The Krein-Šmulian Theorem and its Extensions

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This paper is a survey of the series of talks given by the author in the 36th Winter School in Abstract Analysis under the title "The Krein-Šmulian Theorem and its extensions". Some results of this work are new but the mam part of them is taken from the papers [12] – [19]. We investigate here whether, given a Banach space X and a convex subset C of the dual X^*, the distance $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), C) := \sup\{\inf\{\|k - c\| : c \in C\} : k \in \overline{\operatorname{co}}^{w^*}(K)\}$ from $\overline{\operatorname{co}}^{w^*}(K)$ to C is controlled by the distance $\operatorname{dist}(K, C)$, that is, if $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), C) \leq M \operatorname{dist}(K, C)$ for some constant $1 \leq M < \infty$ not dependent on K, where K is any weak* compact subset of X^*. Actually, all the results obtained extend in some way the classical Krein-Šmulian Theorem and this fact justifies the title of the present work.

1. Introduction

This paper is a survey of the series of talks given by the author in the 36th Winter School in Abstract Analysis under the title "The Krein-Šmulian Theorem and its extensions". The main part of this work is taken from the papers [12] – [19]. In all these papers we investigate whether, given a Banach space X and a convex subset C of the dual X^*, the distance

$$\operatorname{dist}\left(\overline{\operatorname{co}}^{w^*}(K), C\right) := \sup\left\{\inf\{\|k - c\| : c \in C\} : k \in \overline{\operatorname{co}}^{w^*}(K)\right\}$$

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from $\overline{co}^{w^*}(K)$ to C is controlled by the distance $\operatorname{dist}(K,C)$, that is, if $\operatorname{dist}(\overline{co}^{w^*}(K), C) \leq M \operatorname{dist}(K, C)$ for some constant $1 \leq M < \infty$ not dependent on K, where K is any weak* compact subset of X^*. Actually, all the results obtained in the above papers extend in some way the classical Krein-Šmulian Theorem and this fact justifies the title of the present work. Recall that this theorem, with the terminology of distances, states the following (see [8, p. 29]): if X is a Banach space and K a weak* compact subset of X^{**} such that $\operatorname{dist}(K, X) = 0$ (that is, K is a weak compact subset of X), then $\operatorname{dist}(\overline{co}^{w^*}(K), X) = 0$, that is, $\overline{co}^{w^*}(K) \subset X$ and so $\overline{co}^{w^*}(K)$ is a weak compact subset of X and $\overline{co}^{w^*}(K) = \overline{co}(K)$. Thus, looking at the Krein-Šmulian Theorem with the terminology of distances, it is natural to ask the following:

Question 1. If X is a Banach space and K a weak* compact subset of X^{**}, does the equality $\operatorname{dist}(\overline{co}^{w^*}(K), X) = \operatorname{dist}(K, X)$ always hold?

The answer is negative. Actually, we construct in Section 3 counterexamples such that $\operatorname{dist}(\overline{co}^{w^*}(K), X) \geq 3\operatorname{dist}(K, X) > 0$.

Question 2. Does there exist a universal constant $1 < M < \infty$ such that always $\operatorname{dist}(\overline{co}^{w^*}(K), X) \leq M \operatorname{dist}(K, X)$ for every weak* compact subset $K \subset X^{**}$?

The answer is affirmative. Actually, it holds true the following result, which extends the Krein-Šmulian Theorem: if K is a weak* compact subset of X^{**} and Z a convex subset of X, then $\operatorname{dist}(\overline{co}^{w^*}(K), Z) \leq 5\operatorname{dist}(K,Z)$; moreover, if $Z \cap K$ is weak* dense in K, then $\operatorname{dist}(\overline{co}^{w^*}(K), Z) \leq 2\operatorname{dist}(K, Z)$. However, for many Banach spaces X the equality $\operatorname{dist}(\overline{co}^{w^*}(K), Z) = \operatorname{dist}(K, Z)$ holds true for every convex subset $Z \subset X$ and every weak* compact subset K of X^{**} as we will see later on.

We go a step further and investigate the control of the distance $\operatorname{dist}(\overline{co}^{w^*}(K), C)$ by the distance $\operatorname{dist}(K, C)$ when C is a convex subset of a dual Banach spaces X^* and K is a weak* compact subset of X^*. The behavior of the distance $\operatorname{dist}(\overline{co}^{w^*}(K), C)$ with respect to the distance $\operatorname{dist}(K, C)$ varies. If C is a weak* closed convex subset of X^*, it is very easy to see that $\operatorname{dist}(\overline{co}^{w^*}(K), C) =$ $= \operatorname{dist}(K, C)$. However, if $C \subset X^*$ is not weak* closed, all situations are possible. In any case, as we will see later, the control of C inside X^* and the existence in C of a copy of the basis of $\ell_1(\mathfrak{c})$ are closely connected.

The paper is organized as follows.

- In Section 2 we study the control of the convex subsets C of a Banach space X inside its bidual X^{**}.
- In Section 3 we construct some counterexamples, namely, two weak* compact subsets K_1, K_2 of a bidual Banach space X^{**} such that: (i) $K_1 \cap X$ is weak* dense in K_1, $\operatorname{dist}(K_1, X) = \frac{1}{2}$ and $\operatorname{dist}(\overline{co}^{w^*}(K_1), X) = 1$; (ii) $\operatorname{dist}(K_2, X) = \frac{1}{3}$ and $\operatorname{dist}(\overline{co}^{w^*}(K_2), X) = 1$.

- In Section 4 we study the control of convex subsets of a dual Banach space X^* inside X^*.
- The Section 5 is devoted to study the class of universally Krein-Šmulian Banach spaces.
- In Section 6 we study the convex weak*-closures versus the convex norm-closures in dual Banach spaces.
- The section 7 is devoted to study the control of X inside its bidual X^{**} when X is an 1-unconditional direct sums of Banach spaces and a Banach lattice.
- In section 8 we study the control of some convex subsets of the dual space $\ell_\infty(I)$.

Our notation is standard. If A and I are sets, $a \in A^I$ and $i \in I$ then a_i (or $a(i)$) denotes the i-th coordinate of a and $\pi_i : A^I \to A$ the i-th. projection mapping such that $\pi_i(a) = a_i$. $|I|$ is the cardinality of I and $\mathfrak{c} := |\mathbb{R}|$. βI denotes the Stone-Čech compactification of I (the set I is endowed with the discrete topology) and $I^* := \beta I \backslash I$. If $f : I \to \mathbb{R}$ is a bounded function, then $\check{f} \in C(\beta I)$ is the Stone-Čech continuous extension of f to the all βI.

We shall consider only Banach spaces over the real field. If X is a Banach space, let $B(a;r) := \{x \in X : \|x - a\| \leq r\}$ be the closed ball with center at $a \in X$ and radius $r \geq 0$. $B(X)$ and $S(X)$ will be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. If A is a subset of X, then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A, respectively. A subset A of the Banach space X is said to have a copy of the basis of $\ell_1(\mathfrak{c})$ if A contains a family of vectors $\{a_i : i < \mathfrak{c}\}$ which is equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. The weak* topology of the dual Banach space X^* is denoted by w^* and the weak topology of X by w. If A is a subset of X^*, $\operatorname{co}(A)$ denotes the convex hull of the set A, $\overline{\operatorname{co}}(A)$ is the $\|\cdot\|$-closure of $\operatorname{co}(A)$ and $\overline{\operatorname{co}}^{w^*}(A)$ the w^*-closure of $\operatorname{co}(A)$. Given $1 \leq M < \infty$, a convex subset C of X^* *is said to have M-control inside* X^* if $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), C) \leq$ $\leq M \operatorname{dist}(K, C)$ for every w^*-compact subset K of X^*. *C is said to have control inside X^* if C has M-control inside* X^*, for some constant $1 \leq M < \infty$.

If K is a w^*-compact subset of a dual Banach space X^* and μ a Radon Borel probability on K, then $r(\mu)$ will denote the barycenter or resultant of μ (see [7, p. 115]). Recall that: (i) $r(\mu) \in \overline{\operatorname{co}}^{w^*}(K)$; (ii) $x^* \in \overline{\operatorname{co}}^{w^*}(K)$ if and only if there exists a Radon Borel probability μ on K such that $r(\mu) = x^*$; (iii) $r(\mu)(x) =$ $= \int_K x^*(x)\, d\mu(x^*)$ for all $x \in X$.

We refer the reader to the book [10] for the definition and properties of weakly compactly generated (*WCG*) and weakly Lindelöf determined (*WLD*) Banach spaces.

2. The control of convex subsets of X inside X**

The convex subsets of a bidual Banach space X^{**}, in general, fail to have control inside X^{**}. For example, if X is a Banach space such that X^* contains a copy of ℓ_1, then there exists a w^*-compact subset H of X^{**} such that

$\mathrm{dist}\left(\overline{\mathrm{co}}^{w^*}(H), \overline{\mathrm{co}}(H)\right) > 0$ (see [20]). However, when we restrict ourself to the convex subsets C of the Banach space X, we will see in this section that there exists control inside X^{**}. We begin with the calculation of the distance $\mathrm{dist}(x, C)$, when C is a convex subset of a Banach space X and $x \in X$.

Lemma 2.1. *Let X be a Banach space, C a convex subset of X and $x \in X$. Then the distance $dist(x, C)$ from x to C satisfies*

$$dist(x, C) = \sup_{\varphi \in S(X^*)} \inf\{|\varphi(x - c)| : c \in C\}.$$

Moreover, if $x \notin \overline{C}$, then even $dist(x, C) = \sup_{\varphi \in S(X^*)} \inf \varphi(x - C)$.

Proof. If we assume that $x \notin \overline{C}$, the proof of the statement is a simple application of Banach separation theorem. If $x \in \overline{C}$, then for every $\varphi \in S(X^*)$ we have $\inf\{|\varphi(x - c)| : c \in C\} = 0$, whence

$$\mathrm{dist}(x, C) = 0 = \sup_{\varphi \in S(X^*)} \inf\{|\varphi(x - c)| : c \in C\}. \qquad \square$$

The following lemmas are basic for the proofs of next propositions.

Lemma 2.2. *Let X be a Banach space and D a convex subset of X. Then for every $z \in \overline{D}^{w^*} \subset X^{**}$ we have:*

$$dist(z, D) \leq 2dist(z, X).$$

Proof. Suppose that $\mathrm{dist}(z, D) > 2\mathrm{dist}(z, X)$. Then

(i) for some $a > 0$ we have $\mathrm{dist}(z, D) > 2a > 2\mathrm{dist}(z, X)$ and

(ii) there exists a vector $w \in X$ such that $\|w - z\| < a$ (because $\mathrm{dist}(z, X) < a$) and so $\mathrm{dist}(w, D) > a$ (otherwise, if $\mathrm{dist}(w, D) \leq a$, we would get $\mathrm{dist}(z, D) \leq$ $\leq \|w - z\| + \mathrm{dist}(w, D) < 2a$, a contradiction).

Since $\mathrm{dist}(w, D) > a$, by Lemma 2.1 there exists $x^* \in S(X^*)$ such that $\inf\{x^*(w - d) : d \in D\} > a$. Let $\{d_i\}_{i \in I} \subset D$ be a net such that $d_i \xrightarrow{w^*} z$. Then $w - d_i \xrightarrow{w^*} w - z$ and so $x^*(w - d_i) \longrightarrow x^*(w - z)$. Hence $x^*(w - z) > a$ and so $\|w - z\| > a$, a contradiction. Thus, we get $\mathrm{dist}(z, D) \leq 2\mathrm{dist}(z, X)$. $\square$

Lemma 2.3. *Let X be a Banach space, C a convex subset of X^*, K a w^*-compact subset of X^* and assume there exist two numbers $a, b > 0$ such that:*

$$dist(K, C) < a < b < dist\left(\overline{co}^{w^*}(K), C\right).$$

Then there exist $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ with $\inf \psi(z_0 - C) > b$ such that, if μ is a Radon probability on K with barycenter $r(\mu) = z_0$ and $H = \mathrm{supp}(\mu)$ is the support of μ, for every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ such that $\inf \psi(\xi - C) > b$.

Proof. Without loss of generality, we suppose that $K \subset B(X^*)$. Choose $z \in \overline{\mathrm{co}}^{w^*}(K)$ such that $\mathrm{dist}(z, C) > b$. By Lemma 2.1 there exists $\psi \in S(X^{**})$ such that $\inf \psi(z - C) > b + \varepsilon$ for some $\varepsilon > 0$, that is, $\psi(z) > b + \varepsilon + \sup \psi(C)$. By the Bishop-Phelps Theorem, there exists a vector $\phi \in S(X^{**})$ with $\|\psi - \phi\| \leq \varepsilon/4$ such that ϕ attains its maximum on $\overline{\text{co}}^{w^*}(K)$ at some point $z_0 \in \overline{\text{co}}^{w^*}(K)$. So:

$$\begin{aligned} \phi(z_0) \geq \phi(z) = \psi(z) + (\phi - \psi)(z) > \sup \psi(C) + b + \varepsilon - \frac{1}{4}\varepsilon = \\ = \sup \psi(C) + b + \frac{3}{4}\varepsilon, \end{aligned} \tag{2.1}$$

whence we get

$$\psi(z_0) = \phi(z_0) + (\psi - \phi)(z_0) > \sup \psi(C) + b + \frac{3}{4}\varepsilon - \frac{1}{4}\varepsilon = \sup \psi(C) + b + \frac{1}{2}\varepsilon,$$

that is,

$$\inf \psi(z_0 - C) > b + \frac{1}{2}\varepsilon. \tag{2.2}$$

Thus $\text{dist}(z_0, C) > b + \frac{1}{2}\varepsilon$ and so $z_0 \notin \overline{C}$ and $z_0 \notin K$ (because $\text{dist}(K, C) < a < b$). Let μ be a Radon probability on K with barycenter $r(\mu) = z_0$ and let $H := \text{supp}(\mu)$ be the support of μ. Assume that there exists a w^*-open subset V of X^* with $V \cap H \neq \emptyset$ such that $\inf \psi(\xi - C) \leq b$ (that is, $\psi(\xi) \leq b + \sup \psi(C)$) for every $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$. Let $\mu_1 = \mu \restriction V \cap H$ denote the restriction of μ to $V \cap H$, that is, $\mu_1(B) = \mu(B \cap V \cap H)$ for every Borel subset $B \subset K$. Let $\mu_2 := \mu - \mu_1$. Observe that μ_1 and μ_2 are positive measures such that

(i) $\mu_1 \neq 0$, because $\emptyset \neq V \cap H = V \cap \text{supp}(\mu)$, and

(ii) $\mu_2 \neq 0$ because, if we assume $\mu_2 = 0$ (that is, $\mu = \mu_1 = \mu \restriction V \cap H$), then $z_0 = r(\mu) \in \overline{\text{co}}^{w^*}(V \cap H)$ and so $\inf \psi(z_0 - C) \leq b$, a contradiction to (2.2).

Thus, we have the decomposition $\mu = \mu_1 + \mu_2$ such that $1 = \|\mu\| = \|\mu_1\| + \|\mu_2\|$ with $\|\mu_1\| \neq 0 \neq \|\mu_2\|$. So, we can write:

$$z_0 = r(\mu) = \|\mu_1\| \cdot r\left(\frac{\mu_1}{\|\mu_1\|}\right) + \|\mu_2\| \cdot r\left(\frac{\mu_2}{\|\mu_2\|}\right).$$

Since $r\left(\frac{\mu_1}{\|\mu_1\|}\right) \in \overline{\text{co}}^{w^*}(V \cap H)$, then $\psi\left(r\left(\frac{\mu_1}{\|\mu_1\|}\right)\right) \leq b + \sup \psi(C)$ by hypothesis. Hence $\phi\left(r\frac{\mu_1}{\|\mu_1\|}\right) \leq b + \frac{1}{4}\varepsilon + \sup \psi(C)$ (because $\|\psi - \phi\| \leq \varepsilon/4$). Thus, taking into account that $r\left(\frac{\mu_2}{\|\mu_2\|}\right) \in \overline{\text{co}}^{w^*}(K)$, $\phi\left(r\left(\frac{\mu_2}{\|\mu_2\|}\right)\right) \leq \phi(z_0)$ and (2.1), we get

$$\begin{aligned} \phi(z_0) = \|\mu_1\| \, \phi\left(r\left(\frac{\mu_1}{\|\mu_1\|}\right)\right) + \|\mu_2\| \phi\left(r\left(\frac{\mu_2}{\|\mu_2\|}\right)\right) \leq \\ \leq \|\mu_1\| \left(b + \frac{1}{4}\varepsilon + \sup \psi(C)\right) + \|\mu_2\| \phi(z_0) < \|\mu_1\| \phi(z_0) + \|\mu_2\| \phi(z_0) = \phi(z_0), \end{aligned}$$

a contradiction, and this completes the proof. □

Proposition 2.4. *Let X be a Banach space, C a convex subset of X and K a w^*-compact subset of X^{**}. Then*

$$\textit{dist}\left(\overline{\text{co}}^{w^*}(K), C\right) \leq 5\textit{dist}(K, C).$$

Proof. Without loss of generality, we assume that $0 \in C$. Suppose that the statement is not true and try to get a contradiction. So, assume that there exists a w^*-compact subset K of X^{**} and two real numbers $a, b > 0$ such that

$$\operatorname{dist}\left(\overline{\mathrm{co}}^{w^*}(K), C\right) > b > 5a > 5\operatorname{dist}(K, C).$$

From Lemma 2.3 we have the following Fact:

Fact. There exists a functional $\psi \in S(X^{***})$ and a w^*-compact subset $\emptyset \neq H \subset K$ such that for every w^*-open subset V with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ with $\inf \psi(\xi - C) > b$.

Now we do the following construction step by step:

Step 1. Let $D_0 = \{0\}$. Applying the Fact to the w^*-open subset $V_0 := X^{**}$, we choose a vector $\xi_1 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $\inf \psi(\xi_1 - C) > b$. So, $\psi(\xi_1) > b + \sup \psi(D_0) = b$. As $B(X^*)$ is w^*-dense in $B(X^{***})$, there exists $x_1^* \in S(X^*)$ such that $x_1^*(\xi_1) > b + \max x_1^*(D_0) = b$. Let $W_1 := \{u \in X^{**} : \langle u, x_1^* \rangle > b + \max x_1^*(D_0) = b\}$. Clearly, W_1 is a w^*-open halfspace of X^{**} such that $\xi_1 \in W_1 \cap \overline{\mathrm{co}}^{w^*}(H)$. Thus, $W_1 \cap H \neq \emptyset$ and so we can find a vector $\eta_1 \in W_1 \cap H$. Since $\operatorname{dist}(\eta_1, C) < a$, we have the decomposition $\eta_1 = \eta_1^1 + \eta_1^2$ such that $\eta_1^1 \in C$ and $\eta_1^2 \in aB_{X^{**}}$.

Step 2. Let $D_1 = \{\eta_1^1\} \cup D_0 \subset C$ and $V_1 := W_1 \cap V_0 = W_1$. As V_1 is a w^*-open subset with $V_1 \cap H \neq \emptyset$, by the Fact there exists a vector $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$ such that $\inf \psi(\xi_2 - C) > b$ and also $\inf \psi(\xi_2 - D_1) \geq \inf \psi(\xi_2 - C) > b$ because $D_1 \subset C$. Since D_1 is finite and $\min \psi(\xi_2 - D_1) > b$, there exists a vector $x_2^* \in S(X^*)$ such that $\min x_2^*(\xi_2 - D_1) > b$, that is, $x_2^*(\xi_2) > b + \max x_2^*(D_1)$. Let $W_2 := \{u \in X^{**} : \langle u, x_2^* \rangle > b + \max x_2^*(D_1)\}$. Clearly, W_2 is a w^*-open halfspace of X^{**} such that $\xi_2 \in W_2 \cap \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$. Thus $W_2 \cap V_1 \cap H \neq \emptyset$ and we can find $\eta_2 \in W_2 \cap V_1 \cap H$. So, $x_2^*(\eta_2) > b + \max x_2^*(D_1)$, that is, $\min x_2^*(\eta_2 - D_1) > b$. Moreover, $\min x_1^*(\eta_2 - D_0) > b$ because $\eta_2 \in V_1$. Since $\operatorname{dist}(\eta_2, C) < a$, we have the decomposition $\eta_2 = \eta_2^1 + \eta_2^2$ such that $\eta_2^1 \in C$ and $\eta_2^2 \in aB(X^{**})$.

