} \int_\Sigma \rho(\theta)^n d\theta < \infty$$

if and only if $\rho \in L^n(\Sigma)$ (i.e. $\Omega \in L^{n/(n-\mu)}(\Sigma)$).

Denote $k(x) = \Omega(x)/|x|^{n-\mu}$. Then $|\{k > 1\}| = |S|$, and for $\lambda > 0$,

$$|\{k > \lambda\}| = |\{r\theta \colon 0 \le r < \lambda^{-1/(n-\mu)} \rho(\theta)\}| = \lambda^{-n/(n-\mu)} |S|$$

by dilation.

Thus, if $|S| < \infty$, then

$$|\{k > \lambda\}| \le C \lambda^{-n/(n-\mu)},$$

and it follows from known results that

$$I_{\Omega,\mu} \colon L^p \to L^q, \quad \text{if} \quad \frac{1}{q} = \frac{1}{p} - \frac{\mu}{n}, \quad 1 < p < \frac{n}{\mu},$$

i.e., $I_{\Omega,\mu}$ satisfies the same unweighted estimates as the usual Riesz fractional integral. (The restriction $|S| < \infty$ in the model case $\rho(\theta) = |\theta_n|^{-1/(\gamma+1)}$ amounts to $\gamma + 1 > n$, or $\beta > \alpha/(n-1)$.)

If $E$ is a set in $\mathbb{R}^n$, $t > 0$, then we denote

$$tE = \{tx\colon x \in E\},$$
$$A_{E,t}f(x) = t^{-n} \int\limits_{tE} f(x-y)\,dy = \int\limits_{E} f(x-ty)\,dy.$$

We can prove by the use of polar coordinates that

$$I_{\Omega,\mu}f(x) = (n-\mu) \int\limits_0^\infty [t^\mu A_{S,t}f(x)] \frac{dt}{t}.$$

Now it is natural to define the *maximal operator* (centered at $x$)

$$M_{S,\mu}f(x) = \sup_{t>0} t^\mu A_{S,t}f(x) = \sup_{t>0} t^{\mu-n} \int\limits_{x-tS} f(y)\,dy, \quad f \geq 0.$$

For the maximal operator, we allow $\mu = 0$: $0 \leq \mu < n$.

In the case $\mu = 0$ and $|S| < \infty$, there are unweighted estimates for $M_{S,0}$ due to C. P. Calderón [Ca] as well as M. Christ [Ch] and M. Christ and J. L. Rubio de Francia [ChR]. There are some weighted results due to D. Watson [Wa1,2].

For $\mu > 0$ we have the relation

$$M_{S,\mu}f(x) \leq c \cdot I_{\Omega,\mu}f(x).$$

In fact, if $r < t$ then $rS \subset tS$ by the starlike nature, so

$$\begin{aligned} A_{S,r}f(x) &= r^{-n} \int\limits_{rS} f(x-y)\,dy \leq r^{-n}t^n \cdot t^{-n} \int\limits_{tS} f(x-y)\,dy \\ &= \left(\frac{t}{r}\right)^n A_{S,t}f(x), \end{aligned}$$

and therefore by using the representation above, for all $r > 0$,

$$\begin{aligned} I_{\Omega,\mu}f(x) &\geq (n-\mu) \int\limits_r^{2r} t^\mu \cdot \left(\frac{t}{r}\right)^{-n} A_{S,r}f(x) \cdot \frac{dt}{t} \\ &\geq cr^\mu A_{S,r}f(x). \end{aligned}$$

Now take the supremum over all $r > 0$.

We define the *uncentered maximal operator* (by a set $E$):

$$M_{S,E,\mu} f(x) = \sup_{\substack{t,z \\ t>0, z\in\mathbb{R}^n \\ x\in z-tE}} t^{\mu-n} \int\limits_{z-tS} f.$$

If $E$ is empty, this reduces to $M_{S,\mu} f$. Typically, $E$ would be the central portion of $S$. The results for $M_{S,\mu}$ and $M_{S,E,\mu}$ are very different. In the case $\mu = 0$, $M_{S,E,\mu}$ (with a different normalization) was studied by A. Cordoba [Cor].

