COMPLEXITY OF THE METHOD OF AVERAGING*

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Abstract

The general method of averaging for the superapproximation of an arbitrary partial derivative of a smooth function in a vertex a of a simplicial triangulation $\mathcal{T}$ of a bounded polytopic domain in $\Re^d$ for any $d \geq 2$ is described and its complexity is analysed.

1 Introduction

We reserve the symbol $\mathcal{P}_d^{(m)}$ for the space of (real) polynomials in $d \geq 1$ (real) variables whose degree is less than or equal to m for any $m \geq 1$, Ω for a bounded polytopic domain of dimension $d \geq 2$ and consider meshes of Ω consisting of d-dimensional simplices. For any simplex T, we put

$$h_T = \operatorname{diam}(T) \quad \text{and} \quad \varrho_T = \sup\{\operatorname{diam}(B) \,|\, B \subset T \text{ is a sphere}\}.$$

If a is an inner vertex of a mesh $\mathcal{T}$ and $T_1, \dots, T_n$ are the $\mathcal{T}$-simplices with vertex a then we call $\Theta(a) = T_1 \cup \ldots \cup T_n$ a *neighbourhood* of a and set $h(a) = \max\{h_{T_1}, \dots, h_{T_n}\}$.

A *Lagrange finite element* $e = e_d^{(m)}$ of degree m consists of

a) the simplex $T = \overline{a^1 \ldots a^{d+1}}$,

b) the *local space* $\mathcal{L}^{(m)}$ of restrictions of the polynomials from $\mathcal{P}_d^{(m)}$ to T,

c) the "set of parameters" relating the values $p(n^{i_1 \dots i_d})$ to every $p \in \mathcal{L}^{(m)}$

$$\text{in the} \quad \begin{pmatrix} d+m \\ m \end{pmatrix} \quad \textit{nodes} \quad n^{i_1 \dots i_d} = \sum_{j=1}^{d+1} \frac{i_j}{m} a^j$$

for the non-negative integers $i_1, \dots, i_d$ and i_{d+1} such that $i_1 + \ldots + i_{d+1} = m$. (The fractions $i_1/m, \dots, i_{d+1}/m$ are the barycentric coordinates of the node $n^{i_1 \dots i_d}$ in T.)

If m is a positive integer, T a d-dimensional simplex and $u \in C(T)$ then we denote by $\mathrm{P}_{T,m}[u]$ the $\mathcal{L}^{(m)}$–interpolant of u in the nodes of $e_d^{(m)}$.

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For any integer m, multiindex ϱ with length $r = |\varrho|$ such that $m \geq r \geq 1$, function $u \in C^{m+2}(\overline{\Omega})$ and inner vertex a of a mesh $\mathcal{T}$ it is well-known that the $\mathcal{T}$-simplices $T_1, \ldots, T_n$ with vertex a satisfy

$$\frac{\partial^r (\mathrm{P}_{T_i,m}[u] - u)}{\partial x^\varrho}(a) = O\left((h_{T_i})^{m+1-r}\right).$$

The (general) method of averaging consists in the solution of the problem to construct a vector $f = [f_1, \ldots, f_n]^\top$ such that

$$\frac{\partial^r \left(f_1 \mathrm{P}_{T_1,m}[u] + \ldots + f_n \mathrm{P}_{T_n,m}[u] - u\right)}{\partial x^\varrho}(a) = O\left(h(a)^{m+2-r}\right). \tag{1}$$

The special method of averaging, related to the *special case* $d = 2, m = 1 = r$, is an old problem formulated already in [9], 1967, with the aim to get an accurate approximation of the strain tensor in the postprocessing of the elasticity problem. In many papers including [7], [10], [6], [3], various approaches to the solution of this special case are presented. They can be applied in the constructions of a posteriori error estimators of the finite element solutions of the second–order partial differential problems in the plane, see [3] and [1], in the sensitivity analysis of optimization problems and in other areas. Of course, the applicability of the solution of the general problem is essentially more extensive. A solution of an analogously general problem appeared in [8].

In Section 2, the vector f satisfying (1) is shown to be the minimal 2-norm solution of a small underdetermined system of linear equations. In Section 3, we study the way in which the complexity of these linear equations depends on the given multiindex ϱ. In the last Section 4, the general method of averaging is applied to a concrete problem and an agreement of the order of error with (1) is illustrated numerically.