Further, we proceed by iteration. We get the sequences $\{x_n^*\}_{n \geq 1} \subset S(X^*)$, $\{\eta_k\}_{k \geq 1} \subset H$, $D_k = \{\eta_k^1\} \cup D_{k-1}$ with $\eta_k = \eta_k^1 + \eta_k^2$, $\eta_k^1 \in C$ and $\eta_k^2 \in aB(X^{**})$ $k \geq 1$, such that $\min x_i^*(\eta_k - D_{i-1}) > b$, for every $k \geq i$.

Let $D = \overline{\mathrm{co}}\left(\cup_{k \geq 1} D_k\right) \subset \overline{C}$ and:

$$K_1 = \overline{\{\eta_i^1 : i \geq 1\}}^{w^*} \subset \left(K + aB(X^{**})\right) \cap \overline{D}^{w^*}.$$

Let η_0 be a w^*-cluster point of $\{\eta_k\}_{k \geq 1}$.

Claim 1. $\operatorname{dist}(\eta_0, D) < 5a$.

Indeed, clearly $\eta_0 \in H \cap \left(K_1 + aB(X^{**})\right)$. Observe that:

(i) Since $K_1 \subset K + aB(X^{**})$, we get $\operatorname{dist}(K_1, C) \leq \operatorname{dist}(K, C) + a < 2a$.

(ii) Since $K_1 \subset \overline{D}^{w^*}$, by Lemma 2.2 we get $\operatorname{dist}(K_1, D) \leq 2 \operatorname{dist}(K_1, X) \leq 2\operatorname{dist}(K_1, C) < 4a$.

Thus, as $\eta_0 \in K_1 + aB(X^{**})$, finally we get $\operatorname{dist}(\eta_0, D) < 5a$.

Claim 2. $\operatorname{dist}(\eta_0, D) \geq b$.

Indeed, let $\phi \in B(X^{***})$ be a w^*-cluster point of $\{x_n^*\}_{n\geq 1}$. Since $\min x_n^*(\eta_k - D_{n-1}) > b$ for every $k \geq n$, then $\min x_n^*(\eta_0 - D_{n-1}) \geq b$, $\forall n \geq 1$. Hence $\inf \phi(\eta_0 - D) \geq b$ and so $\operatorname{dist}(\eta_0, D) \geq b$ by Lemma 2.1.

Since $b > 5a$ we get a contradiction and this completes the proof. □

Proposition 2.5. *Let X be a Banach space, $C \subset X$ a convex subset of X and K a w^*-compact subset of X^{**} such that $K \cap C$ is w^*-dense in K. Then $dist(\overline{\text{co}}^{w^*}(K), C) \leq 2dist(K, C)$.*

Proof. Suppose that $\operatorname{dist}(\overline{\text{co}}^{w^*}(K), C) > b > 2a > 2\operatorname{dist}(K, C)$ for some numbers $a, b > 0$. We follow the proof of Proposition 2.4 with the following changes. As $C \cap K$ is w^*-dense in K and $V_k \cap H \neq \emptyset$, $k \geq 0$, then $V_k \cap C \cap K \neq \emptyset$, $\forall k \geq 0$. Thus, we choose $\eta_k \in V_k \cap K \cap C$, $k \geq 1$, and put $\eta_k^1 = \eta_k$ and $\eta_k^2 = 0$. Hence, now $K_1 = \overline{\{\eta_k^1 : k \geq 1\}}^{w^*} = \overline{\{\eta_k : k \geq 1\}}^{w^*}$ satisfies $K_1 \subset K$ and so $\operatorname{dist}(K_1, C) \leq \operatorname{dist}(K, C) < a$, whence we obtain $\operatorname{dist}(K_1, D) < 2a$. Finally, every w^*-cluster point η_0 of $\{\eta_k : k \geq 1\}$ satisfies $\eta_0 \in K_1$, $\operatorname{dist}(\eta_0, D) < 2a$ and $\operatorname{dist}(\eta_0, D) \geq b$, a contradiction. □

3. Counterexapmles

In this Section 3 we construct a Banach space X and a w^*-compact subset $H \subset X^{**}$ such that $\operatorname{dist}(\overline{\text{co}}^{w^*}(H), X) \geq 3\operatorname{dist}(H, X) > 0$. This example together with Proposition 2.4 show that the optimal constant $1 \leq M < \infty$ such that $\operatorname{dist}(\overline{\text{co}}^{w^*}(W), Z) \leq M \operatorname{dist}(W, Z)$, for every Banach space X, every convex subset $Z \subset X$ and every w^*-compact subset $W \subset X^{**}$, satisfies $3 \leq M \leq 5$. We also construct a w^*-compact subset $K \subset X^{**}$ with $K \cap X$ w^*-dense in K such that $\operatorname{dist}(\overline{\text{co}}^{w^*}(K), X) \geq 2\operatorname{dist}(K, X)$. So, this counterexample together with Proposition 2.5 show that $M = 2$ is the optimal constant M such that $\operatorname{dist}(\overline{\text{co}}^{w^*}(W), Z) \leq M \operatorname{dist}(W, Z)$ for every Banach space X, every convex subset $Z \subset X$ and every w^*-compact subset $W \subset X^{**}$ with $W \cap Z$ w^*-dense in W.

Proposition 3.1. *There exists a Banach space X fulfilling the following facts:*

(A) There exists a w^*-compact subset $K \subset B(X^{**})$ such that $K \cap X$ is w^*-dense in K and $dist(K, X) = \frac{1}{2}$ but $dist(\overline{\text{co}}^{w^*}(K), X) = 1$.

(B) There exists a w^*-compact subset $H \subset B(X^{**})$ such that $dist(H, X) = \frac{1}{3}$ but $dist(\overline{\text{co}}^{w^*}(H), X) = 1$.

Proof. Let $\mathscr{C} = \{0, 1\}^{\mathbb{N}}$ be the Cantor compact set and $\mathscr{S} := \{0,1\}^{<\mathbb{N}} = \{0,1\} \cup \{0,1\}^2 \cup \{0,1\}^3 \cup \ldots$. Let λ be the Haar probability on $\{0,1\}^{\mathbb{N}}$. If

$\sigma = (\sigma_1, \sigma_2, \ldots) \in \mathscr{C}$ and $n \in \mathbb{N}$, we put $\sigma_{\restriction n} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathscr{S}$. If $A \subset \{0,1\}^n$, let $f_A : \mathscr{C} \to \{0,1\}$ be the continuous mapping

$$\forall \sigma \in \mathscr{C},\ f_A(\sigma) = \begin{cases} 1, & \text{if } \sigma_{\restriction n} \in A, \\ 0, & \text{if } \sigma_{\restriction n} \notin A. \end{cases}$$

For each $n \in \mathbb{N}$ we define I_n as

$$I_n := \{f_A \subset \{0,1\}^n \text{ with } |A| = 2^n - n\}.$$

Observe that I_n is finite and $\int_{\mathscr{C}} f_A d\lambda = 1 - n2^{-n}$ for each $f_A \in I_n$. Let $I := \bigcup_{n \geq 1} I_n$. Clearly, $|I| = \aleph_0$ and so we can put $I = \{f_{A_m} : m \geq 1\}$. We shall identify I with $\mathbb{N}$ by means of the identification of m and f_{A_m}. So, instead of $\ell_\infty(\mathbb{N})$ we also write $\ell_\infty(I)$. Note that:

(1) I separates points in $\mathscr{C}$.

(2) Since each I_n is finite and $\int_{\mathscr{C}} f_A d\lambda = 1 - n2^{-n}$ for each $f_A \in I_n$, then $\lim_{m\to\infty} \int_{\mathscr{C}} f_{A_m}(\sigma)\, d\lambda(\sigma) = 1$.

(3) Let $\{\sigma_j : j = 1, \ldots, k\}$ be a finite subset of $\mathscr{C}$. Then for each $n \geq k$, there are $f_A, f_B \in I_n$ such that $f_A(\sigma_j) = 0$ and $f_B(\sigma_j) = 1$ for each $j = 1, \ldots, k$. Thus, if for every $\sigma \in \mathscr{C}$ we define $\mathcal{O}(\sigma) = \{f_A \in I : f_A(\sigma) = 0\}$, then $|\bigcap_{i=1}^k (\varepsilon_i) \mathcal{O}(\sigma_i)| = \aleph_0$, where $\varepsilon_i = \pm 1$, $(+1)\mathcal{O}(\sigma_i) = \mathcal{O}(\sigma_i)$ and $(-1)\mathcal{O}(\sigma_i) = I \backslash \mathcal{O}(\sigma_i)$.

(4) For every $f_A \in I$ there exists $\sigma \in \mathscr{C}$ such that $f_A(\sigma) = 1$.

From (3) and (4) we get that the compact set $\mathcal{O} = \bigcap_{\sigma \in \mathscr{C}} \overline{\mathcal{O}(\sigma)}^{\beta I}$ satisfies $\emptyset \neq \mathcal{O} \subset I^* := \beta I \backslash I$. Let $\psi : \mathscr{C} \to \{0,1\}^I \subset B(\ell_\infty(I))$ be the mapping

$$\forall i = f_A \in I,\ \forall \sigma \in \mathscr{C},\ \psi(\sigma)(i) = f_A(\sigma).$$

Clearly ψ is an injective continuous mapping for the w^*-topology of $\{0,1\}^I \subset \ell_\infty(I)$, which coincides with the product topology of $\{0,1\}^I$. Thus $D := \psi(\mathscr{C}) \subset \{0,1\}^I$ is a compact subset homeomorphic with $\mathscr{C}$ such that $\check{d} \restriction \mathcal{O} = 0$, $\forall d \in D$. Let $\mu := \psi(\lambda)$ be the Radon Borel probability on D that is the image of the Haar probability λ under the continuous mapping ψ, and let $r(\mu) =: z_0 \in \overline{\text{co}}^{w^*}(D)$ be the barycenter of μ. Clearly, $z_0 \in [0,1]^I$ and so $0 \leq \check{z}_0(p) \leq 1$ for every $p \in \beta I$ (recall that $\check{z}_0$ is the Stone-Čech continuous extension of z_0 to the all βI).

<u>**Claim 0.**</u> $\check{z}_0(p) = 1$ for every $p \in I^* := \beta I \backslash I$.

Indeed, we know that for every $i = f_A \in I$ we have

$$1 \geq z_0(i) = \pi_i(r(\mu)) = \int_D \pi_i(x) d\mu(x) = \int_{\mathscr{C}} \pi_i \circ \psi(\sigma)\, d\lambda(\sigma) =$$
$$= \int_{\mathscr{C}} \psi(\sigma)(i)\, d\lambda(\sigma) = \int_{\mathscr{C}} f_A(\sigma)\, d\lambda(\sigma).$$

On the other hand, by (2) $\lim_{m\to\infty} \int_{\mathscr{C}} f_{A_m}(\sigma)\, d\lambda(\sigma) = 1$ and this implies that $\check{z}_0(p) = 1$ for every $p \in I^*$.

For each $m \in \mathbb{N}$ (which is associated with $f_{A_m} \in I$) we define

$$D_m^1 = \{d \in D : \pi_m(d) = 1\}, D_m^0 = \{d \in D : \pi_m(d) = 0\}, m \geq 1,$$

$\pi_m : \ell_\infty \to \mathbb{R}$ being the canonical m-th projection. We have $\mu(D_m^1) \to 1$ and so $\mu(D_m^0) = \mu(D \setminus D_m^1) \to 0$ when $m \to \infty$. Indeed, if $m \in \mathbb{N}$, we have

$$\mu(D_m^1) = \int_D \pi_m(x)\, d\mu(x) = \int_{\mathscr{C}} \pi_m \circ \psi(\sigma)\, d\lambda(\sigma) =$$

$$= \int_{\mathscr{C}} \psi(\sigma)(f_{A_m})\, d\lambda(\sigma) = \int_{\mathscr{C}} f_{A_m}(\sigma)\, d\lambda(\sigma).$$

By (2) we know that $\lim_{m\to\infty} \int_{\mathscr{C}} f_{A_m}(\sigma)\, d\lambda(\sigma) = 1$. Thus $\mu(D_m^1) \to 1$ when $m \to \infty$.

Let $X := \{f \in \ell_\infty(I) : \check{f} \upharpoonright \mathcal{O} = 0\}$. The dual space X^* is

$$X^* = \ell_1(I) \oplus_1 M_R(I^*, \mathcal{O}),$$

$M_R(I^*, \mathcal{O})$ being the space of Radon measures ν on I^* such that $|\nu|(\mathcal{O}) = 0$ ($\oplus_1$ means the ℓ_1-sum). Actually, $\ell_1(I) \oplus_1 M_R(I^*, \mathcal{O})$ is a closed complemented subspace of $(\ell_\infty(I))^* = \ell_1(I) \oplus_1 M_R(\beta I \setminus I)$.

The bidual of X is $X^{**} = \ell_\infty(I) \oplus_\infty M_R(I^*, \mathcal{O})^*$, $\oplus_\infty$ meaning the ℓ_∞-sum. Let $\pi_1, \pi_2 : X^{**} \to X^{**}$ be the canonical projections on the summands $\ell_\infty(I)$ and $M_R(I^*, \mathcal{O})^*$, respectively. Observe that the subspaces $\pi_1(X^{**}) = \ell_\infty(I)$ and $\pi_2(X^{**}) = M_R(I^*, \mathcal{O})^*$ are w^*-closed in X^{**}. Moreover, the w^*-topology $\sigma(X^{**}, X^*)$ coincides on $\pi_1(X^{**}) = \ell_\infty(I)$ with the $\sigma(\ell_\infty(I), \ell_1(I))$-topology. If $x \in X^{**}$ we put $x = (x_1, x_2)$, with $\pi_1(x) = x_1 \in \ell_\infty(I)$ and $\pi_2(x) = x_2 \in M_R(I^*, \mathcal{O})^*$. So, if $J : X \to X^{**}$ is the canonical embedding and $f \in X$, we put $J(f) = (f_1, f_2)$, where $f_1 = \pi_1(f) = f$, and $\pi_2(f) = f_2$ satisfies $f_2(\nu) = \nu(\check{f}) = \int_{I^* \setminus \mathcal{O}} \check{f}\, d\nu$, for every $\nu \in M_R(I^*, \mathcal{O})$. Note that the space $(\mathscr{B}_{ob}(I^*, \mathcal{O}), \|\cdot\|_\infty)$ of bounded Borel functions $h : I^* \to \mathbb{R}$ vanishing on $\mathcal{O}$, with the $\|\cdot\|_\infty$-norm, may be considered isometric and isomorphically embedded into $\pi_2(X^{**}) = M_R(I^*, \mathcal{O})^*$. Actually, if $f \in X$, then $\pi_2(f) = f_2 = \check{f} \in \mathscr{B}_{ob}(I^*, \mathcal{O})$.

(A) The mapping $\phi : \ell_\infty(I) \to X^{**}$ such that $\phi(f) = (f, 0)$, $\forall f \in \ell_\infty(I)$, is an isometric isomorphism between $\ell_\infty(I)$ and $\pi_1(X^{**})$, and also an isomorphism for the $\sigma(\ell_\infty(I), \ell_1(I))$-topology of $\ell_\infty(I)$ and the w^*-topology of $\pi_1(X^{**})$. Thus $\phi(D) = \{(d, 0) : d \in D\} \subset B(X^{**})$ is a w^*-compact subset of $B(X^{**})$ homeomorphic with $\mathscr{C}$. Let

$$K := \{(f, 0) \in B(X^{**}) : 0 \leq f \leq d \text{ for some } d \in D\}.$$

Clearly K is a w^*-compact subset of $B(\ell_\infty(I)) \subset B(X^{**})$ such that $\phi(D) \subset K$, and $\overline{K \cap J(X)}^{w^*} = K$.

Claim 1. $\operatorname{dist}(K, J(X)) = \frac{1}{2}$

Indeed, let $(f, 0) \in K$. Then $\|(f, 0) - \frac{1}{2}J(f)\| = \|(\frac{1}{2}f, -\frac{1}{2}\check{f})\| \leq \frac{1}{2}$. Therefore $\operatorname{dist}(K, J(X)) \leq \frac{1}{2}$. On the other hand, given $\tau \in \mathscr{C}$, let $\psi(\tau) =: d_\tau \in D$. Clearly

$\operatorname{supp}(d_\tau) = \{i \in I : d_\tau(i) = 1\} =: A_\tau$ is an infinite subset. We claim that $\operatorname{dist}((d_\tau, 0), J(X)) \geq \frac{1}{2}$. Indeed, otherwise there would exist $h \in X$ such that $\|(d_\tau, 0) - J(h)\| = \|(d_\tau, 0) - (h, \check{h})\| < \frac{1}{2}$. Thus $\|d_\tau - h\| < \frac{1}{2}$ in $\ell_\infty(I)$, and this implies $\frac{1}{2} < h$ on A_τ. Hence $\check{h} \geq \frac{1}{2}$ on $\overline{A_\tau}^{\beta I}$. Since A_τ is infinite, $\emptyset \neq \overline{A_\tau}^{\beta I} \setminus I \subset I^*$ and every $p \in \overline{A_\tau}^{\beta I} \setminus I$ satisfies $\check{h}(p) \geq \frac{1}{2}$. Let δ_p be the Dirac probability with mass 1 on p for some $p \in \overline{A_\tau}^{\beta I} \setminus I$. Observe that $\delta_p \in M_R(I^*, \mathcal{O})$ because $\mathcal{O} \cap \overline{A_\tau}^{\beta I} = \emptyset$. Then

$$|((d_\tau, 0) - (h, \check{h}))(\delta_p)| = |0 - \check{h}(\delta_p)| = |-\check{h}(p)| \geq \tfrac{1}{2},$$

whence $\|(d_\tau, 0) - J(h)\| \geq \frac{1}{2}$, a contradiction. Thus $\operatorname{dist}(K, J(X)) \geq \frac{1}{2}$.

Claim 2. $\operatorname{dist}(\overline{\mathrm{co}}^{w^*}(K), J(X)) = 1$.

Indeed, first $\operatorname{dist}(\overline{\mathrm{co}}^{w^*}(K), J(X)) \leq 1$ because $\overline{\mathrm{co}}^{w^*}(K) \subset B(X^{**})$. On the other hand, let $\nu := \phi(\mu)$ be the probability on $\phi(D) \subset K$ image of μ under the continuous linear mapping ϕ. Then the barycenter $r(\nu)$ of ν belongs to $\overline{\mathrm{co}}^{w^*}(K)$ and satisfies $r(\nu) = (z_0, 0)$, where $z_0 = r(\mu) \in B(\ell_\infty(I))$. We claim that $\operatorname{dist}((z_0, 0), J(X)) \geq 1$. Indeed, given $h \in X$, we have $\check{h} \restriction \mathcal{O} = 0$. On the other hand, $\check{z}_0 \restriction \mathcal{O} = 1$. Thus for $\varepsilon > 0$ there exists an open neighborhood V of $\mathcal{O}$ in βI such that

$$\forall v \in V,\ \check{h}(v) \leq \frac{\varepsilon}{2} \text{ and } \check{z}_0(v) \geq 1 - \frac{\varepsilon}{2}.$$

In particular, $\forall v \in V \cap I$, $h(v) \leq \frac{\varepsilon}{2}$ and $z_0(v) \geq 1 - \frac{\varepsilon}{2}$, whence we get $\|z_0 - h\| \geq 1 - \varepsilon$, that is, $\|(z_0, 0) - (h, \check{h})\| \geq 1$ because $\varepsilon > 0$ is arbitrary, and this proves that $\operatorname{dist}((z_0, 0), J(X)) \geq 1$.