**Some Results.** Consider first the model case $S = S_\gamma$. There is a class of rectangles naturally associated with $S$. For $a \geq 1$, consider the linear transformation $\delta_a$ defined by

$$\delta_a = (ax_1, \ldots, ax_{n-1}, a^{-\gamma} x_n) \quad \text{when} \quad x = (x_1, \ldots, x_n),$$

and the rectangle $R_a = \delta_a Q_1$ where $Q_1$ is the unit cube with center 0. Note that $|R_a| = a^{n-1-\gamma}$; $|S| < \infty$ if $n - 1 - \gamma < 0$. Denote

$$\mathcal{B}(R_a) = \{z + tR_a \colon z \in \mathbb{R}^n, t > 0\}$$

the class of all translates and dilates of $R_a$.

**Theorem 1.** *Let $1 \leq p \leq q < \infty$, $0 \leq \mu < n$, and $w$, $v$ be weights.*

(I) *If the weak type estimate*

$$w(\{x \colon M_{S,\mu} f(x) > \lambda\}) \leq \left( \frac{B \|f\|_{L^p_v}}{\lambda} \right)^q$$

*holds, then there exists a constant $c > 0$ such that for all $a \geq 1$ and all $R \in \mathcal{B}(R_a)$,*

$$|R|^{\frac{\mu}{n}-1} \Big( \int\limits_R w \Big)^{\frac{1}{q}} \Big( \int\limits_R v^{-\frac{1}{p-1}} \Big)^{\frac{1}{p'}} \leq cB |R_a|^{\frac{\mu}{n-1}} \quad \textit{if} \quad p > 1$$

$$|R|^{\frac{\mu}{n}-1} \Big( \int\limits_R w \Big)^{\frac{1}{q}} \operatorname*{ess\,sup}_{x\in R} \frac{1}{v(x)} \leq cB |R_a|^{\frac{\mu}{n-1}} \quad \textit{if} \quad p = 1.$$

(II) *Conversely, if the last condition holds with "cB" replaced by a monotone function $C(a)$ which satisfies*

$$\int_1^\infty C(a)\frac{da}{a} = b \quad \text{if} \quad q > 1$$

$$\int_1^\infty C(a)\big(1 + \log^+ C(a)\big)\frac{da}{a} = b \quad \text{if} \quad p = q = 1,$$

*then the weak type estimate holds with $B \leq cb$, for some constant $c$ independent of $w$, $v$ and $f$.*

(III) *If $1 < p \leq q < \infty$, the strong type estimate*

$$\|M_{S,\mu} f\|_{L^q_w} \leq B\|f\|_{L^p_v}$$

*holds if there exists $r > 1$ such that*

$$|R|^{\frac{\mu}{n}-\frac{1}{p}}\Big(\int_R w\Big)^{\frac{1}{q}}\Big(\frac{1}{|R|}\int_R v^{-\frac{r}{p-1}}\Big)^{\frac{1}{rp'}} \leq C(a)|R_a|^{\frac{\mu}{n}-1}$$

*for all $a \geq 1$ and all $R \in \mathcal{B}(R_a)$, provided $\int_1^\infty C(a)\,\frac{da}{a} < \infty$.*

There are also both weak and strong results for $I_{\alpha,\beta}$, e.g.

**Theorem 2.** *Let $1 < p \leq q < \infty$. Then the strong type estimate*

$$\|I_{\alpha,\beta} f\|_{L^p_w} \leq C\|f\|_{L^p_v}$$

*holds if $0 < \beta < 1$, $-\beta < \alpha < n-1$, $\gamma = (n-1-\alpha)/(1-\beta)$ and there exists $r > 1$ such that*

$$|R|^{\frac{\mu}{n}-\frac{1}{p}+\frac{1}{q}}\left(\frac{1}{|R|}\int_R w^r\right)^{\frac{1}{qr}}\left(\frac{1}{|R|}\int_R v^{-\frac{r}{p-1}}\right)^{\frac{1}{p'r}} \leq C(a)|R_a|^{\frac{\mu}{n}-1}$$

*for all $R \in \mathcal{B}(R_a)$ and all $a \geq 1$, and with $C(a)$ the same as in Theorem 1.*