2 The general method of averaging

We describe the system of linear equations for the vector f from (1) and conditions guaranteeing the order of error required in (1).

Definition 1. If m is an integer, ϱ a multiindex such that $m \geq r = |\varrho| \geq 1$, a an inner vertex of a mesh $\mathcal{T}$ and $T_1, \ldots, T_n$ are the $\mathcal{T}$-simplices with vertex a then $\mathcal{F}_{m,\varrho}(a)$ denotes the set of vectors $f = [f_1, \ldots, f_n]^\top$ satisfying

$$f_1 \frac{\partial^r \mathrm{P}_{T_1,m}[p]}{\partial x^\varrho}(a) + \ldots + f_n \frac{\partial^r \mathrm{P}_{T_n,m}[p]}{\partial x^\varrho}(a) = \frac{\partial^r p}{\partial x^\varrho}(a) \tag{2}$$

for all $p \in \mathcal{P}_d^{(m+1)}$.

Remark 1. If $p \in \mathcal{P}_d^{(m)}$ then $\mathrm{P}_{T_i,m}[p] = p$ for $i = 1, \ldots, n$. In this case the equation (2) is trivial when $\partial^r p / \partial x^\varrho(a) = 0$ and it is of the form

$$f_1 + \ldots + f_n = 1 \tag{3}$$

when $\partial^r p / \partial x^\varrho(a) \neq 0$. Obviously, the latter case appears for $p = x^\varrho$.

Definition 2. A system $\mathbf{T}$ of meshes of our domain $\Omega \subset \Re^d$ is said to be a *regular family* when the following conditions (a), (b) are satisfied.

(a) For every $\varepsilon > 0$ there is a mesh $\mathcal{T} \in \mathbf{T}$ such that $h_T < \varepsilon$ for all $T \in \mathcal{T}$.

(b) There exists a constant σ such that $\sigma \geq h_T/\varrho_T$ for all simplices T in any mesh from $\mathbf{T}$.

The following hypothesis, related to a regular family $\mathbf{T}$, parameter m and to a multiindex ϱ with $m \geq r = |\varrho| \geq 1$, has been proved in the special case for the regular family of triangulations consisting of triangles without obtuse inner angles in [3].

Hypothesis (H). There exists a constant C_0 such that a vector $f \in \mathcal{F}_{m,\varrho}(a)$ with the 2-norm $\|f\| \leq C_0$ can be found for every inner vertex a of every mesh $\mathcal{T} \in \mathbf{T}$.

The following main statement has been proved in [4], Theorem 4.

Theorem 1. Let us assume that a regular family $\mathbf{T}$, an integer m and a multiindex ϱ such that $m \geq r = |\varrho| \geq 1$ satisfy the hypothesis (H). Then there exists a constant C_1 such that

$$\left| \frac{\partial^r (f_1 \mathrm{P}_{T_1,m}[u] + \ldots + f_n \mathrm{P}_{T_n,m}[u] - u)}{\partial x^\varrho} (a) \right| \leq C_1 |u|_{m+2,\infty} h(a)^{m+2-r}$$

for every function $u \in C^{m+2}(\overline{\Omega})$, all inner vertices a of the meshes $\mathcal{T} \in \mathbf{T}$, the $\mathcal{T}$-simplices $T_1, \ldots, T_n$ with vertex a and for the vectors $f \in \mathcal{F}_{m,\varrho}(a)$ with the property $\|f\| \leq C_0$.

Let us assume that a regular family $\mathbf{T}$, integer m and a multiindex ϱ such that $m \geq r = |\varrho| \geq 1$ satisfy the hypothesis (H). Then, for any inner vertex a of a triangulation $\mathcal{T} \in \mathbf{T}$, the $\mathcal{T}$-simplices $T_1, \ldots, T_n$ with vertex a and any function $u \in C^{m+2}(\overline{\Omega})$, the minimal 2-norm solution $f = [f_1, \ldots, f_n]^\top$ of the system of equations (2) satisfies $\|f\| \leq C_0$ and the related linear combination

$$\mathrm{G}_{m,\varrho}[u](a) \equiv f_1 \frac{\partial^r \mathrm{P}_{T_1,m}[u]}{\partial x^\varrho} (a) + \ldots + f_n \frac{\partial^r \mathrm{P}_{T_n,m}[u]}{\partial x^\varrho} (a) \tag{4}$$