(B) Let $g := \mathbf{1}_{I^* \setminus \mathcal{O}} \in \mathcal{B}_{ob}(I^*, \mathcal{O})$ and let $\Phi : \ell_\infty(I) \to X^{**}$ be such that $\Phi(f) = (f, +\frac{1}{3}g)$, $\forall f \in \ell_\infty(I)$. Φ is an injective affine mapping from $\ell_\infty(I)$ into X^{**}. Moreover, Φ is a continuous mapping for the $\sigma(\ell_\infty(I), \ell_1(I))$-topology of $\ell_\infty(I)$ and the w^*-topology of X^{**}. Thus $\Phi(D) = \{(d, \frac{1}{3}g) : d \in D\} =: H \subset B(X^{**})$ is a w^*-compact subset of $B(X^{**})$ homeomorphic to $\mathscr{C}$.

Claim 3. $\operatorname{dist}(H, J(X)) = \frac{1}{3}$.

Indeed, let $(d, +\frac{1}{3}g) \in H$. Then clearly $\|(d, +\frac{1}{3}g) - \frac{2}{3}J(d)\| = \|(\frac{1}{3}d, +\frac{1}{3}g - \frac{2}{3}\check{d})\| \leq \frac{1}{3}$. Thus $\operatorname{dist}(H, J(X)) \leq \frac{1}{3}$. On the other hand, given $\tau \in \mathscr{C}$, let $\psi(\tau) =: d_\tau \in D$ and $\operatorname{supp}(d_\tau) = \{i \in I : d_\tau(i) = 1\} =: A_\tau$, which is an infinite subset. We claim that $\operatorname{dist}((d_\tau, +\frac{1}{3}g), J(X)) \geq \frac{1}{3}$. Indeed, otherwise there would exist $f \in X$ such that $\|(d_\tau, +\frac{1}{3}g) - J(f)\| = \|(d_\tau - f, +\frac{1}{3}g - \check{f})\| < \frac{1}{3}$. Thus $\|d_\tau - f\| < \frac{1}{3}$ in $\ell_\infty(I)$, and this implies $\frac{2}{3} < f$ on A_τ, whence $\check{f} \geq \frac{2}{3}$ on $\overline{A_\tau}^{\beta I}$. Since A_τ is infinite, $\emptyset \neq \overline{A_\tau}^{\beta I} \setminus I \subset I^*$ and every $p \in \overline{A_\tau}^{\beta I} \setminus I$ satisfies $\check{f}(p) \geq \frac{2}{3}$. Let δ_p be the Dirac probability with mass 1 on some $p \in \overline{A_\tau}^{\beta I} \setminus I$. Observe that $\delta_p \in M_R(I^*, \mathcal{O})$ since $\mathcal{O} \cap \overline{A_\tau}^{\beta I} = \emptyset$ and so $\delta_p(\frac{1}{3}g) = \frac{1}{3}$. Thus

$$\|((d_\tau, +\tfrac{1}{3}g) - (f, \check{f}))(\delta_p)\| = |(\tfrac{1}{3}g - \check{f})(\delta_p)| = \check{f}(p) - \tfrac{1}{3} \geq \tfrac{1}{3},$$

whence $\|(d_\tau, \frac{1}{3}g) - J(f)\| \geq \frac{1}{3}$, and this contradicts our hypothesis. So $\operatorname{dist}(H, J(X)) = \frac{1}{3}$.

Claim 4. $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(H), J(X)) = 1.$

Indeed, first $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(H), J(X)) \leq 1$ because $\overline{\operatorname{co}}^{w^*}(H) \subset B(X^{**})$. On the other hand, let $\varrho := \Phi(\mu)$ be the probability on $\Phi(D) = H$ image of μ under the continuous affine mapping Φ. As in Case (A) we have $r(\varrho) = (z_0, +\frac{1}{3}g)$. We claim that $\operatorname{dist}((z_0, +\frac{1}{3}g), J(X)) \geq 1$. Indeed, given $f \in X$, we have $\check{f} \upharpoonright \mathcal{O} = 0$ and $\check{z}_0 \upharpoonright \mathcal{O} = = +1$. Thus given $\varepsilon > 0$ there exists an open neighborhood V of $\mathcal{O}$ in βI such that

$$\forall v \in V,\ \check{f}(v) \leq \frac{\varepsilon}{2} \text{ and } \check{z}_0(v) \geq 1 - \frac{\varepsilon}{2}.$$

In particular, $\forall v \in V \cap I$, $f(v) \leq \frac{\varepsilon}{2}$ and $z_0(v) \geq 1 - \frac{\varepsilon}{2}$, whence we get $\|z_0 - f\| \geq 1 - \varepsilon$ that is, $\|z_0 - f\| \geq 1$ because $\varepsilon > 0$ is arbitrary. Thus $\|(z_0, +\frac{1}{3}g) - (f, \check{f})\| \geq 1$, and this proves that $\operatorname{dist}((z_0, +\frac{1}{3}g), J(X)) \geq 1$. □

4. Control of convex subsets in the dual X^*

Let X be a Banach space, C a convex subset of X^* and W a w^*-compact subset of X^*. We study in this Section the problem of the control of the distance $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(W), C)$ by the distance $\operatorname{dist}(W, C)$. First, we have the following result of Haydon.

Proposition 4.1. [20] *Let X be a Banach space. The following statements are equivalent:*

(1) X fails to have a copy of ℓ_1.

(2) For every w^*-compact subset $K \subset X^*$ we have

$$\overline{\operatorname{co}}^{w^*}(K) = \overline{\operatorname{co}}(K) = \overline{\operatorname{co}}(Ext(K)).$$

(3) Every convex subset $C \subset X^*$ has 1-control inside X^*.

(4) Every convex subset $C \subset X^*$ has control inside X^*.

An elementary result is the following proposition.

Proposition 4.2. *Let C be a w^*-closed convex subset of the dual Banach space X^*. Then for every subset W of X^* we have $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(W), C) = \operatorname{dist}(W, C)$.*

Proof. Clearly, the statement holds true when $\operatorname{dist}(W, C) = +\infty$. Assume that $\operatorname{dist}(W, C) = a < +\infty$. Since C is w^*-closed, this implies that $W \subset C + aB(X^*)$. As $C + aB(X^*)$ is convex and w^*-closed, we get $\overline{\operatorname{co}}^{w^*}(W) \subset C + aB(X^*)$, which implies $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(W), C) \leq a$ and completes the proof. □

Now we prove the following proposition, that supplies a useful criterion for the 3-control.

Proposition 4.3. *Let X be a Banach space.*

(1) If C is a convex subset of X^* that fails to have a w^*-$\mathbb{N}$-family (in particular, if C fails to have a copy of the basis of $\ell_1(\mathfrak{c})$), then C has 3-control inside X^*, that

is, *for every* w^**-compact subset* K *of* X^* *we have* $dist(\overline{\mathrm{co}}^{w^*}(K),(C) \leq 3dist(K,C)$.

(2) *If* K *is a* w^**-compact subset of* X^* *such that* K *fails to have a* w^*-$\mathbb{N}$*-family (in particular, if* K *fails to have a copy of the basis of* $\ell_1(\mathfrak{c})$*), then* $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.

In order to prove Proposition 4.3 we need to define the notion of w^*-$\mathbb{N}$*-farnily* (see [17, Definition 3.3], [19, Definition 2.1]) and prove the Lemma 4.5.

Definition 4.4. *Let* X *be a Banach space. A subset* $\mathcal{F}$ *of* X^* *is said to be a* w^*-$\mathbb{N}$*-family of width* $d > 0$ *if* $\mathcal{F}$ *is bounded and has the form*

$$\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\},$$

and there exist two sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ *and* $\{x_m : m \geq 1\} \subset B(X)$ *such that for every pair of disjoint subsets* M, N *of* $\mathbb{N}$ *we have*

$$\eta_{M,N}(x_m) \geq r_m + d,\ \forall m \in M, \text{ and } \eta_{M,N}(x_n) \leq r_n,\ \forall n \in N.$$

Moreover, if $r_m = r_0$, $\forall m \geq 1$, *we say that* $\mathcal{F}$ *is a uniform* w^*-$\mathbb{N}$*-family in* X^*.

Remarks. (1) If Z is a set, a family $(A_i, B_i)_{i\in I}$ of pairs of nonempty subsets of Z is said to be an *independent family* if $A_i \cap B_i = \emptyset$, $\forall i \in I$, and for every finite nonempty subset $F \subset I$ we have $\bigcap_{i\in F}\varepsilon_i A_i \neq \emptyset$, where $\varepsilon_i = \pm 1$, $(+1)A_i = A_i$ and $(-1)A_i = B_i$. In $\mathbb{N}$ there exists an independent family $(M_i, N_i)_{i<\mathfrak{c}}$ with cardinal $\mathfrak{c}$. Indeed, since $\beta\mathbb{N}$ is a Hausdorf compact space extremally disconnected with weight $\mathrm{w}(\beta\mathbb{N}) = \mathfrak{c}$ (see [30, p. 76]), by the Balcar-Franěk Theorem (see [2], [9, p. 120]) there exists a continuous onto mapping $f : \beta\mathbb{N} \to \{0,1\}^{\mathfrak{c}}$. Let $\pi_i : \{0,1\}^{\mathfrak{c}} \to \{0,1\}$, $i < \mathfrak{c}$, be the projection onto the i-factor $\{0,1\}$ and put $M_i := (\pi_i \circ f)^{-1}(1) \cap \mathbb{N}$ and $N_i := (\pi_i \circ f)^{-1}(0) \cap \mathbb{N}$. Clearly, $\{(M_i, N_i) : i < \mathfrak{c}\}$ is an independent family in $\mathbb{N}$.

(2) If $(M_i, N_i)_{i<\mathfrak{c}}$ is an independent family in $\mathbb{N}$ with cardinal $\mathfrak{c}$ and $\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in the dual Banach space X^* associated with the sequence $\{x_m : m \geq 1\} \subset B(X)$, then a standard argument (see [8, p. 206]) proves that the family $\{\eta_{M_i,N_i} : i < \mathfrak{c}\}$ is equivalent to the basis of $\ell_1(\mathfrak{c})$. Moreover, the same argument yields that the sequence $\{x_n : n \geq 1\} \subset B(X)$ associated to $\mathcal{F}$ is equivalent to the basis of ℓ_1. So, if a subset $\mathcal{F}$ of a dual Banach space X^* is a w^*-$\mathbb{N}$-family, then X has an isomorphic copy of ℓ_1 and some subset of $\mathcal{F}$ is equivalent to the canonical basis of $\ell_1(\mathfrak{c})$. And vice versa, if X has a copy of ℓ_1, it is easy to see that X^* contains a w^*-$\mathbb{N}$-family associated with the basis of $\ell_1(\mathfrak{c})$.

Lemma 4.5. *Let* X *be a Banach space and* K *a* w^**-compact subset of* X^* *such that* $dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d > 0$. *Then there exist* $r_0 \in \mathbb{R}$, $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ *and* $\psi \in S(X^{**})$ *with* $\psi(z_0) > r_0 + d$ *and* $\psi(k) < r_0$, $\forall k \in K$, *and such that, if* μ *is a Radon probability on* K *with barycenter* $r(\mu) = z_0$ *and* $H = supp(\mu)$ *is the support of* μ, *then: (i) for every* w^**-open subset* $V \subset X^*$ *with* $V \cap H \neq \emptyset$, *there* *exist* $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$ *such that* $\psi(\xi) > r_0 + d$; *(ii) there exist a sequence* $\{x_n : n \geq 1\} \subset B(X)$ *and, for every pair of disjoint subsets* M, N *of* $\mathbb{N}$, *a point* $\eta_{M,N} \in H$ *such that*

$$\eta_{M,N}(x_m) \geq r_0 + d, \ \forall m \in M, \ \text{and} \ \eta_{M,N}(x_n) \leq r_0, \ \forall n \in N.$$

Proof. Find $\varepsilon > 0$ such that $\text{dist}(\overline{\text{co}}^{w^*}(K), \overline{\text{co}}(K)) > d + \varepsilon > 0 = \text{dist}(K, \overline{\text{co}}(K))$. By Lemma 2.3 there exist $z_0 \in \overline{\text{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ such that $\inf \psi(z_0 - - \overline{\text{co}}(K)) > d + \varepsilon$, that is

$$\psi(z_0) > \sup \psi(\overline{\text{co}}(\text{K})) + d + \varepsilon \geq \sup \psi(K) + \varepsilon + d.$$

So, if $r_0 := \sup \psi(K) + \varepsilon$, then $\psi(z_0) > r_0 + d$ and $\psi(k) < r_0$, $\forall k \in K$. Let μ be a Radon Borel probability on K with barycenter $r(\mu) = z_0$ and let $H := \text{supp}(\mu)$ be the support of μ.

Claim. For every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exist $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$ and $\eta \in \text{co}(V \cap H) \subset \overline{\text{co}}^{w^*}(V \cap H)$ such that $\psi(\xi) > r_0 + d$ and $\psi(\eta) < r_0$.

Indeed, by Lemma 2.3 there exists $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$ such that $\inf \psi(\xi - \overline{\text{co}}(K)) > d + \varepsilon$, that is, $\psi(\xi) > r_0 + d$. On the other hand, as $\psi(k) < r_0$, $\forall k \in K$, then $\psi(\eta) < r_0$ for every $\eta \in \text{co}(V \cap H)$. Thus, by the Claim and the proof of [20, 2. Lemma] we can find a sequence $\{x_n : n \geq 1\} \subset S(X)$ such that, if we define

$$A_n = \{\xi \in H : \xi(x_n) > r_0 + d\} \ \text{and} \ B_n = \{\eta \in H : \eta(x_n) < r_0\}, \ \forall n \geq 1,$$

then, for every pair of disjoint finite subsets M, N of $\mathbb{N}$, the w^*-open subset $V(M,N) := (\bigcap_{m \in M} A_m) \cap (\bigcap_{n \in N} B_n)$ of H is nonempty. So for every pair of disjoint finite subsets M, N of $\mathbb{N}$

$$\emptyset \neq V(M,N) \subset \Big(\bigcap_{m \in M} \overline{A_m}^{w^*}\Big) \cap \Big(\bigcap_{n \in N} \overline{B_n}^{w^*}\Big) \subset H.$$

Since H is a w^*-compact subset, we conclude that for every pair of disjoint (finite or infinite) subsets M, N *of* $\mathbb{N}$ then

$$\emptyset \neq \Big(\bigcap_{m \in M} \overline{A_m}^{w^*}\Big) \cap \Big(\bigcap_{n \in N} \overline{B_n}^{w^*}\Big) \subset H.$$

Since $\overline{A_m}^{w^*} \subset \{\xi \in H : \xi(x_m) \geq r_0 + d\}$ and $\overline{B_n}^{w^*} \subset \{\eta \in H : \eta(x_n) \leq r_0\}$, finally we deduce that for every pair of disjoint (finite or infinite) subsets M, N of $\mathbb{N}$ there exists $\eta_{M,N} \in H$ such that

$$\eta_{M,N}(x_m) \geq r_0 + d, \ \forall m \in M, \ \text{and} \ \eta_{M,N}(x_n) \leq r_0, \ \forall n \in N. \qquad \square$$

Proof of Proposition 4.3. (1) Suppose that C fails to have 3-control inside X^*. Then there exist a w^*-compact subset K of X^* and two real numbers $a, b > 0$ such that $\operatorname{dist}(\overline{\mathrm{co}}^{w^*}(K), C) > b > 3a > 3\operatorname{dist}(K, C)$. So, as $\operatorname{dist}(\overline{\mathrm{co}}(K, C) = \operatorname{dist}(K, C) < a$, then $\operatorname{dist}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > b - a > 0$. By Lemma 4.5 there exist a real number $r_0 \in \mathbb{R}$, a sequence $\{x_n : n \geq 1\} \subset B(X)$ and, for every pair of disjoint subsets M, N of $\mathbb{N}$, a vector $\eta_{M,N} \in K$ such that

$$\eta_{M,N}(x_m) \geq r_0 + b - a,\ \forall m \in M, \text{ and } \eta_{M,N}(x_n) \leq r_0,\ \forall n \in N.$$

As $\operatorname{dist}(K, C) < a$, for each pair of disjoint subsets M, N of $\mathbb{N}$ there is $z_{M,N} \in C$ so that $\|z_{M,N} - \eta_{M,N}\| < a$. Thus, the family $\{z_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ is bounded and satisfies

$$z_{M,N}(x_m) \geq r_0 + b - 2a,\ \forall m \in M, \text{ and } z_{M,N}(x_n) \leq r_0 + a,\ \forall n \in \mathbb{N}.$$

Since $r_0 + b - 2a = r_0 + a + (b - 3a) > r_0 + a$, then the set $\{z_{M,N} : M, N$ disjoint subsets of $\mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in C, a contradiction.

(2) Otherwise, there exists $d > 0$ such that $\operatorname{dist}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d > 0$. By Lemma 4.5 there exist a sequence $\{x_n : n \geq 1\} \subset B(X)$, a real number $r_0 \in \mathbb{R}$ and, for every pair of disjoint subsets M, N of $\mathbb{N}$, a vector $\eta_{M,N} \in K$ such that

$$\eta_{M,N}(x_m) \geq r_0 + d,\ \forall m \in M, \text{ and } \eta_{M,N}(x_n) \leq r_0,\ \forall n \in \mathbb{N}.$$

Thus there exists in K a w^*-$\mathbb{N}$-family, a contradiction. □

The following result is due to M. Talagrand [29, Theorem 4].

Proposition 4.6. *Let X be a Banach space and A a subset of X. If τ is a cardinal with cofinality $cf(\tau) > \aleph_0$, we have that A contains a copy of the basis of $\ell_1(\tau)$ if and only if $\overline{[A]}$ has a copy of $\ell_1(\tau)$.*

This result of Talagrand allows us to prove the following corollaries.

Corollary 4.7. *Let X be a Banach space and A a subset of X^* that fails to have a copy of the basis of $\ell_1(\mathfrak{c})$. Then:*

(1) For every w^*-compact subset $K \subset \overline{[A]}$ we have $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.

(2) Every convex subset $C \subset \overline{[A]}$ has 3-control inside X^*.

Proof. First, observe that $\overline{[A]}$ fails to have a copy of the basis of $\ell_1(\mathfrak{c})$ by the above result of Talagrand and by the fact that $\mathrm{cf}(\mathfrak{c}) > \aleph_0$. Now it is enough to apply Proposition 4.3. □

Corollary 4.8. *Let X be a Banach space and let W be a subset of X^* which is either weakly Lindelöf or is closed, convex and has the property (C) of Corson. Then*

(i) Every convex subset C of $\overline{[W]}$ has 3-control inside X^*, and

(ii) For every w^*-compact subset K of $\overline{[W]}$ we have $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$.

Proof. In both cases W cannot have a copy of the basis of $\ell_1(\mathfrak{c})$ and so (i) and (ii) follow from Corollary 4.7. □

Now we consider the control of convex subsets $C \subset X^*$ such that $C \subset Y \subset X^*$ and Y is a closed subspace of X^* with w^*-angelic closed dual unit ball. If Y is a Banach space, the closed dual unit ball $B(Y^*)$ is said to be *w^*-angelic* if given a subset A of $B(Y^*)$ and $a \in \overline{A}^{w^*}$, there exists a sequence $\{a_n : n \geq 1\} \subset A$ such that $a_n \xrightarrow{w^*} a$. A subset B of a w^*-compact subset K of X^* is said to be a *boundary* if every $x \in X$ attains on B its maximum on K; and $B \subset K$ is said to be *a strong boundary* if B is a boundary and $\overline{\text{co}}^{w^*}(K) = \overline{\text{co}}(B)$.