*Remark 1.* The conditions on the weights can be rephrased in terms of cubes $Q$ instead of rectangles $R$, but with altered weights. For example, defining

$$\delta_a w(x) = (\det \delta_a) w(\delta_a x) = |R_a| w(\delta_a x),$$

the condition just above for strong type can be written as

$$|Q|^{\frac{\mu}{n}-\frac{1}{p}+\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q(\delta_a w)^r\right)^{\frac{1}{qr}}\left(\frac{1}{|Q|}\int_Q(\delta_a v)^{-\frac{r}{p-1}}\right)^{\frac{1}{p'r}}\le C(a)|R_a|^{-1}.$$

In the case $p<q$, the condition
(∗)

$$|Q|^{-\left(\frac{\mu}{n}-1\right)}\left(\int_{\mathbb{R}^n}\delta_a w(x)\cdot s_Q(x)^q dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n}\left(\delta_a v(x)\right)^{-\frac{1}{p-1}}\cdot s_Q(x)^{p'}dx\right)^{\frac{1}{p'}}$$
$$\le C(a)|R_a|^{-1},$$

is sufficient, where $\int_1^\infty C(a)\frac{da}{a}<\infty$ and $s_Q(x)=(|Q|^{1/n}+|x-x_Q|)^{\mu-n}$ (here $x_Q$ denotes the center of $Q$), and the same condition, but with $C(a)$ independent of $a$, is necessary.

*Remark 2* (stronger necessary condition). In the earlier condition (∗), if we change the variable, $x=\delta_a^{-1}y$, then (∗) becomes

$$|Q|^{1-\frac{\mu}{n}}\left(\int_{\mathbb{R}^n}w(y)s_Q(\delta_a^{-1}y)^q dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n}v(y)^{-\frac{1}{p-1}}s_Q(\delta_a^{-1}y)^{p'}dy\right)^{\frac{1}{p'}}\le C_a.$$

However, one can show that

$$s_Q(\delta_a^{-1}y)\le c\tilde{s}_R(y),\quad R=\delta_a Q,$$

and where if $R=I\times J$ (where $I$ is a cube in $\mathbb{R}^{n-1}$ and $J$ is an interval in $\mathbb{R}^1$), then

$$\tilde{s}_R(y)=\frac{1}{(|I|^{\frac{1}{n-1}}+|y-y_R|)^{n-1-\alpha}(|J|+|y_n-y_{R,n}|)^{1-\beta}}.$$

The corresponding stronger condition ($|\delta_a^{-1}R|=|Q|$)

$$\left(\frac{|R|}{|R_a|}\right)^{1-\frac{\mu}{n}}\left(\int_{\mathbb{R}^n}w\,\tilde{s}_R^q dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n}v^{-\frac{1}{p-1}}\,\tilde{s}_R^{p'}dy\right)^{\frac{1}{p'}}\le C$$

for $R\in\mathcal{B}(R_a)$, $a\ge1$, is still necessary. But *any* rectangle $R$ of the form $I\times J$ with $I$ a cube in $\mathbb{R}^{n-1}$ and $J$ an interval in $\mathbb{R}^1$ with $|I|^{\frac{1}{n-1}}\ge|J|$ belongs to $\mathcal{B}(R_a)$ for some $a\ge1$. Moreover, then

$$\frac{|R|}{|R_a|}=|R|\left(\frac{|J|}{|I|^{1/(n-1)}}\right)^{-\frac{\gamma+1-n}{\gamma+1}}.$$

Thus, the condition can be rewritten in terms of a general $R$ of this type, with a factor involving the eccentricity. Could this be sufficient if $q > p$?