approximates $\partial^r u / \partial x^\varrho (a)$ with an error $O(h(a)^{m+2-r})$ due to Theorem 1. As both sides of (2) are linear, the equations (2) for all $p \in \mathcal{P}_d^{(m+1)}$ are equivalent to the $\dim \mathcal{P}_d^{(m+1)}$ equations (2) for all p from the basis

$$\begin{gathered} 1, x_1 - a_1, \ldots, x_d - a_d, (x_1 - a_1)^2, (x_1 - a_1)(x_2 - a_2), \ldots, (x_d - a_d)^2, \\ \ldots, (x_1 - a_1)^{m+1}, (x_1 - a_1)^m (x_2 - a_2), \ldots, (x_d - a_d)^{m+1}. \end{gathered} \tag{5}$$

Due to Remark 1, these equations are equivalent to the *reduced system* of

$$1 + \begin{pmatrix} m + d \\ d - 1 \end{pmatrix}$$

m	1	2	3	4	5
$d=2$	4 (6)	5 (10)	6 (15)	7 (21)	8 (28)
$d=3$	7 (10)	11 (20)	16 (35)	22 (57)	29 (84)

Tab. 1: *The numbers of equations in the reduced systems and the dimensions of* $\mathcal{P}_d^{(m+1)}$ *(in brackets).*

equations consisting of the equation (3) and the equations (2) for the polynomials p of degree $m+1$ from (5). In Table 1, the numbers of equations from the reduced systems are compared with the dimensions of the spaces $\mathcal{P}_d^{(m+1)}$ in brackets for $m = 1, \dots, 5$ and $d = 2, 3$. The right-hand sides of the equations (2) for the polynomials of degree $m+1$ from (5) are equal to zero. In [3], the reduced systems of four equations in the special case are analysed completely and efficient procedures for their solution are suggested.

3 Complexity of the general method of averaging

Theorem 1 says that the order of error of approximation of any partial derivative of degree r is proportional to the difference $m - r$ and the method of averaging increases this order from $m + 1 - r$ to $m + 2 - r$. In the special case there is $m = 1 = r$, i.e. the degree of the interpolants used on the triangles surrounding the given vertex a is the least possible. The cases $m = r$ appear, among others, for the following reasons: The data necessary for the higher degree interpolants need not be available and, in the case $m = r$, the calculations of the method of averaging are most simple. In what follows, we restrict our analysis to the special case $m = r$ only. We investigate simplifications of the general method of averaging based on the following identities:

Problem. For a given simplex T and non-zero multiindex ϱ find non-zero multiindices σ, τ with lengths s, t such that $\varrho = \sigma + \tau$ and

$$\frac{\partial^r \mathrm{P}_{T,r}[p]}{\partial x^\varrho} = \frac{\partial^s \mathrm{P}_{T,s}\left[\partial^t p / \partial x^\tau\right]}{\partial x^\sigma} \quad \forall\, p \in \mathcal{P}_d^{(r+1)}. \tag{6}$$

These identities give us the following information about the reduced systems of equations: If the multiindices σ, τ create a solution of the Problem then, as the partial derivatives $\partial^t p / \partial x^\tau$ of all polynomials p of degree $r+1$ are just all polynomials of degree $s+1$, the system of equations (2) for all polynomials p of degree $m = r+1$ is in fact the system of equations (2) for all polynomials $\partial^t p / \partial x^\tau$ of the smaller degree $m = s+1$. Hence the reduced system of $1 + \begin{pmatrix} r+d \\ d-1 \end{pmatrix}$ equations is in fact a simpler reduced system of $1 + \begin{pmatrix} s+d \\ d-1 \end{pmatrix}$ equations.

Identity (6) can be equivalently formulated by means of the space

$$\mathcal{Q}_T^{(r+1)} = \{q \in \mathcal{P}_d^{(r+1)} |\, \mathrm{P}_{T,r}[q] = o\}$$

in the following way.