Proposition 4.9. *Let X be a Banach space and Y a closed subspace of X^* with w^*-angelic closed dual unit ball $(B(Y^*), w^*)$. If C is a convex subset of Y, then $dist(\overline{\text{co}}^{w^*}(K), C) = dist(B, C)$ for every w^*-compact subset K of X^* and every boundary $B \subset K$. Moreover, $\overline{\text{co}}^{w^*}(K) = \overline{\text{co}}(B)$ for every w^*-compact subset K of X^* such that Y contains some boundary B of K.*

Proof. Let C be a convex subset of Y and suppose that there exist a w^*-compact subset K of X^*, a boundary $B \subset K$ and two real numbers $0 < a, b < 1$ such that

$$\text{dist}(\overline{\text{co}}^{w^*}(K), C) > b > a > \text{dist}(B, C) = \text{dist}(\overline{\text{co}}(B), C).$$

Let $w_0 \in \overline{\text{co}}^{w^*}(K)$ and $\varepsilon > 0$ be such that $\text{dist}(w_0, C) > b + \varepsilon$. By Lemma 2.1 there exists $\varphi_0 \in S(X^{**})$ such that $\inf \varphi_0(w_0 - C) > b + \varepsilon$, that is, $\varphi_0(w_0) > \sup \varphi_0(C) + b + \varepsilon$. Denote

$$U := \{\varphi \in B(X^{**}) : \langle \varphi, w_0 \rangle \geq \langle \varphi_0, w_0 \rangle - \varepsilon\} \text{ and}$$
$$V := \{x \in B(X) : \langle w_0, x \rangle \geq \langle \varphi_0, w_0 \rangle - \varepsilon\}.$$

Observe that $\varphi_0 \in U$ and also $U = \overline{V}^{w^*}$. If $i : Y \to X^*$ is the canonical inclusion, then $i^* : X^{**} \to Y^*$ satisfies $i^*(\varphi_0) \in i^*(U) = \overline{i^*(V)}^{w^*} \subset B(Y^*)$. Since $(B(Y^*), w^*)$ is angelic, there exists a sequence $\{x_n : n \geq 1\} \subset V$ such that $i^*(x_n) \xrightarrow{w^*} i^*(\varphi_0)$ in the w^*-topology $\sigma(Y^*, Y)$. Thus, for every $y \in Y$ we have $y(x_n) = i^*(x_n)(y) \to i^*(\varphi_0)(y) = \varphi_0(y)$.

Claim. For every $\beta \in B$,

$$\limsup_{n\to\infty} x_n(\beta) \leq \sup \varphi_0(C) + a < \varphi_0(w_0) - \varepsilon + (a - b).$$

Indeed, as $\text{dist}(B, C) < a$, there exists $y \in C \subset Y$ such that $\|\beta - y\| < a$. Thus

$$\limsup_{n\to\infty} x_n(\beta) = \limsup_{n\to\infty} [x_n(y) + x_n(\beta - y)] =$$
$$= \varphi_0(y) + \limsup_{n\to\infty} x_n(\beta - y) \leq \sup \varphi_0(C) + a.$$

Finally, as $b + \varepsilon + \sup \varphi_0(C) < \varphi_0(w_0)$, we get $\sup \varphi_0(C) + a < \varphi_0(w_0) - \varepsilon + (a - b)$.

By Simons inequality [28, 2. Lemma] we have:

$$\sup_{\beta\in B} [\limsup_{n\to\infty} x_n(\beta)] \geq \inf \Big[\sup_{k\in\overline{\text{co}}^{w^*}(K)} g(k) : g \in co((x_n)_{n>1}) \Big].$$

Thus there exists $g \in co((x_n)_n) \subset V$ such that

$$\sup_{k \in \overline{co}^{w^*}(K)} g(k) < \varphi_0(w_0) - \varepsilon + (a - b).$$

On the other hand, as $g \in V$ and $w_0 \in \overline{co}^{w^*}(K)$, we have $\varphi_0(w_0) - \varepsilon \leq \sup_{k \in \overline{co}^{w^*}(K)} g(k)$, whence we get $\varphi_0(w_0) - \varepsilon < \varphi_0(w_0) - \varepsilon + (a - b)$, a contradiction because $0 < b - a$.

Finally, suppose that Y contains some boundary B of a w^*-compact subset K of X^*. Let $C := \overline{co}(B) \subset Y$. By the above results dist $(\overline{co}^{w^*}(K), C) = \text{dist}(B, C) = 0$. Thus $\overline{co}^{w^*}(K) = \overline{co}(B) = \overline{co}(K)$. □

5. Universally Krein-Šmulian Banach spaces

In this Section we deal with the class $\mathscr{F}$ of Banach spaces that fail to have a copy of $\ell_1(\mathfrak{c})$. Let us introduce our terminology. If Y is a Banach space we adopt the following definitions:

(1) Let Z be a subspace of Y^* and let $\sigma(Y, Z)$ denote the topology of Y of pointwise convergence on Z. Then $(Y, \sigma(Y, Z))$ is said to satisfy the *Krein-Šmulian Theorem* if and only if $\overline{co}^{\sigma(Y,Z)}(K)$ is $\sigma(Y, Z)$-compact whenever K is a norm-bounded $\sigma(Y, Z)$-compact subset of Y. If, moreover, $\overline{co}^{\sigma(Y,Z)}(K) = \overline{co}(K)$, then $(Y, \sigma(Y, Z))$ is said to satisfy the *strong Krein-Šmulian Theorem.*

(2) Y is said to be *universally Krein-Šmulian* if $(Y, \sigma(Y, Z))$ satisfies the Krein-Šmulian Theorem for every norming subspace Z of Y^*. If $(Y, \sigma(Y, Z))$ satisfies the strong Krein-Šmulian Theorem for every norming subspace Z of Y^*, then Y is said to be *strongly universally Krein-Šmulian.*

The following elementary proposition gives some equivalences for the just defined notions.

Proposition 5.1. *If Y is a Banach space, then:*

(a) Y is universally Krein-Šmulian if and only if, for every Banach space X and every subspace Z of X^* isomorphic to Y, the space (Z, w^*) satisfies the Krein-Šmulian Theorem.

(b) Y is strongly universally Krein-Šmulian if and only if, for every Banach space X and every subspace Z of X^* isomorphic to Y, the space (Z, w^*) satisfies the strong Krein-Šmulian Theorem.

Proof. (a) Assume that Y is universally Krein-Šmulian. Let X be a Banach space, $i : Y \to X^*$ be an isomorphic embedding and $i(Y) =: Z \subset X^*$ be the isomorphic copy of Y into X^*. So, $i^*(X) \subset Y^*$ is a subspace of Y^* norming on Y such that (Z, w^*) and $(Y, \sigma(Y, i^*(X)))$ are isomorphic. Thus (Z, w^*) satisfies the Krein-Šmulian Theorem because $(Y, a(Y, i^*(X)))$ does.

To prove the converse implication, let V be a subspace of Y^* norming on Y. Then there exists an isomorphic embedding $i: Y \to V^*$ so that $(i(Y), \sigma(V^*, V))$ and $(Y, \sigma(Y, V))$ are isomorphic. By hypothesis $(i(Y), \sigma(V^*, \overline{V}))$ satisfies the Krein-Šmulian Theorem. Since the topologies $\sigma(V^*, \overline{V})$ and $\sigma(V^*, V)$ coincide on bounded subsets of V^*, we conclude that $(Y, \sigma(Y, V))$ satisfies the Krein-Šmulian Theorem.

(b) This proof is analogous to the one of (a). □

(3) A subspace Z of a dual Banach space X^* is said to have *M-control inside* X^*, for some constant $1 \leq M < \infty$, if $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), Z) \leq M \operatorname{dist}(K, Z)$ for every w^*-compact subset K of X^*. A subspace Z of X^* is said to have *control inside* X^* if Z has M-control inside X^*, for some $1 \leq M < \infty$. Clearly, if a closed subspace Z of X^* has control inside X^*, then (Z, w^*) satisfies the Krein-Šmulian Theorem.

(4) Y is said to have *universal M-control*, for some constant $1 \leq M < \infty$, if for every Banach space X and every subspace Z of X^* isomorphic to Y, Z has M-control inside X^*. Y is said to have *universal control* if for every Banach space X and every subspace Z of X^* isomorphic to Y, Z has control inside X^*.

In this Section we show that the class of universally Krein-Šmulian Banach spaces, the class of strongly universally Krein-Šmulian Banach spaces, the class of Banach spaces that have universal control and the class of Banach spaces that have universal 3-control coincide with the class $\mathscr{F}$ of Banach spaces that do not contain a copy of $\ell_1(\mathfrak{c})$. The class $\mathscr{F}$ is very large. It contains, for instance, the class of Banach spaces X with w^*-angelic closed dual unit ball $B(X^*)$, the class of Banach spaces with the property (C) of Corson, etc. This class $\mathscr{F}$ has been studied by many authors: by Talagrand, by Cascales, Manjabacas, Vera and Shvydkoy, etc. In [5], [6] it is proved that, if a Banach space Y belongs to the class $\mathscr{F}$, then Y is strongly universally Krein-Šmulian.

We start with the connection between the class $\mathscr{F}$ and the properties universal 3-control and strongly universally Krein-Šmulian.

Proposition 5.2. *If Y is a Banach space that fails to have a copy of $\ell_1(\mathfrak{c})$, then Y has universal 3-control and is strongly universally Krein-Šmulian.*

Proof. This follows from Proposition 4.3. □

For the particular class of Banach spaces Y with w^*-angelic closed dual unit ball $B(Y^*)$, we obtain the following stronger result.

Proposition 5.3. *If Y is a Banach space with w^*-angelic closed dual unit ball $B(Y^*)$, then Y has universal 1-control and is strongly universally Krein-Šmulian.*

Proof. Y has universal 3-control and is strongly universally Krein-Šmulian by Proposition 5.2, because a Banach space Y fails to have a copy of $\ell_1(\mathfrak{c})$ whenever $(B(Y^*), w^*)$ is angelic. Moreover, Y has universal 1-control by Proposition 4.9. $\square$

The following result is a converse of Proposition 5.2.

Proposition 5.4. *If X is a universally Krein-Šmulian Banach space, then X does not contain a copy of $\ell_1(\mathfrak{c})$.*

In order to prove this result we need the following elementary lemma.

Lemma 5.5. *$\ell_1(\mathfrak{c})$ is not universally Krein-Šmulian.*

Proof. Consider the Banach space $C([0,1])$ whose dual $C([0,1])^*$ is the Banach space $M_R([0,1])$ of Borel Radon measures on the compact space [0,1]. It is well known that there exists in $(B(M_R([0,1])), w^*)$ a canonical homeomorphic copy K of the compact space [0, 1]. In fact, $K = \{\delta_t : t \in [0,1]\}$, where δ_t is the measure on $[0,1]$ such that $\delta_t(f) = f(t)$ for all $f \in C([0,1])$. Let $\phi : \ell_1([0,1]) \to$ $\to M_R([0,1])$ be the natural isometry given by $\phi((\lambda_t)_{t\in[0,1]}) = \sum_{t\in[0,1]} \lambda_t \delta_t$ for every $(\lambda_t)_{t\in[0,1]} \in \ell_1([0,1])$. Observe that $Z := \phi(\ell_1([0,1]))$ is actually the subspace of purely atomic measures on [0,1]. Clearly, $K \subset B(Z)$ and $\overline{\text{co}}^{w^*}(K)$ is the subset $\mathscr{P}_1([0,1])$ of $M_R([0,1])$ consisting of the Borel Radon probabilities on [0,1], which satisfies $\mathscr{P}_1([0,1]) \setminus Z \neq \emptyset$. So, $\ell_1(\mathfrak{c})$ is not universally Krein-Šmulian. $\square$

Proof of Proposition 5.4. We suppose that X is a Banach space containing a subspace Y isomorphic to $\ell_1([0,1])$ and we shall prove that X is not universally Krein-Šmulian. Let $T : \ell_1([0,1]) \to X$ be an isomorphism into X such that $T(\ell_1([0,1])) = Y$. The space $C([0,1])$, considered as a subspace of $\ell_\infty([0,1]) = \ell_1([0,1])^*$ (that is, $C([0,1]) = \{f \in \ell_\infty([0,1]) : f$ continuous on $[0,1]\}$), is 1-norming on $\ell_1([0,1])$. Let E_1 be the subspace of X^* defined by $E_1 := T^{*-1}(C([0,1]))$. It is easy to see that E_1 is λ_0-norming on Y, for some $0 < \lambda_0 \leq 1$ depending on T (in fact, $\lambda_0 = \|T^{-1}\|^{-1} \cdot \|T\|^{-1}$ holds). Moreover, if τ is the $\sigma(\ell_1([0,1]), C([0,1]))$-topology of $\ell_1([0,1])$, then $T : (\ell_1([0,1]), \tau) \to$ $\to (Y, \sigma(Y, E_1))$ is an isomorphism.

Let $E_2 = Y^\perp = \{z \in X^* : z(y) = 0, \forall y \in Y\} \subset X^*$ and $E = E_1 + E_2$.

Claim 1. E is $\frac{\lambda_0}{3}$-norming on X.

Indeed, pick $u \in S(X)$.

(a) Suppose that $\text{dist}(u, Y) < \frac{\lambda_0}{3}$ and let $y_0 \in Y$ be such that $\|u - y_0\| < \frac{\lambda_0}{3}$. Then $\|y_0\| > 1 - \frac{\lambda_0}{3} \geq \frac{2}{3}$. Since E_1 is λ_0-norming on Y, we can find an element $e_1 \in S(E_1)$ such that $e_1(y_0) > \frac{2}{3}\lambda_0$, whence we get $e_1(u) > \frac{1}{3}\lambda_0$.

(b) Suppose that $\text{dist}(u, Y) \geq \frac{\lambda_0}{3}$. Then

$$\sup\{e(u) : e \in B(E)\} \geq \sup\{e(u) : e \in B(E_2)\} =$$

$$= \sup\{z(u) : z \in B(Y^\perp)\} = \text{dist}(u, Y) \geq \frac{\lambda_0}{3}.$$

Therefore, E is $\frac{\lambda_0}{3}$-norming on X.

Claim 2. Y is $\sigma(X,E)$-closed in $(X,\sigma(X,E))$ and $\sigma(X,E) \upharpoonright Y = \sigma(Y,E_1)$.

Indeed, Y is $\sigma(X,E)$-closed in $(X,\sigma(X,E))$ because $Y = \bigcap_{e\in E_2} Ker(e)$ and $\sigma(X,E) \upharpoonright Y = \sigma(Y,E_1)$ because $E = E_1 + E_2$ and $E_2 = Y^{\perp}$.

By Lemma 5.5 there exists a subset $K \subset B(\ell_1([0,1]))$ such that K is τ-compact but $\overline{\text{co}}^{\tau}(K)$ is not τ-compact in $(\ell_1([0,1]),\tau)$. Let $H := T(K) \subset Y$. By Claim 2, H is a norm-bounded $\sigma(X,E)$-compact subset of Y. Moreover, by Claim 2, $\overline{\text{co}}^{\sigma(X,E)}(H) = \overline{\text{co}}^{\sigma(Y,E_1)}(H) \subset Y$ and, so, $\overline{\text{co}}^{\sigma(X,E)}(H)$ is not $\sigma(X,E)$-compact because it is homeomorphic to $\overline{\text{co}}^{\tau}(K)$, which is not τ-compact. Thus X is not universally Krein-Šmulian. □

Combining all the above results we obtain the following proposition.

Proposition 5.6. *For a Banach space Y the following statements are equivalent:*

(0) Y is universally Krein-Šmulian.

(0′) If X is a Banach space and Z a subspace of X^* isomorphic to Y, (Z,w^*) satisfies the Krein-Šmulian Theorem.

(1) Y is strongly universally Krein-Šmulian.

(1′) If X is a Banach space and Z a subspace of X^* isomorphic to Y, (Z,w^*) satisfies the strong Krein-Šmulian Theorem.

(2) Y has universal 3-control, that is, for every Banach space X and every subspace Z of X^* isomorphic to Y we have $dist(\overline{\text{co}}^{w^*}(K),Z) \leq 3dist(K,Z)$ for every w^*-compact subset K of X^*.

(3) Y has universal control, that is, if X is any Banach space and Z is a subspace of X^* isomorphic to Y, there exists a constant $1 \leq M < \infty$ such that $dist(\overline{\text{co}}^{w^*}(K),Z) \leq M\, dist(K,Z)$ for every w^*-compact subset K of X^*.

(4) Y fails to have a copy of $\ell_1(\mathfrak{c})$.

Proof. By Proposition 5.1 we have $(0) \Leftrightarrow (0')$ and $(1) \Leftrightarrow (1')$. Clearly, $(1) \Rightarrow (0)$ and $(2) \Rightarrow (3) \Rightarrow (0')$. From Proposition 5.2 we get $(4) \Rightarrow (1) + (2)$. Finally, $(0) \Rightarrow (4)$ by Proposition 5.4.

6. Convex w^*-closures vs convex $\|\cdot\|$-closures

A subset Y of a dual Banach space X^* is said to *have the property* (P) if $\overline{\text{co}}^{w^*}(H) = \overline{\text{co}}(H)$ for every w^*-compact subset H of Y, that is, every w^*-compact subset $H \subset Y$ is a strong boundary. The purpose of this section is to give an inner characterization of the property (P) for subsets of the dual Banach space X^*.

Haydon [20] characterized the property (P) for a whole dual Banach space X^* as follows: X^* has the property (P) if and only if X fails to have a copy of ℓ_1 if and only if every $z \in X^{**}$ is universally measurable on (X^*,w^*).

The fragmentability is a useful notion related with the property (P). Namioka proved that a subset $Y \subset X^*$ has the property (P) whenever (Y, w^*) is norm-fragmented ([24, 2.3. Theorem]). So, norm-fragmentability implies the property (P). The converse is not true. Indeed, let X be the James Tree space JT (see [21]), which is a non-Asplund separable Banach space without a copy of ℓ_1. So, JT^* has the property (P) by a result of Haydon [20], but the closed unit ball $B(JT^*)$ of JT^* is not norm-fragmentable, because the norm-fragmentability of $B(X^*)$ is equivalent to the asplundness of X (see [24, 1.3. Theorem]).

Let (X, T) be a Hausdorff topological space, Y a subset of X and μ a finite positive Borel Radon measure on X.

- $\mathscr{B}_0(X)$ will denote the σ-algebra of Borel subsets of X.
- The positive Radon measure μ *is carried by* Y if there exist a sequence of compact subsets $\{K_n : n \geq 1\}$ of Y such that $K_n \subset K_{n+1}$ and $\mu(K_n) \uparrow \mu(X)$.
- Y is said to be a *universally measurable* subset of X if Y is μ-measurable for every finite positive Borel Radon measure μ on X.
- A mapping $f : X \to \mathbb{R}$ is said to be *μ-measurable* if $f^{-1}(G)$ is μ-measurable for all open subset G of $\mathbb{R}$.
- If (Z, T) is another topological space, a mapping $f : X \to Z$ is said to be *Lusin μ-measurable* if for each $\varepsilon > 0$ there exists a compact subset K of X such that $\mu(X \backslash K) \leq \varepsilon$ and $f \upharpoonright K$ is continuous. Recall that by Lusin's Theorem a mapping $f : X \to \mathbb{R}$ is μ-measurable if and only f is Lusin μ-measurable.
- A mapping $f : X \to Z$ is said to be *universally measurable on* Y if and only if f is Lusin μ-measurable for every positive finite Radon Borel measure μ carried by Y, which is equivalent to say that, for every w^*-compact subset $K \subset Y$ and for every Radon Borel probability μ on K, f is Lusin μ-measurable.