*Remark 3.* One can find conditions involving only cubes or balls and the original weights, but they are not very sharp. Here is one for $\mu = 0$:

$$\|M_{S,0}f\|_{L^p_w} \leq c\|f\|_{L^p_w}$$

if $1 < p < \infty$, $\gamma + 1 > n$, and either

$$w \in A_{p(1-\frac{n}{\gamma+1})} \quad \text{if} \quad p(1 - \tfrac{n}{\gamma+1}) > 1$$

or

$$w \in A_{p'(1-\frac{n}{\gamma+1})} \quad \text{if} \quad p'(1 - \tfrac{n}{\gamma+1}) > 1.$$

**The off-centered function.**

Define

$$M_{S,Q_1,\mu}f(x) = \sup_{\substack{t,z \\ t>0, z\in\mathbb{R}^n \\ x\in z+tQ_1}} t^{\mu-n} \int\limits_{z+tS} f.$$

This amounts to taking the sup over all regions which contain $x$ in their central portions but not necessarily as their center.

Define $\delta_a^* x = (ax_1, \ldots, ax_{n-1}, x_n)$ and $R_a^* = \delta_a^* Q_1$. Note $R_a^*$ is the smallest rectangle which contains both $R_a$ and $Q_1$. Consider pairs $(R, R^*)$ which are joint translates and dilates of $(R_a, R_a^*)$.

**Theorem 3.** *Let $1 \leq p \leq q < \infty$, $0 \leq \mu < n$. If the weak type estimate*

$$w(\{M_{S,Q_1,\mu}f > \lambda\}) \leq \Big(\frac{B\|f\|_{L^p_v}}{\lambda}\Big)^q$$

*holds, then*

$$|R|^{\frac{\mu}{n}-1}\Big(\int\limits_{R^*} w\Big)^{\frac{1}{q}} \int\limits_{R} v^{-\frac{1}{p-1}} \leq cB|R_a|^{\frac{\mu}{n}-1} \quad \textit{if} \quad p > 1,$$

$$|R|^{\frac{\mu}{n}-1}\Big(\int\limits_{R^*} w\Big)^{\frac{1}{q}} \operatorname*{ess\,sup}_{x\in R} \frac{1}{v(x)} \leq cB|R_a|^{\frac{\mu}{n}-1} \quad \textit{if} \quad p = 1,$$

*for all $a \geq 1$ and all pairs $(R, R^*)$ with $R \in \mathcal{B}(R_a)$. Conversely, if the condition holds with $cB$ replaced by $C(a)$ (as in Theorem 1), then the weak type estimate holds.*

*Remark.* The behaviors of the centered and off-centered operators are very different, even for unweighted results: if $w = v = 1$, one needs $\gamma + 1 > n$

and $1/q = 1/p - \mu/n$ for either the centered or off-centered operators, and also

$$1 < p < n/\mu$$

for *centered operator* to be of strong type; for the *off-centered* operator the condition

$$p \geq \frac{\gamma}{\gamma - (n-1)\left(1 - \frac{\mu}{n}\right)} \quad (> 1)$$

is necessary to be of weak type, while the condition

$$p > \frac{\gamma}{\gamma - (n-1)\left(1 - \frac{\mu}{n}\right)}$$

is sufficient.

**Theorem 4.** *If $1 < p \leq q < \infty$, the off-centered maximal function satisfies the strong type estimate*

$$\|M_{S,Q_1,\mu} f\|_{L^q_w} \leq B \|f\|_{L^p_v}$$

*if there exists $r > 1$ such that*

$$|R|^{\frac{\mu}{n} - \frac{1}{p}} \Big( \int\limits_{R^*} w \Big)^{\frac{1}{q}} \Big( \frac{1}{|R|} \int\limits_{R} v^{-\frac{r}{p-1}} \Big)^{\frac{1}{rp'}} \leq C(a) |R_a|^{\frac{\mu}{n} - 1}$$

*with $C(a)$ as in Theorem 1.*

The following result used in the proof of Theorem 4 may be of interest in itself.