Theorem 2. For all simplices T and non-zero multiindices σ, τ with $\varrho = \sigma + \tau$, (6) is equivalent to the condition

$$\frac{\partial^s \mathrm{P}_{T,s}\left[\partial^t q / \partial x^\tau\right]}{\partial x^\sigma} = 0 \quad \forall\, q \in \mathcal{Q}_T^{(r+1)}. \tag{7}$$

Proof. Let us assume that the multiindices σ, τ satisfy condition (7) and consider a polynomial $p \in \mathcal{P}_d^{(r+1)}$. If we set $q = p - \mathrm{P}_{T,r}[p]$ then $q \in \mathcal{Q}_T^{(r+1)}$ so that q satisfies (7) by assumption. But then

$$\frac{\partial^s \mathrm{P}_{T,s}\left[\partial^t p / \partial x^\tau\right]}{\partial x^\sigma} = \frac{\partial^s \mathrm{P}_{T,s}\left[\partial^t \left(\mathrm{P}_{T,r}[p] + q\right) / \partial x^\tau\right]}{\partial x^\sigma} = \frac{\partial^r \mathrm{P}_{T,r}[p]}{\partial x^\varrho}.$$

If (6) is true then we obtain (7) by inserting the polynomials $q \in \mathcal{Q}_T^{(r+1)}$ into (6).

The following solution of an analogy of our Problem in dimension $d = 1$ appears to be usefull in what follows.

Theorem 3. Let $r > 1$, $p \in \mathcal{P}_1^{(r+1)}$ and $a = x_0 < x_1 < \ldots < x_r = b$ be equidistant nodes. Then the Lagrange interpolant $\mathrm{P}_r[p] \in \mathcal{P}_1^{(r)}$ of p in the nodes $a = x_0, x_1, \ldots, x_r = b$ and the Lagrange interpolant $\mathrm{P}_1\left[p^{(r-1)}\right] \in \mathcal{P}_1^{(1)}$ of $p^{(r-1)}$ in the nodes a, b satisfy

$$\frac{d^r \mathrm{P}_r[p]}{dx^r} = \frac{1}{b-a}\int_a^b p^{(r)}(x)dx = \frac{d\mathrm{P}_1\left[p^{(r-1)}\right]}{dx}. \tag{8}$$

Proof. Of course,

$$\frac{d\mathrm{P}_1\left[p^{(r-1)}\right]}{dx} = \frac{p^{(r-1)}(b) - p^{(r-1)}(a)}{b-a} = \frac{1}{b-a}\int_a^b p^{(r)}(x)dx. \tag{9}$$

On the other hand, for every $x \in \langle a, b\rangle$ there is $\xi \in (a,b)$ such that

$$p(x) - \mathrm{P}_r[p](x) = \frac{p^{(r+1)}(\xi)}{(r+1)!}\,(x - x_0)(x - x_1)\ldots(x - x_r)$$

due to [2], Section 2.3. As $p \in \mathcal{P}_1^{(r+1)}$, there exists a constant C such that $p^{(r+1)}(\xi) = C$ for all $\xi \in (a,b)$. This and the comparison of the r-th derivatives of both sides of the last identity lead to

$$\begin{aligned} \frac{d^r \mathrm{P}_r[p]}{dx^r} &= p^{(r)}(x) - \frac{C}{(r+1)!}\left[(r+1)!\,x - r!\,(x_0 + \ldots + x_r)\right] \\ &= p^{(r)}(x) - Cx + \frac{C}{r+1}\,(x_0 + \ldots + x_r) \end{aligned}$$

for all $x \in \langle a, b\rangle$. Integrating both sides of this identity over $\langle a, b\rangle$, dividing by $b-a$ and using the fact that $d^r \mathrm{P}_r[p]/dx^r$ is a constant, we obtain

$$\frac{d^r \mathrm{P}_r[p]}{dx^r} = \frac{1}{b-a}\int_a^b p^{(r)}(x)dx + C\left[\frac{x_0+\ldots+x_r}{r+1} - \frac{a+b}{2}\right].$$

As the nodes $a = x_0, x_1, \ldots, x_r = b$ are equidistant, this identity means

$$\frac{d^r \mathrm{P}_r[p]}{dx^r} = \frac{1}{b-a}\int_a^b p^{(r)}(x)dx.$$

Lemma 1. Under the assumptions of Theorem 3,

$$\frac{d^r \mathrm{P}_r[p]}{dx^r} = \frac{1}{h^r}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i} p\left(x_i\right) \quad \text{for} \quad h = \frac{b-a}{r}.$$

Proof. If we express the Lagrange interpolant $\mathrm{P}_r[p](x)$ in the Newton form for equidistant nodes then we obtain

$$\frac{d^r \mathrm{P}_r[p]}{dx^r} = \frac{\Delta^r p(0)}{h^r}.$$

The statement can be proved by induction using the recursive definition of the r-th forward difference $\Delta^r p(0)$.