In the following Proposition 6.3 we characterize the property (P) for an arbitrary subset Y of a dual Banach space X^* by means of w^*-$\mathbb{N}$-families (see Definition 4.4) and Cantor skeletons. Let us give the definition of a Cantor skeleton.

Definition 6.1. *A subset $\mathscr{A}$ of a dual Banach space X^* is said to be a Cantor skeleton of width $\delta > 0$ if $\mathscr{A}$ is a bounded set of the form $\mathscr{A} = \{k_\sigma : \sigma \in \mathscr{C}\}$ and there exist sequences $\{a_n : n \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$ such that, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and for every $m \geq 1$, we have $\langle k_\sigma, x_m \rangle \leq a_m$, if $\sigma(m) = 0$, and $\langle k_\sigma, x_m \rangle \geq a_m + \delta$, if $\sigma(m) = 1$. Moreover, if $a_n = a$, $\forall n \geq 1$, we say that $\mathscr{A}$ is a uniform Cantor skeleton. A w^*-compact subset* K of *X^* is said to be endowed with a Cantor skeleton $\mathscr{K}$ if $\mathscr{K}$ is a Cantor skeleton and $\overline{\mathscr{K}}^{w^*} = K$.*

Remark 6.2. (0) w^*-$\mathbb{N}$-families and Cantor skeletons are almost the same thing. Actually, if $\mathscr{F}$ is a w^*-$\mathbb{N}$-family, there exists a subset $\mathscr{K}$ of $\mathscr{F}$ which is a Cantor skeleton. And vice versa, if $\mathscr{K}$ is a Cantor skeleton, there exists a subset $\mathscr{F}$ of

$\mathscr{K}$ which is a w^*-$\mathbb{N}$-family. Indeed, suppose that $\mathscr{F} := \{\eta_{M,N}$ disjoint subsets of $\mathbb{N}\}$ is a w-$\mathbb{N}$-family in X^* such that

$$\eta_{M,N}(x_m) \geq r_m + \delta, \ \forall m \in M, \text{ and } \eta_{M,N}(x_n) \leq r_n, \ \forall_n \in \mathbb{N}.$$

For each $\sigma \in \{0,1\}^{\mathbb{N}}$, let $M := \{n \in \mathbb{N} : \sigma(n) = 1\}$ and $N := \mathbb{N} \setminus M$, and define $h_\sigma := \eta_{M,N}$. Then, it is easy to see that $\mathscr{K} := \{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of width δ in X^*. Of course, $\mathscr{K}$ is uniform if $\mathscr{F}$ is uniform. The converse is also true: if $\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of width $\delta > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$ for each pair of disjoint subset M, N of $\mathbb{N}$ choose $\sigma_{M,N} \in \mathscr{C}$ such that $\sigma_{M,N}(m) = 1$, $\forall m \in M$ and $\sigma_{M,N} = 0$, $\forall n \in N$. So, if for each pair of disjoint subset M, N of $\mathbb{N}$ we define $\eta_{M,N} = k_{\sigma_{M,N}}$, then $\{\eta_{M,N} : M, N$ disjoint subsets of $\mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in X^*.

(1) Let K be a w^*-compact subset endowed with a Cantor skeleton $\mathscr{A} = \{k_\sigma : \sigma \in \mathscr{C}\}$ of width $\delta > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$. Then we have:

(11) For every $k \in K$ and every $m \geq 1$ either $\langle k, x_m \rangle \leq a_m$ or $\langle k, x_m \rangle \geq a_m + \delta$. Moreover, if we define the mapping $\Phi : K \to \mathscr{C} = \{0,1\}^{\mathbb{N}}$ as

$$\forall k \in K, \ \forall m \geq 1, \ \Phi(k)(m) = \begin{cases} 1, & \text{if } \langle k, x_m \rangle \geq a_m + \delta, \\ 0, & \text{if } \langle k, x_m \rangle \leq a_m, \end{cases}$$

we have that Φ is a continuous mapping that satisfies $\Phi(K) = \mathscr{C}$.

(12) In general, K may not be homeomorphic to $\mathscr{C}$, even K may not contain a subspace homeomorphic to $\mathscr{C}$. Indeed, pick the compact space $\beta\mathbb{N}$ considered homeomorphically embedded into $(B(C(\beta\mathbb{N})^*), w^*)$. It is clear that $\overline{\mathrm{co}}(\beta\mathbb{N}) \subsetneqq \overline{\mathrm{co}}^{w^*}(\beta\mathbb{N})$ because $\overline{\mathrm{co}}(\beta\mathbb{N})$ is the set of purely atomic probabilities on $\beta\mathbb{N}$ and $\overline{\mathrm{co}}^{w^*}(\beta\mathbb{N})$ is the set of all Radon probabilities on $\beta\mathbb{N}$. This fact implies (by the next Proposition 6.3) that there exists a w^*-compact subset K of $\beta\mathbb{N}$ endowed with a uniform Cantor skeleton with respect to $C(\beta\mathbb{N})^*$. However, K cannot contain a homeomorphic copy of $\mathscr{C}$ because $\beta\mathbb{N}$ fails to contain non-trivial convergent sequences.

(13) For every $0 < \eta < \delta$ there exist an infinite subset $\mathbb{N}_\eta \subset \mathbb{N}$, a real number b_η, and a subset $\mathscr{A}_\eta \subset \mathscr{A}$ such that $\mathscr{A}_\eta$ is a uniform Cantor skeleton of width η associated to the number b_η and the sequence $\{x_m : m \in \mathbb{N}_\eta\} \subset B(X)$. Indeed, since the family $\{a_n : n \geq 1\} \subset \mathbb{R}$ is bounded, there exists $b_\eta \in \mathbb{R}$ such that $\mathbb{N}_\eta := \{m \in \mathbb{N} : b_\eta + \eta - \delta \leq a_m \leq b_\eta\}$ is infinite. Let $\pi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}_\eta}$ be the canonical projection and for each $\tau \in \{0,1\}^{\mathbb{N}_\eta}$ choose $\sigma(\tau) \in \pi^{-1}(\tau)$. Define $h_\tau := k_{\sigma(\tau)}$ for each $\tau \in \{0,1\}^{\mathbb{N}_\eta}$. Then it is easy to see that $\mathscr{A}_\eta := \{h_\tau : \tau \in \{0,1\}^{\mathbb{N}_\eta}\}$ is a uniform skeleton of width $\eta > 0$ associated with $b_\eta \in \mathbb{R}$ and the sequence $\{x_m : m \in \mathbb{N}_\eta\} \subset B(X)$.

Proposition 6.3. *Let X be a Banach space and Y a subset of X^*. The following statements are equivalent:*

(1) Y does not have the property (P).

(2) *There exist a* w^**-compact subset* H *of* Y *and two real numbers* $a < b$ *such that for every finite family* $\mathscr{F}$ *of* w^**-open subsets of* X^* *with* $V \cap H \neq \emptyset$, $\forall V \in \mathscr{F}$, *there exists* $x_{\mathscr{F}} \in B(X)$ *fulfilling that*

$$\inf \langle V \cap H, x_F \rangle < a < b < \sup < V \cap H, x_{\mathscr{F}} \rangle, \ \forall V \in \mathscr{F} .$$

(3) *There exists a* w^**-compact subset* K *of* Y *endowed with a uniform Cantor skeleton.*

(4) *There exists a functional* $\psi \in X^{**}$ *which is not universally measurable on* Y.

(5) *There exists a* w^**-compact subset* H of Y *which is uniformly non fragmentable, that is, there exists* $\delta > 0$ *such that for every finite family* $\mathscr{F}$ *of* w^**-open subsets of* X^* *with* $V \cap H \neq \emptyset$, $\forall V \in \mathscr{F}$, *there exist* $x_{\mathscr{F}} \in B(X)$ *and* $r_{\mathscr{F}} \in \mathbb{R}$ *such that*

$$\inf \langle V \cap H, x_{\mathscr{F}} \rangle < r_{\mathscr{F}} < r_{\mathscr{F}} + \delta < \sup \langle V \cap H, x_{\mathscr{F}} \rangle, \ \forall V \in \mathscr{F} .$$

(6) *There exists a* w^**-compact subset* H *of* Y *that contains a* w^**-*$\mathbb{N}$*-family.*

Proof. (1) $\Rightarrow$ (2). Since Y does not have the property (P), there exists a w^*-compact subset $K \subset Y$ such that $\operatorname{dist}(\overline{\text{co}}^{w^*}(K), \overline{\text{co}}(K)) > d + \varepsilon > 0 = \operatorname{dist}(K, \overline{\text{co}}\,(K))$ for some $d, \varepsilon > 0$. By Lemma 2.3 there exist $z_0 \in \overline{\text{co}}^{w^*}(K)$ and $\psi \in S(X^{**})$ such that $\inf \psi(z_0 - \overline{\text{co}}(K)) > d + \varepsilon$. Thus

$$\psi(z_0) > \sup \psi(\overline{\text{co}}(K)) + d + \varepsilon \geq \sup \psi(K) + \varepsilon + d .$$

Moreover, there exists a nonempty w^*-compact subset $H \subset K$ such that for every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$ with $\inf \psi(\xi - \overline{\text{co}}(K)) > d + \varepsilon$. Thus $\psi(\xi) > \sup \psi(K) + d + \varepsilon$. So, if we put $r_0 := \sup \psi(K) + \varepsilon$, then $\psi(\xi) > r_0 + d$ and $\psi(k) < r_0$, $\forall k \in K$. Therefore, if $\mathscr{F}$ is a finite family of w^*-open subsets of X^* such that $V \cap H \neq \emptyset$, $\forall V \in \mathscr{F}$, there exist $k_V \in V \cap H$ and $\xi_V \in \overline{\text{co}}^{w^*}(V \cap H)$ so that $\psi(k_V) < r_0$ and $\psi(\xi_V) > r_0 + d$ for every $V \in \mathscr{F}$. Thus, as $B(X)$ is w^*-dense in $B(X^{**})$, we can find a vector $x_{\mathscr{F}} \in B(X)$ such that

$$\inf \langle V \cap H, x_{\mathscr{F}} \rangle < r_0 < r_0 + d < \sup \langle \overline{\text{co}}^{w^*}(V \cap H), x_{\mathscr{F}} \rangle, \ \forall V \in \mathscr{F} .$$

Since $x_{\mathscr{F}} \in X$, then $\sup \langle \overline{\text{co}}^{w^*}(V \cap H), x_{\mathscr{F}} \rangle = \sup \langle V \cap H, x_{\mathscr{F}} \rangle$ and so (2) holds with $a := r_0$ and $b := r_0 + d$.

(2) $\Rightarrow$ (3). Let H be a w^*-compact subset of Y fulfilling (2). First, we construct an independent sequence $\{(A_m, B_m) : m \geq 1\}$ of subsets of H.

Step 1. By (2) there exists $x_1 \in B(X)$ such that

$$\inf \langle H, x_1 \rangle < a < b < \sup \langle H, x_1 \rangle .$$

Define $V_{11} = \{h \in X^* : \langle h, x_1 \rangle < a\}$ and $V_{12} = \{h \in X^* : \langle h, x_1 \rangle > b\}$. Observe that $V_{1i} \cap H \neq \emptyset$, $i = 1, 2$.

Step 2. By (2) there exists $x_2 \in B(X)$ such that

$$\inf \langle V_{1i} \cap H, x_2 \rangle < a < b < \sup \langle V_{1i} \cap H, x_2 \rangle, \; i = 1,2.$$

Let $V_{21} = \{h \in X^* : \langle h, x_2 \rangle < a\}$ and $V_{22} = \{h \in X^* : \langle h, x_2 \rangle > b\}$. Observe that $V_{1i} \cap V_{2j} \cap H \neq \emptyset$, $i,j = 1,2$.

Further, we proceed by iteration. We obtain a sequence $\{V_{n1}, V_{n2} : n \geq 1\}$ of w^*-open subsets of X^* such that $V_{1i_1} \cap \ldots \cap V_{ni_n} \cap H \neq \emptyset$, $i_j \in \{1,2\}, n \geq 1$. Thus, if we define

$$A_m = \{h \in H : \langle h, x_m \rangle \geq b\} \text{ and } B_m = \{h \in H : \langle h, x_m \rangle \leq a\}, \; m \geq 1,$$

then it is easy to verify that $\{(A_m, B_m) : m \geq 1\}$ is an independent sequence of w^*-closed subsets of H. Now, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and each $n \in \mathbb{N}$, let $C_{(\sigma,n)} = A_n$, if $\sigma(n) = 1$, and $C_{(\sigma,n)} = B_n$, if $\sigma(n) = 0$. By compactness, it is clear that $\bigcap_{n \geq 1} C_{(\sigma,n)} \neq \emptyset$, $\forall \sigma \in \{0,1\}^{\mathbb{N}}$. So, we can choose $h_\sigma \in \bigcap_{n \geq 1} C_{(\sigma,n)}$, $\forall \sigma \in \{0,1\}^{\mathbb{N}}$. Let $K := \overline{\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*}$. It is easy to see that $\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a uniform Cantor skeleton of K of width $b - a$.

<u>(3) ⇒ (4).</u> Let K be a w^*-compact subset of Y endowed with a uniform Cantor skeleton $\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}$ of width $\delta > 0$ associated with the number $r_0 \in \mathbb{R}$ and the sequence $\{x_m : m \geq 1\} \subset B(X)$. So, $K = \overline{\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*}$. Let $T : \ell_1 \to X$ be the continuous operator such that $T(e_n) = x_n$, $\forall n \geq 1$, $\{e_n : n \geq 1\}$ being the canonical basis of ℓ_1. So, its adjoint $T^* : X^* \to \ell_\infty$ fulfills $T^*(x^*) = (x^*(x_m))_m$, $\forall x^* \in X^*$. Define the mapping $\Phi : \ell_\infty \to \ell_\infty$ as follows

$$\forall (a_n)_n \in \ell_\infty, \; \Phi((a_n)_n) = \frac{1}{\sigma}(((a_n - r_0) \vee 0) \wedge \delta)_n.$$

The mapping Φ is w^*-w^*-continuous and satisfies $\Phi \circ T^*(K) = \{0,1\}^{\mathbb{N}} = \mathscr{C}$. Let λ be the Haar probability on $\mathscr{C}$ and μ a Radon probability on K such that $\Phi \circ T^*(\mu) = \lambda$, that is, λ is the image of μ under the w^*-w^*-continuous mapping $\Phi \circ T^*$. By a well known Sierpinski's argument ([27], [26, 14.5.1]), for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$ the point mass $\delta_p \in S(\ell_\infty^*)$ is not λ-measurable. By [25, Theorem 9, p. 35] the mapping $\delta_p \circ \Phi \circ T^* : K \to \mathbb{R}$ is not μ-measurable on K, which actually means that $\{x^* \in K : \delta_p \circ \Phi \circ T^*(x^*) \geq 1\}$ is not μ-measurable (because for every $c \in \mathscr{C}$ either $\delta_p(c) = 1$ or $\delta_p(c) = 0$). As

$$\{x^* \in K : \delta_p \circ \Phi \circ T^*(x^*) \geq 1\} = \{x^* \in K : \delta_p \circ T^*(x^*) \geq r_0 + \delta\},$$

we conclude that $\delta_p \circ T^* \in X^{**}$ is not μ-measurable. So, $\delta_p \circ T^* \in X^{**}$ is a functional which is not universally measurable on Y.

<u>(4) ⇒ (5).</u> Let K be a w^*-compact subset of Y and μ a Radon Borel probability on K such that there exists a functional $\psi \in X^{**}$ which fails to be μ-measurable on K. For every subset $A \subset K$ we define the "inner measure $\mu_*(A)$" as follows

$$\mu_*(A) = \sup \{\mu(L) : L \text{ a } w^*\text{-Borel subset of } K \text{ with } L \subset A\}.$$

It is easy to see that: (i) μ_* is monotone and $0 < \mu_*(A) \leq 1$, $\forall A \subset K$; (ii) if $A \subset K$, there exists a Borel subset $L \subset A$ such that $\mu(L) = \mu_*(A)$; (iii) if $\{A_n : n \geq 1\}$ is a sequence of subsets of K with $A_{n+1} \subset A_n$, then $\mu_*(\cap_{n\geq 1} A_n) = \inf_{n\geq 1}\mu_*(A_n)$; (iv) a subset $A \subset K$ is not μ-measurable if and only if $\mu_*(A) + \mu_*(K \setminus A) < 1$. For every $r \in \mathbb{R}$ we define

$$A_r = \{\xi \in K : \psi(\xi) > r) \text{ and } B_r = \{\xi \in K : \psi(\xi) < r\}.$$

Since ψ fails to be μ-measurable, there exists $r_0 \in \mathbb{R}$ such that A_{r_0} is not μ-measurable, that is, $\mu_*(A_{r_0}) + \mu_*(K \setminus A_{r_0}) < 1$. As $K \setminus A_{r_0} = \bigcap_{n\geq 1} B_{r_0+\frac{1}{n}}$, we get $\mu_*(K \setminus A_{r_0}) = \inf_{n\geq 1}\mu_*(B_{r_0+\frac{1}{n}})$ and so there is some $\delta_0 > 0$ such that $\mu_*(A_{r_0}) + \mu_*(B_{r_0+\delta_0}) < 1$.

Claim. There exists a nonempty w^*-compact subset $H \subset K$ such that, if V is a w^*-open subset of X^* with $V \cap H \neq \emptyset$, then $V \cap H$ intersects simultaneously $K \setminus A_{r_0}$ and $K \setminus B_{r_0+\delta_0}$.

Indeed, let $L \subset A_{r_0}$ and $M \subset B_{r_0+\delta_0}$ be Borel subsets such that $\mu(L) = \mu_*(A_{r_0})$ and $\mu(M) = \mu_*(B_{r_0+\delta_0})$. Clearly, $\mu(L \cup M) \leq \mu(L) + \mu(M) = \mu_*(A_{r_0}) + \mu_*(B_{r_0+\delta_0}) < 1$, whence $\mu(K \setminus (L \cup M)) > 0$. Let $H \subset K \setminus (L \cup M)$ be any w^*-compact subset such that, if $\nu := \mu \restriction H$, then $\nu > 0$ and $\operatorname{supp}(\nu) = H$. Let V be a w^*-open subset with $V \cap H \neq \emptyset$. Then $\mu(V \cap H) > 0$. Assume that $V \cap H \subset A_{r_0}$. Put $L' = L \cup (V \cap H)$. Clearly, $\mu_*(A_{r_0}) \geq \mu(L') = \mu(L) + \mu(V \cap H) > \mu_*(A_{r_0})$, a contradiction that proves that $(K \setminus A_{r_0}) \cap (V \cap H) \neq \emptyset$. In a similar way one can prove that $(K \setminus B_{r_0+\delta_0}) \cap (V \cap H) \neq \emptyset$.