Consider any fixed rectangle (e.g., $R_a$) and let $\mathcal{B}$ be the family of all its translates and dilates. To each $R \in \mathcal{B}$ associate a *set* $R^*$ (not necessarily a rectangle) and assume that

$$R_1, R_2 \in \mathcal{B} \quad \text{and} \quad R_1 \subset R_2 \Rightarrow R_1^* \subset R_2^*.$$

Define

$$M_\alpha f(x) = \sup_{\substack{R \in \mathcal{B} \\ R^* \ni x}} \frac{1}{|R|^{1-\alpha}} \int\limits_{R} f, \quad 0 \leq \alpha < 1.$$

**Theorem 5.** *Let $1 < p \le q < \infty$ and $0 \le \alpha < 1$. If there exists $r > 1$ such that*

$$|R|^{\alpha-\frac{1}{p}}\Big(\int\limits_{R^*} w\Big)^{\frac{1}{q}}\Big(\frac{1}{|R|}\int\limits_{R} v^{-\frac{r}{p-1}}\Big)^{\frac{1}{rp'}} \le c_1$$

*with $c_1$ independent of $R, R^*$, then*

$$\|M_\alpha f\|_{L^q_w} \le c_2\|f\|_{L^p_v}$$

*with $c_2 = cc_1$ and $c$ independent of $\mathcal{B}$, $f$.*

*Remark.* In case $R = R^* =$ cube, this was proved by C. Perez [P2]. The problem is to find a necessary and sufficient condition when $q > p$.

**Results for more general starlike sets.** *If $S$ is starlike around 0, then there exist rectangles $\{R_j\}_{j=1}^{\infty}$ satisfying*

(1) *$R_j$ contains 0 on its major axis,*
(2) *$S \subset \bigcup_j R_j$,*
(3) *$\sum_j |R_j| \le C|S|$.*

*If $S$ is also open then in addition*

(5) *there exists $c > 0$ such that $cR_j \subset S$.*

Given a rectangle $R$ containing 0, we can associate a linear transformation $\delta_R$ with $\det \delta_R = |R|$ and $R = \delta_R Q_R$ where $Q_R$ is a cube of edgelength 1 containing 0. Let

$$\delta_R w(x) = (\det \delta_R) w(\delta_R x).$$

Here is a typical result, analogous to the earlier strong type estimates for $I_{\alpha,\beta}$.

**Theorem 6.** *Let $0 < \mu < n$, $\Omega \ge 0$ be homogeneous of degree 0, $S$ be the starlike set corresponding to $r = \rho(\theta)$ for $\rho(\theta) = \Omega(\theta)^{1/(n-\mu)}$, and suppose $1 < p \le q < \infty$.*

(a) Necessity*: If $S$ is open and symmetric about 0 and if*

$$\|I_{\Omega,\mu} f\|_{L^q_w} \le B\|f\|_{L^p_v},$$

*then for every rectangle $R \subset S$ with center 0 and every cube $Q$ (not related to $R$)*

$$|Q|^{1-\frac{\mu}{n}}\Big(\int\limits_{\mathbb{R}^n}(\delta_R w)s_Q^q dx\Big)^{\frac{1}{q}}\Big(\int\limits_{\mathbb{R}^n}(\delta_R v)^{-\frac{1}{p-1}}s_Q^{p'}dx\Big)^{\frac{1}{p'}} \le \frac{cB}{|R|}.$$

(b) Sufficiency: *If $S$ is starlike and $\{R_j\}$ is a cover satisfying* (1), (2), (3) *and there exists $r > 1$ such that for all cubes $Q$*

$$|Q|^{\frac{\mu}{n}-\frac{1}{p}+\frac{1}{q}}\Big(\frac{1}{|Q|}\int\limits_Q(\delta_{R_j}w)^r dx\Big)^{\frac{1}{qr}}\Big(\frac{1}{|Q|}\int\limits_Q(\delta_{R_j}v)^{-\frac{r}{p-1}}dx\Big)^{\frac{1}{p'r}} \le \frac{C_j}{|R_j|}$$

*with $\sum_j C_j < \infty$, then the strong type estimate holds.*

(c) *If $p < q$ and if for all cubes $Q$*

$$|Q|^{1-\frac{\mu}{n}}\Big(\int\limits_{\mathbb{R}^n}(\delta_{R_j}w)s_Q^q dx\Big)^{\frac{1}{q}}\Big(\int\limits_{\mathbb{R}^n}(\delta_{R_j}v)^{-\frac{1}{p-1}}s_Q^{p'}dx\Big)^{\frac{1}{p'}} \le \frac{C_j}{|R_j|}$$

*with $\sum_j C_j < \infty$, then the strong type estimate holds.*

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