In the following Theorem 4 we describe all solutions of our Problem in the special case of the partial derivatives in the variables $\xi_1, \ldots, \xi_d$ given by the directions of the catheti of the *unit simplices* $\hat{T} = \overline{a^1 \ldots a^{d+1}}$ with $a^1 = [0, 0, \ldots, 0]$, $a^2 = [1, 0, \ldots, 0]$, $\ldots$, $a^{d+1} = [0, 0, \ldots, 1]$ of the reference finite elements $\hat{e}_d^{(r)}$ with the *discretization step* $h = 1/r$. For the indices $i_1 = 0, \ldots, r$, $i_2 = 0, \ldots, r - i_1$, $\ldots$, $i_d = 0, \ldots, r - i_1 - \ldots - i_{d-1}$,

$$\hat{n}^{i_1 \ldots i_d} = [i_1 h, i_2 h, \ldots, i_d h] \tag{10}$$

are the nodes of $\hat{e}_d^{(r)}$. In Fig. 1, the black circles illustrate the nodes of the finite element $\hat{e}_2^{(r)}$.

Theorem 4. Let $\hat{T}$ be a unit simplex and ϱ a non-zero multiindex with length r. The non-zero multiindices σ, τ of lengths s, t create a solution of the Problem if and only if $\sigma = \tau \cdot s/t$.

Proof. Let us consider arbitrary indices

$$\begin{array}{rcl} i_1 &=& 0, \ldots, r+1, \\ i_k &=& 0, \ldots, r+1-i_1-\ldots-i_{k-1} \quad \text{for} \quad k = 2, \ldots, d-1 \quad \text{and} \\ i_d &=& r+1-i_1-\ldots-i_{d-1} \end{array} \tag{11}$$

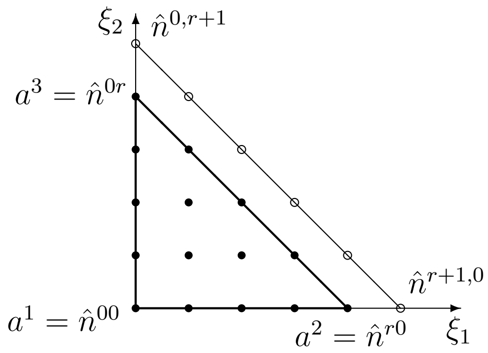

Fig. 1: *The nodes of the finite element* $\hat{e}_2^{(r)}$.

and set

$$f_{i_k}(\xi_k) = \textstyle\prod_{\iota=0}^{i_k-1}(\xi_k - \iota h) \quad \text{for} \quad k = 1, \dots, d,$$
$$q_{i_1 \dots i_d}(\xi_1, \dots, \xi_d) = f_{i_1}(\xi_1) \dots f_{i_d}(\xi_d). \tag{12}$$

As a matter of fact, f_{i_k} is a polynomial of degree i_k in the variable ξ_k such that $f_{i_k}(\iota h) = 0$ for all indices ι, $0 \le \iota < i_k$. Consequently, $\deg(q_{i_1 \dots i_d}) = r + 1$ and $q_{i_1 \dots i_d}$ is equal to zero in all nodes (10) of the finite element $\hat{e}_d^{(r)}$ as well as in the additional nodes $\hat{n}^{j_1 \dots j_d}$ with the indices $j_1 \dots j_d$ of the form (11) except the node $\hat{n}^{i_1 \dots i_d}$ itself. The additional nodes are indicated by the white circles in the case $d = 2$ in Fig. 1. These facts lead to the conclusion that the polynomials (12) create a basis in the space $\mathcal{Q}_{\hat{T}}^{(r+1)}$. This and the linearity of condition (7) mean that (7) is valid for all $q \in \mathcal{Q}_{\hat{T}}^{(r+1)}$ if and only if (7) is valid for the polynomials (12) related to all indices $i_1 \dots i_d$ of the form (11).