Let $\varepsilon > 0$ be such that $r_0 + \varepsilon < r_0 + \delta_0 - \varepsilon$ and define $r_1 := r_0 + \varepsilon$ and $\delta := \delta_0 - 2\varepsilon$. Then $\delta > 0$. By the Claim, if $\mathscr{F}$ is a finite family of w^*-open subsets of X^* such that $V \cap H \neq \emptyset$, $\forall V \in \mathscr{F}$, for each $V \in \mathscr{F}$ we can find vectors $\xi_V, \eta_V \in V \cap H$ if so that

$$\psi(\eta_V) < r_1 < r_1 + \delta < \psi(\xi_V).$$

Since $B(X)$ is w^*-dense in $B(X^{**})$, we can find a vector $x_{\mathscr{F}} \in B(X)$ such that

$$\langle \eta_V, x_{\mathscr{F}} \rangle < r_1 < r_1 + \delta < \langle \xi_V, x_{\mathscr{F}} \rangle, \ \forall V \in \mathscr{F}.$$

<u>(5) ⇒ (6).</u> Let H be a w^*-compact subset of Y, which is uniformly non fragmentable for some $\delta > 0$. By using an argument similar to the one of the implication <u>(2) ⇒ (3)</u>, we find two sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$ such that, if

$$A_m = \{h \in H : \langle h, x_m \rangle \geq r_m + \delta\}$$

and

$$B_m = \{h \in H : \langle h, x_m \rangle \leq r_m\}, \ m \geq 1,$$

then $\{(A_m, B_m) : m \geq 1\}$ is an independent sequence of w^*-closed subsets of H. By an argument of compactness, for each pair of disjoint subsets M, N of $\mathbb{N}$ we have $(\bigcap_{m\in M} A_m) \cap (\bigcap_{n\in N} B_n) \neq \emptyset$. So, we can choose $\eta_{M,N} \in (\bigcap_{m\in M} A_m) \cap$ $\cap (\bigcap_{n\in N} B_n)$. Clearly, $\{\eta_{M,N} : M, N$ disjoint subsets of $\mathbb{N}\}$ is a w^*-$\mathbb{N}$-family in H such that

$$\eta_{M,N}(x_m) \geq r_m + \delta, \ \forall m \in M, \text{ and } \eta_{M,N}(x_n) \leq r_n, \ \forall n \in N.$$

In order to prove the implication <u>(6) ⇒ (1)</u> we use the following lemmas.

Lemma 6.4. *Let $\mathscr{C} := \{0,1\}^{\mathbb{N}}$ be the Cantor compact set considered as a subset of the compact space $(B(\ell_\infty(\mathbb{N})), w^*)$. There exists a w^*-compact subset $D \subset \mathscr{C}$, homeomorphic to $\mathscr{C}$, such that $\overline{\mathrm{co}}(D) \subsetneqq \overline{\mathrm{co}}^{w^*}(D)$. Actually, there exists $z_0 \in \overline{\mathrm{co}}^{w^*}(D)$ such that $dist(z_0, \overline{\mathrm{co}}(D)) = 1 = dist(\overline{\mathrm{co}}^{w^*}(D), \overline{\mathrm{co}}(D))$.*

Proof. Let us recall the notation introduced in the proof of Proposition 3.1: $\mathscr{C} = \{0,1\}^{\mathbb{N}}$, $\mathscr{S} := \{0,1\}^{<\mathbb{N}} = \{0,1\} \cup \{0,1\}^2 \cup \{0,1\}^3 \cup \ldots$, the Haar probability λ on $\{0,1\}^{\mathbb{N}}$, $I_n := \{f_A : A \subset \{0,1\}^n \text{ with } |A| = 2^n - n\}$, $I := \bigcup_{n\geq 1} I_n$, $\mathscr{O}(\sigma) =$ $= \{f_A \in I : f_A(\sigma) = 0\}$, $\mathscr{O} := \bigcap_{\sigma\in\mathscr{C}} \overline{\mathscr{O}(\sigma)}^{\beta I}$, the mapping $\psi : \mathscr{C} \to \{0,1\}^I \subset B(\ell_\infty(I))$, $D := \psi(\mathscr{C}) \subset \{0,1\}^I$, $\mu := \psi(\lambda)$, $r(\mu) =: z_0 \in \overline{\mathrm{co}}^{w^*}(D)$, etc. Recall that $\check{z}_0(p) = 1$ for every $p \in I^* := \beta I \setminus I$.

Take $p \in \mathscr{O}$ and let $\delta_p \in \ell_\infty(I)^*$ be such that $\delta_p(f) = \check{f}(p)$, $\forall f \in \ell_\infty(I)$. Clearly, $\delta_p(z_0) = \check{z}_0(p) = +1$, but $\delta_p(d) = \check{d}(p) = 0$, $\forall d \in D$. Thus $1 \leq \mathrm{dist}(z_0, \overline{\mathrm{co}}(D)) \leq$ $\leq \mathrm{dist}(\overline{\mathrm{co}}^{w^*}(D), \overline{\mathrm{co}}(D))$. As $\overline{\mathrm{co}}^{w^*}(D) \subset [0,1]^I$ and $\mathrm{diam}([0,1]^I) \leq 1$, finally we get $1 = \mathrm{dist}(z_0, \overline{\mathrm{co}}(D)) = \mathrm{dist}(\overline{\mathrm{co}}^{w^*}(D), \overline{\mathrm{co}}(D))$. □

Lemma 6.5. *Let K be a w^*-compact subset of a dual Banach space X^* such that K contains a Cantor skeleton of width $\delta > 0$. Then there exists a w^*-compact subset H of K such that $dist(\overline{\mathrm{co}}^{w^*}(H), \overline{\mathrm{co}}(H)) \geq \delta$.*

Proof. Let $\mathscr{A} := \{k_\sigma : \sigma \in \mathscr{C}\}$ be a Cantor skeleton of width $\delta > 0$ inside K. Without loss of generality, we suppose that $K = \overline{\mathscr{A}}^{w^*}$.

(A) First, we assume that K is a w^*-compact subset of ℓ_∞ and $\mathscr{A}$ a uniform Cantor skeleton of width $\delta = 1$ of K so that, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and for every $m \geq 1$, we have $\pi_m(k_\sigma) \leq 0$, if $\sigma(m) = 0$, *and* $\pi_m(k_\sigma) \geq 1$, if $\sigma(m) = 1$. Consider the continuous mapping $\Phi : K \to \mathscr{C}$ such that, $\forall k \in K$, $\Phi(k)(m) = 1$, if $k_m \geq 1$, and $\Phi(k)(m) = 0$, if $k_m \leq 0$. Clearly, $\Phi(K) = \mathscr{C}$. By the proof of Lemma 6.4 there exist a w^*-compact subset $D \subset \mathscr{C} \subset \ell_\infty(I)$, a Radon probability μ on D so that $\mu = \psi\lambda$, λ being Haar probability on $\mathscr{C}$, such that, if $z_0 = r(\mu)$ is the barycenter of μ, then $\mathrm{dist}(z_0, \overline{\mathrm{co}}(D)) = 1$. Let

$$D_m^1 = \{d \in D : \pi_m(d) = 1\} \text{ and } D_m^0 = \{d \in D : \pi_m(d) = 0\},\ m \geq 1.$$

By the proof of Proposition 3.1 we have $\mu(D_m^1) \to 1$ and so $\mu(D_m^0) = \mu(D \setminus D_m^1) \to 0$ for $m \to \infty$.

Claim. If $\Phi^{-1}(D) =: H \subset K$, then there exists $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $d(u_0, \overline{\mathrm{co}}(H)) \geq 1$.

Indeed, since $\Phi(H) = D$ and Φ is w^*-w^*-continuous, there exists a Radon Borel probability ν on H such that $\Phi\nu = \mu$. Let $u_0 := r(\nu)$ be the barycenter of ν, that satisfies $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$.

Sub-Claim. Given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\pi_m(u_0) \geq 1 - \varepsilon$, $\forall m \geq n_\varepsilon$.

Indeed, observe that $\pi_m(u_0) = \pi_m(r(\nu)) = \int_H \pi_m(h)\,d\nu(h)$, $\forall m \geq 1$. Let $0 \leq M < \infty$ be such that $\|h\| \leq M$, $\forall h \in H$, and choose $\eta > 0$ with $\varepsilon \geq \eta(1 + M)$. Now we choose $n_\varepsilon \in \mathbb{N}$ such that $\mu(D_m^1) \geq 1 - \eta$, $\forall m \geq n_\varepsilon$, (and $\mu(D_m^0) \leq \eta$). Then for $m \geq n_\varepsilon$ we have

$$\int_H \pi_m(h)\,d\nu(h) = \int_{\Phi^{-1}(D_m^1)} \pi_m(h)\,d\nu(h) + \int_{\Phi^{-1}(D_m^0)} \pi_m(h)\,d\nu(h) \geq$$
$$\geq \int_{\Phi^{-1}(D_m^1)} 1\,d\nu(h) + \int_{\Phi^{-1}(D_m^0)} (-M)\,d\nu(h) = \nu(\Phi^{-1}(D_m^1)) - M\nu(\Phi^{-1}(D_m^0)) =$$
$$= \mu(D_m^1) - M\mu(D_m^0) \geq 1 - \eta - M\eta \geq 1 - \varepsilon.$$

In order to show that $d(u_0, \overline{\mathrm{co}}(H)) \geq 1$, it is sufficient to show that $\|u_0 - p\| \geq 1$ for each $p \in \mathrm{co}(H)$. Let $p = \sum_{j=1}^k t_j h_j$, where $t_j \in [0,1]$, $\sum_{j=1}^k t_j = 1$, $h_j \in H$ and $\Phi(h_j) =: d_j \in D$ for each j. By (3) of the proof of Proposition 3.1 there exists a sequence of integers $m_1 < m_2 < \ldots$ such that $\pi_{m_r}(d_j) = 0$ for $r \geq 1$ and $j = 1,\ldots, k$. So, by the definition of Φ we have $\pi_{m_r}(h_j) \leq 0$ for $r \geq 1$ and $j = 1,\ldots k$, that is, $\pi_{m_r}(p) \leq 0$ for $r \geq 1$. Thus from the Sub-Claim we obtain $\|u_0 - p\| \geq 1$. So, this proves the Claim and completes the proof of the statement in this case (A).

(B) Now, we suppose that K is a w^*-compact subset of ℓ_∞-endowed with a Cantor skeleton $\mathscr{A} := \{k_\sigma : \sigma \in \mathscr{C}\}$ of width $\delta > 0$ associated with the numbers $(a_n)_{n\geq 1} \in \ell_\infty$ and the sequence of canonical projections $\{\pi_m : m \geq 1\}$, where $\pi_m(k) = k_m$, $\forall k \in \ell_\infty$. Let $T: \ell_\infty \to \ell_\infty$ be the mapping such that $T(x)(n) = (x_n - a_n)/\delta$, $\forall n \in \mathbb{N}$. Then T is an affine mapping which is w^*-w^*-continuous and $\|\cdot\|$-continuous. If $L = T(K)$, then L *is a* w^*-compact subset endowed with a uniform Cantor skeleton $T(\mathscr{A})$, which satisfies the requirements of case (A). So, there exists a w^*-compact subset $W \subset L$ and a point $w_0 \in \overline{\mathrm{co}}^{w^*}(W)$ such that $\mathrm{dist}(w_0, \overline{\mathrm{co}}(W)) \geq 1$. Let $H := T^{-1}(W)$. Clearly, H is a w^*-compact subset of K such that $T(H) = W$, $T(\overline{\mathrm{co}}(H)) \subset \overline{\mathrm{co}}(W)$ and $T(\overline{\mathrm{co}}^{w^*}(H)) = \overline{\mathrm{co}}^{w^*}(W)$. Thus, if $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ satisfies $T(u_0) = w_0$, then $\mathrm{dist}(u_0, \overline{\mathrm{co}}(H)) \geq \delta$, by the form of the mapping T.

(C) Finally, we suppose that K is a w^*-compact subset of an arbitrary dual Banach space X^* endowed with a Cantor skeleton $\mathscr{A} := \{k_\sigma : \sigma \in \mathscr{C}\}$ of width $\delta > 0$ associated with the numbers $(a_n)_{n\geq 1} \in \ell_\infty$ and the sequence $\{x_n : n \geq 1\} \subset$

$\subset B(X)$. Consider the continuous operator $T:\ell_1 \to X$ such that, $\forall (\lambda_n)_{n\geq 1} \in \ell_1$, $T((\lambda_n)_{n\geq 1}) = \sum_{n\geq 1}\lambda_n x_n \in X$. Observe that $\|T\| \leq 1$. Then, $T^*(K)$ is a w^*-compact subset of ℓ_∞ and $\{T^*(k_\sigma) : \sigma \in \mathscr{C}\}$ is a Cantor skeleton of $T^*(K)$ of width $\delta > 0$, that satisfies the requirements of case (B). So, there exists a w^*-compact subset $W \subset T^*(K)$ and a point $w_0 \in \overline{\mathrm{co}}^{w^*}(W)$ such that $\mathrm{dist}(w_0, \overline{\mathrm{co}}(W)) \geq \delta$. Let $H := T^{*-1}(W) \cap K$. Then H is a w^*-compact subset of K such that $T^*(H) = W$ and $T^*(\overline{\mathrm{co}}^{w^*}(H)) = \overline{\mathrm{co}}^{w^*}(W)$. Let $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ be such that $T^*(u_0) = w_0$. Taking into account the fact that $\|T^*\| \leq 1$ and that $\mathrm{co}(W) \subset T^*(\overline{\mathrm{co}}(H)) \subset \overline{\mathrm{co}}(W)$, we get $\mathrm{dist}(u_0, \overline{\mathrm{co}}(H)) \geq \mathrm{dist}(T^*(u_0), \; T^*(\overline{\mathrm{co}}(H))) = \mathrm{dist}(w_0, \overline{\mathrm{co}}(W)) \geq \delta$ and this completes the proof of the Lemma. □

Proof of (6) $\Rightarrow$ (1). Let $\{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\}$ be a w^*-$\mathbb{N}$-family in some w^*-compact subset H of Y. For each $\sigma \in \{0,1\}^{\mathbb{N}}$, let $M := \{n \in \mathbb{N} : \sigma(n) = 1\}$ and $N := \mathbb{N} \setminus M$, and define $h_\sigma := \eta_{M,N}$. Then, it is easy to see that $\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of the w^*-compact subset $\overline{\{h_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*} =: K \subset H$. Now it is enough to apply Lemma 6.5. □

Remark. By Proposition 6.3, if Y is a w^*-compact subset of a dual Banach space X^*, then Y fulfills the property (P) if and only if Y does not contain a Cantor skeleton. Actually, this equivalence holds true for the class of $\mathscr{K}$-analytic subsets of (X^*, w^*) (see [19, Proposition 3.8]). On the other hand, in [17, Corollary 12] we have constructed subspaces Y (non w^*-$\mathscr{K}$-analytic) of a dual Banach space X^* that simultaneously have the property (P) but Y fails to have 3-control inside X^*. Thus, Y contains a w^*-$\mathbb{N}$-family and so a Cantor skeleton by Proposition 4.3.

7. The control for 1-unconditional direct sums and Banach lattices

In order to find classes of Banach spaces with a control in the bidual better than in the general case, we examine in this Section the class of 1-unconditional direct sums of Banach spaces and the class of Banach lattices. First, we have the following remark: the counterexamples we have constructed in Section 2 (a Banach space X and two w^*-compact subsets $K_1, K_2 \subset B(X^{**})$ such that $\mathrm{dist}(K_1, X) = \frac{1}{3}$, $\mathrm{dist}(K_2, X) = \frac{1}{2}$ but $\mathrm{dist}(\overline{\mathrm{co}}^{w^*}(K_1), X) = 1 = \mathrm{dist}(\overline{\mathrm{co}}^{w^*}(K_2), X))$ are Banach lattices. So, concerning the control inside the bidual, the class of Banach lattices behaves as in the general case. However, as we see in the sequel, the behavior of some classes of Banach lattices (as the order-continuous Banach lattices, Banach spaces with an 1-symmetric basis, etc.) is better than in the general case. Let us begin with the definition of 1-unconditional direct sums of Banach spaces.

Definition 7.1. *A Banach space X is said to be an 1-unconditional direct sum of a family of Banach subspaces $\{X_\alpha : \alpha \in \mathscr{A}\}$ of X, for short, $X = \sum_{\alpha\in\mathscr{A}} \otimes X_\alpha$ 1-unconditional, when $X = \overline{[\bigcup_{\alpha\in\mathscr{A}} X_\alpha]}$ and, if $x_\alpha \in X_\alpha$, $\varepsilon_\alpha = \pm 1$, $\alpha \in \mathscr{A}$, and A is a finite subset of $\mathscr{A}$, then $\|\sum_{\alpha\in A} \varepsilon_\alpha x_\alpha\| \leq \|\sum_{\alpha\in A} x_\alpha\|$.*

Remarks. Let $X = \sum_{\alpha\in\mathscr{A}} \bigoplus X_\alpha$ be an 1-unconditional direct sum of Banach spaces. We have:

(1) For each subset $A \subset \mathscr{A}$ there exists a projection $P_A : X \to X$ such that $\|P_A\| = 1$ and $P_A(X) = \sum_{\alpha\in A} \bigoplus X_\alpha$.

(2) Every $x \in X$ has a unique representation of the form $x = \sum_{\alpha\in\mathscr{A}} x_\alpha$ with $x_\alpha \in X_\alpha$ such that the subset $\{\alpha \in \mathscr{A} : x_\alpha \neq 0\}$ is countable, the above series converges unconditionally and $\|\sum_{\alpha\in\mathscr{A}} \varepsilon_\alpha x_\alpha\| = \|x\|$, where $\varepsilon_\alpha = \pm 1$, $\forall \alpha \in \mathscr{A}$.

(3) If $u \in X^*$, the α-th coordinate u_α of u will be the restriction $u_\alpha := u \restriction X_\alpha \in X_\alpha^*$ of u to X_α. We will identify u with the family $(u_\alpha)_{\alpha\in\mathscr{A}}$ of its coordinates.

(4) We consider each dual X_α^* canonically and isometrically embedded into X^* as follows. If $P_\alpha : X \to X_\alpha$ is the projection associated to X_α, then $P_\alpha^*(X_\alpha^*)$ is a subspace of X^* isometric to X_α^*. We identify X_α^* with $P_\alpha^*(X_\alpha^*)$. Consider in X^* the closed subspace $Y_0 := \overline{[\bigcup_{\alpha\in\mathscr{A}} X_\alpha^*]}$, which is actually the 1-unconditional direct sum of the closed subspaces $\{X_\alpha^* : \alpha \in \mathscr{A}\}$, that is, $Y_0 = \sum_{\alpha\in\mathscr{A}} \bigoplus X_\alpha^*$ 1-unconditional. Let Y_0^* be the dual of Y_0. We have the following fact.

Fact. There exists an isometric and isomorphic embedding $h : Y_0^* \to X^{**}$ of Y_0^* into X^{**} so that $X^{**} = h(Y_0^*) \overset{m}{\bigoplus} Y_0^\perp$, that is, X^{**} is the monotone direct sum of $h(Y_0^*)$ and $Y_0^\perp$, which means that every $z \in X^{**}$ has a unique decomposition $z = z_1 + z_2$ with $z_1 \in h(Y_0^*)$ and $z_2 \in Y_0^\perp$ such that $\|z\| \geq \|z_1\| \bigvee \|z_2\|$.

Indeed, if $z \in Y_0^*$, for each $\alpha \in \mathscr{A}$ let $z_\alpha := z \restriction X_\alpha^*$ be the α-th coordinate of z and identify z with the family $(z_\alpha)_{\alpha\in\mathscr{A}}$ of its coordinates. In order to embed Y_0^* into X^{**}, define the mapping $h : Y_0^* \to X^{**}$ as follows:

$$\forall z \in Y_0^*, \quad \forall u \in X^*, \quad h(z)(u) = \sum_{\alpha\in\mathscr{A}} z_\alpha(u_\alpha).$$

It is easy to see that h is an isometric and isomorphic embedding of Y_0^* into X^{**} such that every $z \in X^{**}$ has a unique decomposition $z = z_1 + z_2$ with $z_1 \in h(Y_0^*)$, $z_2 \in Y_0^\perp$ and $\|z\| \geq \|z_1\| \bigvee \|z_2\|$.