Let us now express the partial derivative from (7) for a function $q = q_{i_1 \dots i_d}$. Setting $\sigma = (\alpha_1, \dots, \alpha_d)$ and $\tau = (\beta_1, \dots, \beta_d)$, we obtain

$$\frac{\partial^t q_{i_1 \dots i_d}}{\partial \xi^\tau} = \frac{\partial^t q_{i_1 \dots i_d}}{\partial \xi_1^{\beta_1} \dots \partial \xi_d^{\beta_d}} = f_{i_1}^{(\beta_1)}(\xi_1) \dots f_{i_d}^{(\beta_d)}(\xi_d). \tag{13}$$

Observe that this derivative is different form zero if and only if

$$\beta_1 \le i_1, \dots, \beta_d \le i_d. \tag{14}$$

The next step towards the formulation of condition (7) for the functions $q_{i_1 \dots i_d}$ is to create the interpolant $\mathrm{P}_{\hat{T},s}\left[\partial^t q_{i_1 \dots i_d}/\partial \xi^\tau\right]$. We set $H = 1/s$ and, to every node $\hat{U} = \hat{N}^{u_1 \dots u_d}$ of the finite element $\hat{e}_d^{(s)}$, relate the function

$$L_{\hat{U}}^0(\xi_1, \dots, \xi_d) = F_{u_1}(\xi_1) \dots F_{u_d}(\xi_d) G_{\hat{U}}(\xi_1, \dots, \xi_d)$$

such that

$$F_{u_k}(\xi_k) \;=\; \prod_{\iota=0}^{u_k-1}(\xi_k - \iota H) \quad \text{for} \quad k = 1, \dots, d,$$

$$G_{\hat{U}}(\xi_1,\ldots,\xi_d) \;=\; \prod_{\iota=u_1+\ldots+u_d+1}^{s} (\iota H-\xi_1-\ldots-\xi_d).$$

As $\deg(F_{u_1}) = u_1,\ldots,\deg(F_{u_d}) = u_d$ and $\deg(G_{\hat{U}}) = s-u_1-\ldots-u_d$, we have $\deg(L^0_{\hat{U}}) = s$. Moreover, $L^0_{\hat{U}}(v_1,\ldots,v_d) = 0$ for every node $\hat{N}^{v_1\ldots v_d}$ of $\hat{e}^{(s)}_d$ different from $\hat{U}$. Indeed, if $v_1+\ldots+v_d \le u_1+\ldots+u_d$ then there exists an index $v_k < u_k$ so that $F_{u_k}(v_k) = 0$ and $G_{\hat{U}}(v_1,\ldots,v_n) = 0$ in the case $v_1+\ldots+v_d > u_1+\ldots+u_d$. As

$$L^0_{\hat{U}}(u_1,\ldots,u_d) = H^s u_1!\ldots u_d!(s-u_1-\ldots-u_d)!,$$

we can see that

$$L_{\hat{U}}(\xi_1,\ldots,\xi_d) = \frac{1}{H^s u_1!\ldots u_d!(s-u_1-\ldots-u_d)!}\; L^0_{\hat{U}}(\xi_1,\ldots,\xi_d) \tag{15}$$

is the Lagrange base function in the local space $\hat{\mathcal{L}}^{(s)} = \mathcal{P}^{(s)}_d$ of the reference finite element $\hat{e}^{(s)}_d$ related to the node $\hat{U}$. Then, due to (13),

$$\begin{aligned}
&\mathrm{P}_{\hat{T},s}\left[\frac{\partial^t q_{i_1\ldots i_d}}{\partial\xi^\tau}\right] = \mathrm{P}_{\hat{T},s}\left[f^{(\beta_1)}_{i_1}(\xi_1)\ldots f^{(\beta_d)}_{i_d}(\xi_d)\right]\\
&= \sum_{u_1=0}^{s}\sum_{u_2=0}^{s-u_1}\cdots\sum_{u_d=0}^{s-u_1-\ldots-u_{d-1}} L_{\hat{U}}(\xi_1,\ldots,\xi_d)\, f^{(\beta_1)}_{i_1}(u_1H)\ldots f^{(\beta_d)}_{i_d}(u_dH).
\end{aligned}$$

In order to obtain the σ-th partial derivative of this interpolant, let us analyse the partial derivatives

$$\frac{\partial^s L_{\hat{U}}}{\partial\xi^\sigma} = \frac{\partial^s L_{\hat{U}}}{\partial\xi_1^{\alpha_1}\ldots\partial\xi_d^{\alpha_d}}. \tag{16}$$