(5) Observe that the canonical copy $J(X)$ of X in X^{**} is inside $h(Y_0^*)$ although $J(X) \neq h(Y_0^*)$ in general.

Let us investigate the control inside its bidual of a Banach space which is an 1-unconditional direct sum of *WCG* subspaces. First, we need the following lemma.

Lemma 7.2. *Let X be a Banach space and K a w-compact subset of X^*. Given $z \in B(X^{**})$ and $\varepsilon > 0$, there exists $x \in X$ such that $\|x\| \leq 1 + \varepsilon$ and*

$$\forall k \in K, \quad z(k) - \varepsilon \leq x(k) - \varepsilon \leq x(k) \leq z(k) + \varepsilon.$$

Proof. Without loss of generality, we suppose that K is convex and symmetric with respect to 0 (otherwise, pick $\overline{\mathrm{co}}(K \cup (-K))$ instead of K). Consider the

Banach space $Z = X \bigoplus_1 \mathbb{R}$. Then $Z^* = X^* \bigoplus_\infty \mathbb{R}$ and $Z^{**} = X^{**} \bigoplus_1 \mathbb{R}$. Let $H_1 := \{(k, z(k) - \frac{\varepsilon}{2}) : k \in K\}$ and $H_2 := \{(k, z(k) + \frac{\varepsilon}{2}) : k \in K\}$ be two w-compact convex disjoint subsets of Z^* such that, if $H = H_2 - H_1$, then $H \subset Z^*$ is a w-compact convex subset (and so a w^*-compact subset) of Z^* fulfilling that $H \cap \mathring{B}(0; \frac{\varepsilon}{2}) = \emptyset$. Thus, if we pick $\varrho > 0$ with $\frac{2}{2+\varepsilon} \le \varrho < 1$, then $H \cap B(0; \frac{\varrho\varepsilon}{2}) =$ $= \emptyset$. By the Hahn-Banach Theorem there exists a vector $\varphi \in B(Z)$ such that $\langle h, \varphi \rangle \ge \frac{\varrho\varepsilon}{2}$, $\forall h \in H$. If $\varphi = x_0 + t_0$, with $x_0 \in X$, $t_0 \in \mathbb{R}$ and $\|\varphi\| = \|x_0\| +$ $+ |t_0| \le 1$, then for every $(k_1, z(k_1) - \frac{\varepsilon}{2}) \in H_1$ and every $(k_2, z(k_2) + \frac{\varepsilon}{2}) \in H_2$ we have

$$\varphi\left(\left(k_2, z(k_2) + \frac{\varepsilon}{2}\right)\right) - \varphi\left(\left(k_1, z(k_1) - \tfrac{\varepsilon}{2}\right)\right) \ge \frac{\varrho\varepsilon}{2}.$$

Thus

$$x_0(k_2) + t_0 z(k_2) + t_0\frac{\varepsilon}{2} \ge x_0(k_1) + t_0 z(k_1) - t_0\frac{\varepsilon}{2} + \frac{\varrho\varepsilon}{2}, \tag{7.1}$$

whence choosing $k_1 = k_2$ in (7.1), we get $t_0\varepsilon \ge \frac{\varrho\varepsilon}{2}$, that is, $\frac{\varrho}{2} \le t_0 \le 1$. So, $\|x_0\| \le 1 - \frac{\varrho}{2}$. Putting $k_1 = 0$ in (7.1) we get

$$\forall k \in K, \quad x_0(k) + t_0 z(k) + t_0\frac{\varepsilon}{2} \ge -t_0\tfrac{\varepsilon}{2} + \frac{\varrho\varepsilon}{2}.$$

Thus

$$\forall k \in K, \quad -\frac{1}{t_0}x_0(k) \le z(k) + \frac{\varepsilon}{2}\frac{2t_0 - \varrho}{t_0} \le z(k) + \varepsilon.$$

On the other hand, putting $k_2 = 0$ in (7.1) we obtain

$$\forall k \in K, \quad \frac{t_0}{2}\varepsilon \ge x_0(k) + t_0 z(k) - t_0\frac{\varepsilon}{2} + \frac{\varrho\varepsilon}{2}.$$

Thus

$$\forall k \in K, \quad z(k) - \varepsilon \le z(k) - \frac{\varepsilon}{2}\frac{2t_0 - \varrho}{t_0} \le -\frac{1}{t_0}x_0(k).$$

Therefore, if $x = -\frac{1}{t_0}x_0$, , then x satisfies the statement of the Lemma. □

Proposition 7.3. *Let X be a Banach space, which is an 1-unconditional direct sum of a family $\{X_\alpha : \alpha \in \mathscr{A}\}$ of WCG Banach spaces, we say, $X = \sum_{\alpha\in\mathscr{A}} \bigoplus X_\alpha$. Then*

(A) X has 2-control inside its bidual X^{**}.

(B) If the spaces X_α are reflexive and $X := \sum_{\alpha\in\mathscr{A}} \bigoplus_{\ell_1} X_\alpha$ (that is, X is the direct ℓ_1-sum of the family $\{X_\alpha : \alpha \in \mathscr{A}\}$), then X has 1-control in its bidual X^{**}.

Proof. We adopt the notation of the above paragraphs. So, let $Y_0 = \sum_{\alpha\in\mathscr{A}} \bigoplus$ $\bigoplus X_\alpha^*$, $X^{**} = h(Y_0^*) \overset{m}{\bigoplus} Y_0^\perp$, etc. Observe that in the case (B) we have $Y_0 = \sum_{\alpha\in\mathscr{A}} \bigoplus$ $\bigoplus_{c_0} X_\alpha^*$, that is, Y_0 is the direct c_0-sum of the subspaces $\{X_\alpha^* : \alpha \in \mathscr{A}\}$. Let K_α be a w-compact subset of X_α such that $0 \in K_\alpha$ and $X_\alpha = \overline{[K_\alpha]}$, $\alpha \in \mathscr{A}$. In the case (B) we pick $K_\alpha := B(X_\alpha)$. Suppose that there exist a w^*-compact subset $K \subset B(X^{**})$ and some real numbers $a, b > 0$ such that

(1) $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), X) > b > 2a > 2\operatorname{dist}(K, X) > 0$ in the case (A).

(2) $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), X) > b > a > \operatorname{dist}(K, X) > 0$ in the case (B).

By Lemma 2.3 we have the following fact.

Fact. There exist $\psi \in S(X^{***})$ and $z_0 \in \overline{\operatorname{co}}^{w^*}(K)$ with $\inf \psi(z_0 - X) > b$ (and so $\psi \in S(X^{***}) \cap X^\perp$), and a w^*-compact subset $\emptyset \neq H \subset K$ such that for every w^*-open subset V of X^{**} with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\operatorname{co}}^{w^*}(V \cap H)$ such that $\langle \psi, \xi \rangle > b$.

Now we proceed step by step:

Step 1. By the Fact there exists a vector $\xi_1 \in \overline{\operatorname{co}}^{w^*}(H)$ such that $\langle \psi, \xi_1 \rangle > b$. Since $B(X^*)$ is w^*-dense in $B(X^{***})$, we can find a vector $x_1^* \in B(X^*)$ such that $\langle \xi_1, x_1^* \rangle > b$ and another vector $\eta_1 \in H$ so that $\langle \eta_1, x_1^* \rangle > b$. Let $\eta_1 = v_1 + w_1$ with $v_1 \in h(Y_0^*)$ and $w_1 \in Y_0^\perp$. Then $a > \operatorname{dist}(\eta_1, X) \geq \operatorname{dist}(\eta_1, h(Y_0^*)) = \|w_1\|$, whence

$$\langle v_1, x_1^* \rangle = \langle \eta_1, x_1^* \rangle - \langle w_1, x_1^* \rangle > b - a\,.$$

As $\langle v_1, x_1^* \rangle = \sum_{\alpha \in \mathscr{A}} v_{1\alpha}(x_{1\alpha}^*) > b - a$, we can find a finite subset $\mathscr{A}_1 \subset \mathscr{A}$ such that, if y_1 is the restriction of x_1^* to $\sum_{\alpha \in \mathscr{A}_1} \oplus X_\alpha$ (so $y_1 = \sum_{\alpha \in \mathscr{A}_1} x_{1\alpha}^* \in$ $\in B(\sum_{\alpha \in \mathscr{A}_1} \oplus X_\alpha^*) \subset B(Y_0)$), then $\langle \eta_1, y_1 \rangle = \langle v_1, y_1 \rangle > b - a$.

Step 2. Let $V_1 = \{u \in X^{**} : \langle u, y_1 \rangle > b - a\}$, which is a w^*-open subset of X^{**} with $V_1 \cap H \neq \emptyset$, because $\eta_1 \in V_1 \cap H$. By the Fact there exists $\xi_2 \in \overline{\operatorname{co}}^{w^*}(V_1 \cap H)$ with $\langle \psi, \xi_2 \rangle > b$. Let $0 < 2\varepsilon_1 < 2^{-1} \wedge (\langle \psi, \xi_2 \rangle - b) \wedge$ $\wedge\, (a(\operatorname{dist}(K, X))^{-1} - 1)$. Consider in X^{**} the subset $L_1 := \{\xi_2\} \cup (\sum_{\alpha \in \mathscr{A}_1} K_\alpha)$. Clearly L_1 is a w-compact subset of X^{**}. Moreover, in the case (B), we have $B(\sum_{\alpha \in \mathscr{A}_1} \oplus_1 X_\alpha) \subset L_1$. Now by the above Lemma 7.2 there exists a vector $x_2^* \in X^*$ such that $\|x_2^*\| \leq 1 + \varepsilon_1$ and

$$\forall k \in L_1, \quad \langle \psi, k \rangle - \varepsilon_1 < \langle k, x_2^* \rangle < \langle \psi, k \rangle + \varepsilon_1\,.$$

In particular, $\langle \xi_2, x_2^* \rangle > b + \varepsilon_1$ and $|\langle x_2^*, k \rangle| \leq \varepsilon_1 \leq 2^{-2}$, $\forall k \in \sum_{\alpha \in \mathscr{A}_1} K_\alpha$, because $\psi(k) = 0$. Since $\langle \xi_2, x_2^* \rangle > b + \varepsilon_1$, we can choose $\eta_2 \in V_1 \cap H$ such that $\langle \eta_2, x_2^* \rangle >$ $> b + \varepsilon_1$ and also $\langle \eta_2, y_1 \rangle > b - a$ because $\eta_2 \in V_1$. Let $\eta_2 = v_2 + w_2$ with $v_2 \in h(Y_0^*)$ and $w_2 \in Y_0^\perp$. Observe that $\|w_2\| = \operatorname{dist}(\eta_2, h(Y_0^*)) \leq \operatorname{dist}(\eta_2, X) \leq$ $\leq \operatorname{dist}(K, X) < a$ and $|\langle w_2, x_2^* \rangle| \leq (1 + \varepsilon_1)\operatorname{dist}(K, X) \leq a$. Now we choose y_2 and $\mathscr{A}_2$ in the cases (A) and (B) as follows:

<u>**Case A.**</u> We have

$$\langle v_2, x_2^* \rangle = \langle \eta_2, x_2^* \rangle - \langle w_2, x_2^* \rangle \geq \langle \eta_2, x_2^* \rangle - |\langle w_2, x_2^* \rangle| > b - a\,.$$

Thus, as $\langle v_2, x_2^* \rangle = \sum_{\alpha \in \mathscr{A}} \langle v_{2\alpha}, x_{2\alpha}^* \rangle > b - a$, we can find a finite subset $\mathscr{A}_2$ of $\mathscr{A}$ satisfying $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{A}$ such that, if y_2 is the restriction of x_2^* to $\sum_{\alpha \in \mathscr{A}_2} \oplus X_\alpha$

(so $y_2 = \sum_{\alpha\in\mathscr{A}_2} x^*_{2\alpha} \in \sum_{\alpha\in\mathscr{A}_2} \oplus X^*_\alpha \subset Y_0$ with $\|y_2\| \le 1 + \varepsilon_1$), then $\langle \eta_2, y_2\rangle = \langle v_2, y_2\rangle > b - a$. Observe that for every $k \in \bigcup_{\alpha\in\mathscr{A}_1} K_\alpha$ we have $\psi(k) = 0$, whence

$$|\langle y_2, k\rangle| = |\langle x^*_2, k\rangle| \le \varepsilon_1 \le 2^{-2}.$$

Case B. Let $\gamma_{21} := x^*_2 \upharpoonright \sum_{\alpha\in\mathscr{A}_1} \oplus_1 X_\alpha$ (that is, $\gamma_{21} = \sum_{\alpha\in\mathscr{A}_1} x^*_{2\alpha}$) and $\gamma_{22} = x^*_2 - \gamma_{21}$. Since $|\langle x^*_2, k\rangle| \le \varepsilon_1$, $\forall k \in \sum_{\alpha\in\mathscr{A}_1} K_\alpha$, and $B\left(\sum_{\alpha\in\mathscr{A}_1} \oplus_1 X_\alpha\right) \subset \sum_{\alpha\in\mathscr{A}_1} K_\alpha$, then $\|\gamma_{21}\| \le \varepsilon_1$. So

$$\langle v_2, \gamma_{22}\rangle = \langle \eta_2, x^*_2\rangle - \langle w_2, x^*_2\rangle - \langle v_2, \gamma_{21}\rangle \ge \langle \eta_2, x^*_2\rangle - \varepsilon_1 - a > b - a.$$

Since $\langle v_2, \gamma_{22}\rangle = \sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_1} \langle v_{2\alpha}, x^*_{2\alpha}\rangle > b - a$, we can find a finite subset $\mathscr{A}_2 \subset \mathscr{A}\setminus\mathscr{A}_1$ such that, if y_2 is the restriction of x^*_2 to $\sum_{\alpha\in\mathscr{A}_2} \oplus X_\alpha$ (so $y_2 = \sum_{\alpha\in\mathscr{A}_2} x^*_{2\alpha} \in \sum \alpha\in\mathscr{A}_2 \oplus X^*_\alpha \subset Y_0$ with $\|y_2\| \le 1 + \varepsilon_1$), then $\langle \eta_2, y_2\rangle = \langle v_2, y_2\rangle > b - a$.

Further we proceed by iteration. We obtain the sequences $\{y_k : k \ge 1\} \subset Y_0$, $\{\eta_k : k \ge 1\} \subset K$ and $\{\mathscr{A}_k : k \ge 1\}$, $\mathscr{A}_k \subset \mathscr{A}$, fulfilling the following conditions:

Case A. In this case we have:

(i) The finite subsets $\mathscr{A}_k$ *of* $\mathscr{A}$ satisfy $\mathscr{A}_k \subset \mathscr{A}_{k+1}$ for $k \ge 1$.

(ii) $y_k \in \sum_{\alpha\in\mathscr{A}_k} \oplus X^*_\alpha \subset Y_0$, $\|y_k\| \le 1 + \varepsilon_{k-1}$, $k \ge 2$, and $\langle \eta_j, y_k\rangle > b - a$ for $j \ge k$ with $j, k \in \mathbb{N}$.

(iii) For every $h \in \bigcup_{\alpha\in\mathscr{A}_k} K_\alpha$ we have $|\langle y_{k+1}, h\rangle| \le 2^{-k-1}$, $\forall k \ge 1$.

Let $\mathscr{A}_0 := \bigcup_{n\ge1} \mathscr{A}_n$, $X_0 := \sum_{\alpha\in\mathscr{A}_0} \oplus X_\alpha$ and let $P_0 : X \to X_0$ be the canonical projection on X_0, with norm $\|P_0\| = 1$. The space X admits the monotone decomposition

$$X = X_0 \overset{m}{\oplus} X_1 \quad \text{where} \quad X_1 := \sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_0} X_\alpha.$$

Therefore we get the following monotone decompositions

$$X^* = X^*_0 \overset{m}{\oplus} X^*_1,\ X^{**} = X^{**}_0 \overset{m}{\oplus} X^{**}_1\ X^{***} = X^{***}_0 \overset{m}{\oplus} X^{***}_1, \text{ etc.},$$

with projections $P_0 : X \to X_0$, $P^*_0 : X^* \to X^*_0$, $P^{**}_0 : X^{**} \to X^{**}_0$, $P^{***}_0 : X^{***} \to X^{***}_0$, etc. Observe that $P^*_0(y_k) = y_k$, $\forall k \ge 1$, that is, $y_k \in X^*_0 = P^*_0(X^*)$, $\forall k \ge 1$. Let η_0 be a w^*-cluster point of the sequence $\{\eta_k : k \ge 1\}$ in X^{**}. Obviously $\eta_0 \in K$. Moreover, since $\langle \eta_j, y_k\rangle > b - a$, $\forall j \ge k$, we get $\langle \eta_0, y_k\rangle \ge b - a$, $\forall k \ge 1$. Let φ_0 be a w^*-cluster point of $\{y_k : k \ge 1\}$ in X^{***}. Then

(i) $\varphi_0 \in B(X^{***})$. Actually $\varphi_0 \in P^{***}_0(X^{***}) = X^{***}_0$, that is, $P^{***}_0(\varphi_0) = \varphi_0$.

(ii) By construction $\varphi_0 \upharpoonright K_\alpha = 0$, $\forall \alpha \in \mathscr{A}_0$. Thus $\varphi_0 \in X^\perp_0$, because $\bigcup_{\alpha\in\mathscr{A}_0} K_\alpha$ generates X_0.

(iii) $\langle \varphi_0, \eta_0\rangle \ge b - a$ because $\langle \eta_0, y_k\rangle \ge b - a$, $\forall k \ge 1$.

Let $W := P^{**}_0(K) \subset B(X^{**}_0)$, which is a w^*-compact subset of X^{**}_0, and $w_0 = P^{**}_0(\eta_0)$. Obviously $w_0 \in W$.

Claim 1. $\operatorname{dist}(w_0, X_0) < a$.

Indeed, let $x \in X$ be arbitrary. Then

$$\operatorname{dist}(w_0, X_0) \geq \|w_0 - P_0^{**}x\| = \|P_0^{**}(\eta_0) - P_0^{**}x\| \leq \|\eta_0 - x\|.$$

That is, $\operatorname{dist}(w_0 X_0) \leq \operatorname{dist}(\eta_0, X) \leq \operatorname{dist}(K, X) < a$.

Claim 2. $\operatorname{dist}(w_0, X_0) \geq b - a$.

Indeed, as $\varphi_0 \in B(X^{***}) \cap X_0^\perp$ and

$$\langle \varphi_0, w_0 \rangle = \langle \varphi_0, P_0^{**}\eta_0 \rangle = \langle P_0^{***}\varphi_0, \eta_0 \rangle = \langle \varphi_0, \eta_0 \rangle \geq b - a,$$

we conclude that $\operatorname{dist}(w_0, X_0) \geq b - a$.

As $a < b - a$ we get a contradiction which proves the statement in the case (A).

Case B. In this case we have:

(i) The finite subsets $\mathscr{A}_k$, $k \geq 1$, of $\mathscr{A}$ are disjoint.

(ii) $y_k \in \sum_{\alpha \in \mathscr{A}_k} \oplus_0 X_\alpha^* \subset Y_0$, $\|y_k\| \leq 1 + \varepsilon_{k-1}$, $k \geq 2$, and $\langle \eta_j, y_k \rangle > b - a$ for $j \geq k$ with $j, k \in \mathbb{N}$.

(iii) For every $n \in \mathbb{N}$ we have $\|\sum_{i=1}^n y_i\| \leq 2$.

Let η_0 be a w^*-cluster point of the sequence $\{\eta_k : k \geq 1\}$ in X^{**}. Obviously $\eta_0 \in K$. Moreover, since $\langle \eta_j, y_k \rangle > b - a$, $\forall j \geq k$, we get $\langle \eta_0, y_k \rangle \geq b - a > 0$, $\forall k \geq 1$. Thus $\langle \eta_0, \sum_{i=1}^n y_i \rangle \geq n(b - a)$, $\forall n \geq 1$. Since $\|\sum_{i=1}^n y_i\| \leq 2$, $\forall n \geq 1$, we get a contradiction which proves the statement (B). $\square$

Proposition 7.4. *Let X be a Banach space, which is the 1-unconditional direct sum $X = \sum_{\alpha \in \mathscr{A}} \oplus X_\alpha$ of the family $\{X_\alpha : \alpha \in \mathscr{A}\}$ of WCG Banach spaces. If $K \subset X^{**}$ is a w^*-compact subset such that $K \cap X$ is w^*-dense in K, then $\operatorname{dist}(\overline{\operatorname{co}}^{w^*}(K), X) = \operatorname{dist}(K, X)$.*

Proof. The proof is analogous the the one of Proposition 7.3, but in this case, as $K \cap X$ is w^*-dense in K, we can choose η_{k+1} in $V_k \cap K \cap X$ with $\langle \eta_{k+1}, x_{k+1}^* \rangle > b$ so that $\eta_k = v_k$, $w_k = 0$. $\square$

Definition 7.5. *Let X be a Banach space which admits the decomposition $X = \sum_{\alpha \in \mathscr{A}} \oplus X_\alpha$ as an 1-unconditional direct sum of closed subspaces X_α. We say that the decomposition $X = \sum_{\alpha \in \mathscr{A}} \oplus X_\alpha$ is of countable type if for every $u \in X^*$ the support $\operatorname{supp}(u) := \{\alpha \in \mathscr{A} : u_\alpha \neq 0\}$ of u is countable, $(u_\alpha)_{\alpha \in \mathscr{A}}$ being the set of coordinates of u, that is, $u_\alpha := u \upharpoonright X_\alpha = u \circ P_\alpha$, where $P_\alpha : X \to X_\alpha$ is the canonical projection.*

Lemma 7.6. *Let X be a Banach space which admits a decomposition $X = \sum_{\alpha \in \mathscr{A}} \oplus X_\alpha$ as an 1-unconditional direct sum of the closed subspaces X_α. The following statement are equivalent:*

(1) The decomposition $X = \sum_{\alpha \in \mathscr{A}} \oplus X_\alpha$ is not of countable type.

(2) *X has an isomorphic copy of $\ell_1(\aleph_1)$ disjointly disposed with respect to the decomposition $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$, that is, there exists a subset $\mathscr{A}_1 \subset \mathscr{A}$ with cardinality $|\mathscr{A}_1| = \aleph_1$ and for each $\alpha \in \mathscr{A}_1$ an element $v_\alpha \in X_\alpha$ so that the family $\{v_\alpha : \alpha \in \mathscr{A}_1\}$ is equivalent to the canonical basis of $\ell_1(\aleph_1)$.*

Proof. (1) $\Rightarrow$ (2). If the decomposition $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$ is not of countable type, there exists some $u \in X^*$ such that the subset $\mathscr{A}_0 := \{\alpha\in\mathscr{A} : u_\alpha \neq 0\}$ satisfies $|\mathscr{A}_0| \geq \aleph_1$, where $u_\alpha := u \restriction X_\alpha = u \circ P_\alpha$ and $P_\alpha : X \to X_\alpha$ is the canonical projection. By passing to a subset if necessary, we can find a real number $\varepsilon > 0$, a subset $\mathscr{A}_1 \subset \mathscr{A}_0$ with $|\mathscr{A}_1| = \aleph_1$ and a family $\{v_\alpha : \alpha \in \mathscr{A}_1\}$ with $v_\alpha \in B(X_\alpha)$ so that $\langle u,v_\alpha\rangle = \langle u_\alpha, v_\alpha\rangle > \varepsilon$. This fact proves, by a standard argument, that the family $\{v_\alpha : \alpha \in \mathscr{A}_1\}$ is equivalent to the canonical basis of $\ell_1(\aleph_1)$ and generates a copy of $\ell_1(\aleph_1)$, which is disjointly disposed with respect to the decomposition $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$.

(2) $\Rightarrow$ (1). Let $\mathscr{A}_1 \subset \mathscr{A}$ be a subset with cardinality $|\mathscr{A}_1| = \aleph_1$ and for each $\alpha \in \mathscr{A}_1$ let v_α be an element of X_α so that the family $\{v_\alpha : \alpha \in \mathscr{A}_1\}$ is equivalent to the canonical basis $\{e_\alpha : \alpha \in \mathscr{A}_1\}$ of $\ell_1(\mathscr{A}_1)$. Let $T : \ell_1(\mathscr{A}_1) \to X$ be the isomorphism between $\ell_1(\mathscr{A}_1)$ and the closed subspace generated by $\{v_\alpha : \alpha \in \mathscr{A}_1\}$ so that $T(e_\alpha) = v_\alpha$. Since $T^* : X^* \to \ell_\infty(\mathscr{A}_1)$ is a quotient mapping and so $T^*(X^*) = \ell_\infty(\mathscr{A}_1)$, if $w_0 \in \ell_\infty(\mathscr{A}_1)$ is such that $w_0(\alpha) = 1$, $\forall\alpha \in \mathscr{A}_1$, there exists a vector $u \in X^*$ such that $T^*(u) = w_0$. Then for every $\alpha \in \mathscr{A}_1$ we have

$$\langle u,v_\alpha\rangle = \langle u, Te_\alpha\rangle = \langle T^*u, e_\alpha\rangle = \langle w_0, e_\alpha\rangle = 1,$$

and this proves that u is an element of X^* that does not have countable support with respect to the decomposition $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$. □

Proposition 7.7. *Let X be a Banach space that admits a decomposition of countable type $X = \sum_{i\in I} \oplus X_i$ as an 1-unconditional direct sum of WLD (weakly Lindelöf determined) closed subspaces $\{X_i : i \in I\}$. Then X is WLD and so for every convex subset $C \subset X$, every w^*-compact subset K of X^{**} and every boundary $B \subset K$ we have $dist(\overline{co}^{w^*}(K), C) = dist(\overline{co}(B), C)$.*

Proof. It is well known that the dual unit ball of a *WLD* space is w^*-angelic (see [1]). So by Proposition 4.9 it is enough to prove that X is *WLD*, that is, that for some set J there exists an injective continuous linear operator $T : X^* \to \ell^c_\infty(J) := \{f \in \ell_\infty(J) : \text{supp } (f) \text{ is countable}\}$ which is w^* to pointwise continuous (see [1, Definition 1.1]). Since each X_i is *WLD*, there exist a set J_i and an injective linear operator $T_i : X_i^* \to \ell^c_\infty(J_i)$ which is w^* to pointwise continuous and satisfies $\|T_i\| \leq 1$. We assume that the family of sets $\{J_i : i \in I\}$ is pairwise disjoint and put $J := \bigcup_{i\in I} J_i$. Define $T : X^* \to \ell_\infty(J)$ such that, if $x^* \in X^*$ and $x_i^* \in X_i^*$ is the restriction $x_i^* := x^* \restriction X_i$, then $Tx^* = (T_i(x_i^*))_{i\in I}$. Clearly T is an injective norm-continuous operator which is w^* pointwise continuous. Moreover, as the decomposition of X is of countable type, we have that supp (Tx^*) is countable for every $x^* \in X^*$ and this completes the proof. $\square$

In the sequel we apply the above results to the class of order-continuous Banach lattices. First, we see the well known fact that, if X is an order-continuous Banach lattice, then X is an 1-unconditional direct sum of disjoint closed ideals which are *WCG*.

Lemma 7.8. *Let X be an order-continuous Banach lattice with weak unit $e > 0$. Then X is WCG.*

Proof. It is well known (see [23, p. 28]) that the interval $[0,e] := \{x \in X : 0 \leq \leq x \leq e\}$ is a w-compact subset of X. Let us see that $X = \overline{[[0,e]]}$, that is, X is the closure of the space generated by $[0,e]$. Pick a positive element $x \in X^+$. Then $ne \wedge x \uparrow x$ for $n \to \infty$, whence $\|x - ne \wedge x\| \downarrow 0$ because X is order-continuous. So $\bigcup_{n\geq 1}[0,ne] = \bigcup_{n\geq 1} n[0,e]$ is dense in the positive cone X^+. As $X = X^+ - X^+$, we conclude that X is the closure of the subspace generated by $[0,e]$. $\square$

Lemma 7.9. *If X is an order-continuous Banach lattice, then X is the 1-unconditional direct sum $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$ of a family of closed ideals $\{X_\alpha : \alpha \in \mathscr{A}\}$ mutually disjoint, such that each X_α has weak unit and so it is WCG.*

Proof. By [1.a.9] of [23] X admits the expression $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$ as a direct sum of a family of closed ideals mutually disjoint $\{X_\alpha : \alpha \in \mathscr{A}\}$ (so as an 1-unconditional direct sum), such that each X_α has weak unit. By the previous Lemma 7.8 we get the statement. $\square$

Proposition 7.10. *Let X be an order-continuous Banach lattice. If K is a w^*-compact subset of X^{**}, then $dist(\overline{co}^{w^*}(K), X) \leq 2dist(K, X)$ and, if $K \cap X$ is w^*-dense in K, then $dist(\overline{co}^{w^*}(K), X) = dist(K, X)$.*

Proof. Apply Lemma 7.9, Proposition 7.3 and Proposition 7.4. $\square$

Proposition 7.11. *Let X be an order-continuous Banach lattice that does not have a copy of $\ell_1(\aleph_1)$. Then X is WLD and so for every convex subset $C \subset X$, every w^*-compact subset K of X^{**} and every boundary $B \subset K$ we have $dist(\overline{co}^{w^*}(K), C) = dist(\overline{co}(B), C)$.*

Proof. Clearly, if X is an order-continuous Banach lattice that does not have a copy of $\ell_1(\aleph_1)$, then X admits, by Lemma 7.6 and Lemma 7.9, a decomposition of countable type $X = \sum_{\alpha\in\mathscr{A}} \oplus X_\alpha$ as an 1-unconditional direct sum of *WCG* closed ideals X_α. So, this result follows from Proposition 7.7 $\square$

Proposition 7.12. *Let X be a Banach space with an 1-unconditional basis $\{e_i : i \in I\}$ equivalent to the canonical basis of $\ell_1(I)$. Then X has 1-control in its bidual X^{**}.*

Proof. The proof is analogous to the one of part (B) of Proposition 7.3, putting $X_i = [e_i]$, $i \in I$, and taking into account the fact that X^* and the subspace Y_0 of X^* are canonically isomorphic to $\ell_\infty(I)$ and $c_0(I)$, respectively. $\square$

A Banach space X has an 1-symmetric basis $\{e_i : i \in I\}$ whenever $X = \overline{[\{e_i : i \in I\}]}$ and for every countable subset $J \subset I$ the family $\{e_j : j \in J\}$ is a countable 1-symmetric basis of $\overline{[\{e_j : j \in J\}]}$ (see p. 113 of [22]) for the definition of a countable 1-symmetric basis).

Proposition 7.13. *Let X be a Banach space with an 1-symmetric basis. Then X has 1-control in its bidual X^{**}.*

Proof. **Case 1.** Let every element of the dual X^* have countable support. In this case the result follows from Proposition 7.7.

Case 2. Suppose that there exists a vector $u \in B(X^*)$ with uncountable support. By Proposition 7.12 it is enough to prove the following claim.

Claim. If there exists a vector $u \in B(X^*)$ with uncountable support, then the 1-symmetric basis $\{e_i : i \in I\}$ of X is equivalent to the canonical basis of $\ell_1(I)$.

Indeed, since $\operatorname{supp}(u) := \{i \in I : u(e_i) \neq 0\}$ is uncountable, we can find a real number $\varepsilon > 0$ and an uncountable subset $J \subset \operatorname{supp}(u)$ such that $|u(e_i)| > \varepsilon$, $\forall i \in J$. Let us prove that the family $\{e_i : i \in J\}$ is equivalent to the basis of $\ell_1(J)$. Suppose that the basis $\{e_i : i \in J\}$ is normalized and choose a vector of the form $\sum_{1 \le k \le n} \lambda_k e_{i_k}$, $i_k \in J$. Let $\varepsilon_k = \pm 1$ so that $u(\lambda_k \varepsilon_k e_{i_k}) = |(\lambda_k u(e_{i_k})| \ge \varepsilon |\lambda_k|$, $1 \le k \le n$. Then

$$\sum_{l \le k \le n} |\lambda_k| \ge \| \sum_{1 \le k \le n} \lambda_k e_{i_k} \| = \| \sum_{1 \le k \le n} \lambda_k \varepsilon_k e_{i_k} \| \ge$$

$$\ge |u(\sum_{1 \le k \le n} \lambda_k \varepsilon_k e_{i_k})| \ge \varepsilon \sum_{1 \le k \le n} |\lambda_k|,$$

and this implies that the family $\{e_i : i \in J\}$ is equivalent to the basis of $\ell_1(J)$. As the basis $\{e_i : i \in I\}$ of X is symmetric, finally we conclude that $\{e_i : i \in I\}$ is equivalent to the canonical basis of $\ell_1(I)$, and this proves the Claim and completes the proof of the Proposition. $\square$

8. The control inside $\ell_\infty(I)$

Throughout this Section H will be a Hausdorf completely regular topological space and $C_b(H)$ will denote the Banach space of continuous bounded functions $f : H \to \mathbb{R}$ with the supremum norm. We consider $C_b(H)$ as a closed subspace of $(\ell_\infty(H), \|\cdot\|_\infty)$. What is the control of $C_b(H)$ inside $(\ell_\infty(H), \|\cdot\|_\infty)$? This problem has been studied in [4] and [16]. In this Section we use the Simons inequality to extend Proposition 3.1 of [16].

If $k \in H$ let $\mathscr{V}^k$ denote the family of open neighborhoods of k in H. Now, we define the oscillation $Osc(f,k)$ of $f: H \to \mathbb{R}$ in $k \in H$ as:

$$Osc(f,k) = \lim_{V \in \mathscr{V}^k} \left(\sup \{ f(i) - f(j) : i,j \in V \} \right).$$

The oscillation of f *in* H is:

$$Osc(f) = \sup \{ Osc(f,k) : k \in H \}.$$

If H is a normal topological space and $f \in \ell_\infty(H)$, we have $\operatorname{dist}(f, C_b(H)) = \frac{1}{2} Osc(f)$ (see [3, Proposition 1.18, p. 23]). We say that a topological space H belongs to the class $\mathfrak{F}$ (for short, $H \in \mathfrak{F}$) if for every $A \subset H \times H$ and every $h \in H$, with $(h,h) \in \overline{A}$, there exist $d \in H$ and a sequence $(\alpha_n)_{n \geq 1}$ in A such that $\alpha_n \to (d,d)$ as $n \to \infty$. So, H is in $\mathfrak{F}$ provided: (1) H is metrizable; (2) H satisfies the first axiom of countability; (3) $H \times H$ is a Fréchet-Urysohn space.

Proposition 8.1. *Let H be a normal topological space with* $H \in \mathfrak{F}$, $W \subset \ell_\infty(H)$ *a w*-compact subset and* $B \subset W$ *a boundary for W. Then*

$$dist\left(\overline{\text{co}}^{w^*}(W), C_b(H)\right) = dist\left(B, C_b(H)\right).$$

Proof. Suppose that there exist a w^*-compact subset $W \subset B(\ell_\infty(H))$, a boundary $B \subset W$ and two real numbers $a, b > 0$ *such that*

$$\operatorname{dist}\left(\overline{\text{co}}^{w^*}(W), C_b(H)\right) > b > a > \operatorname{dist}\left(B, C_b(H)\right).$$

Pick $f_0 \in \overline{\text{co}}^{w^*}(W)$ with $\operatorname{dist}(f_0, C_b(H)) > b$. Then there exists a point $k_0 \in H$ such that $\frac{1}{2} Osc(f_0, k_0) > b$. So, there exist $\varepsilon > 0$ and, for every $V \in \mathscr{V}^{k_0}$, two points $i_V, j_V \in V$ such that

$$f_0(i_V) - f_0(j_V) > 2b + \varepsilon.$$

In particular, $(k_0, k_0) \in \overline{\{(i_V, j_V) : V \in \mathscr{V}^{k_0}\}}$. Since $H \in \mathfrak{F}$ there exist a sequence $\{(i_n, j_n) : n \geq 1\} \subset \{(i_V, j_V) : V \in \mathscr{V}^{k_0}\}$ and a point $h_0 \in H$ such that $(i_n, j_n) \to (h_0, h_0)$. For every $n \geq 1$ let $T_n : \ell_\infty(H) \to \mathbb{R}$ be such that $T_n(f) = f(i_n) - f(j_n)$, for all $f \in \ell_\infty(I)$. Clearly, T_n is a linear mapping which is $\|\cdot\|$-continuous weak*-continuous and $\|T_n\| \leq 2$. Moreover, we have $T_n(f_0) > 2b + \varepsilon$, $\forall n \geq 1$, and $\lim_{n\to\infty} T_n(f) = 0$ for every $f \in C_b(H)$.

Claim. For every $\beta \in B$ we have $\limsup_{n\to\infty} T_n(\beta) < 2a$.

Indeed, fix $\beta \in B$ and, as $\operatorname{dist}(B, C_b(H)) < a$, find $f \in C_b(H)$ such that $\|\beta - f\| < a$. We have

$$\begin{aligned}\limsup_{n\to\infty} T_n(\beta) &= \limsup_{n\to\infty} \left(T_n(f) + T_n(\beta - f) \right) = \\ &= \lim_{n\to\infty} T_n(f) + \limsup_{n\to\infty} T_n(\beta - f) < 2a,\end{aligned}$$

where we have applied that $\lim_{n\to\infty} T_n(f) = 0$, $\|T_n\| \leq 2$ and $\|\beta - f\| < a$.

By Simons inequality [28, 2. Lemma] we have

$$\sup_{\beta\in B}\left[\limsup_{n\to\infty} T_n(\beta)\right] \geq \inf\left[\sup_{k\in\overline{\mathrm{co}}^{w^*}(W)} g(k) : g\in co\left((T_n)_{n\geq 1}\right)\right].$$

So there exists some $g\in co\left((T_n)_{n>1}\right)$, we say, $g=\sum_{n=1}^{p}\lambda_n T_n$ with $\lambda_n\geq 0$ and $\sum_{n=1}^{p}\lambda_n=1$, such that $\sup_{k\in\overline{\mathrm{co}}^{w^*}(W)} g(k) < 2a+\varepsilon$. On the other hand, as $f_0\in\overline{\mathrm{co}}^{w^*}(W)$ and $T_n(f_0)=f_0(i_n)-f_0(j_n)>2b+\varepsilon$ we have

$$\sup_{k\in\overline{\mathrm{co}}^{w^*}(W)} g(k) \geq \sum_{n=1}^{p}\lambda_n T_n(f_0) > 2b+\varepsilon,$$

whence we get $2a+\varepsilon>2b+\varepsilon$, a contradiction, and this completes the proof. □

Corollary 8.2. *Let K be a scattered compact Hausdorff space such that $K^{(2)}=\emptyset$. Then for every w^*-compact subset $W\subset\ell_\infty(K)$ and every boundary B of W we have $dist(B,C(K))=dist(\overline{\mathrm{co}}^{w^*}(W),C(K))$.*

Proof. By Proposition 8.1 it is enough to prove that $K\in\mathfrak{F}$. As $K^{(2)}=\emptyset$, then K is the topological sum of a finite number of disjoint clopen subsets, say $K=\bigoplus_{i=1}^{n}K_i$, each K_i being the Alexandrov compactification $K_i=\alpha J_i$ of some discrete set J_i. So, K has property $\mathfrak{F}$ if and only if each αJ_i has. Now apply the trivial fact that the Alexandrov compactification αJ of a discrete set J has property $\mathfrak{F}$. □

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