As $\deg(L_{\hat{U}}) = s$, (16) is a constant depending on the coefficient C of the maximal-order monomial $C\xi_1^{\alpha_1}\ldots\xi_d^{\alpha_d}$ of $L_{\hat{U}}$. Necessarily, this monomial is a product of the maximal-order monomials

$$\xi_1^{u_1},\ldots,\xi_d^{u_d} \tag{17}$$

from the factors F_{u_1},…,F_{u_d} of $L_{\hat{U}}$. But then

$$u_k \le \alpha_k \quad\text{for}\quad k=1,\ldots,d. \tag{18}$$

The nodes $[u_1H,\ldots,u_dH]$ of the finite element $\hat{e}^{(s)}_2$ satisfying (18) are illustrated by the black circles in the case $d=2$ in Fig. 2. A simple consideration tells us that the product of the monomials (17) with the maximal-order monomial

$$\frac{(-1)^{s-u_1-\ldots-u_d}(s-u_1-\ldots-u_d)!}{(\alpha_1-u_1)!\ldots(\alpha_d-u_d)!}\,\xi_1^{\alpha_1-u_1}\ldots\xi_d^{\alpha_d-u_d}$$

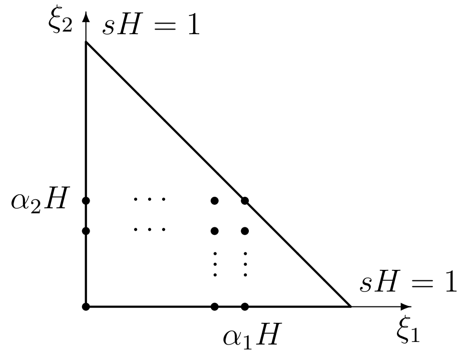

Fig. 2: *The nodes $[u_1H,\dots,u_dH]$ of the finite element $\hat{e}_2^{(s)}$.*

from the factor $G_{\hat{U}}$ appears in $L^0_{\hat{U}}$ and, due to (15),

$$\begin{aligned}\frac{\partial^s L_{\hat{U}}}{\partial \xi^\sigma} &= \frac{(-1)^{s-u_1-\dots-u_d}}{H^s u_1!\dots u_d!(\alpha_1-u_1)!\dots(\alpha_d-u_d)!}\frac{\partial^s}{\partial\xi^\sigma}\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}\\ &= \frac{(-1)^{s-u_1-\dots-u_d}}{H^s}\begin{pmatrix}\alpha_1\\ u_1\end{pmatrix}\dots\begin{pmatrix}\alpha_d\\ u_d\end{pmatrix}.\end{aligned}$$

Hence, by this result, (18) and Lemma 1, $\partial^s \mathrm{P}_{\hat{T},s}\left[\partial^t q_{i_1\dots i_d}/\partial\xi^\tau\right]/\partial\xi^\sigma =$

$$\begin{aligned}&= \sum_{u_1=0}^{s}\sum_{u_2=0}^{s-u_1}\dots\sum_{u_d=0}^{s-u_1-\dots-u_{d-1}}\frac{\partial^s L_{\hat{U}}}{\partial\xi^\sigma} f_{i_1}^{(\beta_1)}(u_1H)\dots f_{i_d}^{(\beta_d)}(u_dH)\\ &= \sum_{u_1=0}^{\alpha_1}\sum_{u_2=0}^{\alpha_2}\dots\sum_{u_d=0}^{\alpha_d}\frac{(-1)^{s-u_1-\dots-u_d}}{H^s}\begin{pmatrix}\alpha_1\\ u_1\end{pmatrix}\dots\begin{pmatrix}\alpha_d\\ u_d\end{pmatrix} f_{i_1}^{(\beta_1)}(u_1H)\dots f_{i_d}^{(\beta_d)}(u_dH)\\ &= \prod_{k=1}^{d}\frac{1}{H^{\alpha_k}}\sum_{u_k=0}^{\alpha_k}(-1)^{\alpha_k-u_k}\begin{pmatrix}\alpha_k\\ u_k\end{pmatrix} f_{i_k}^{(\beta_k)}(u_kH)\\ &= \prod_{k=1,\alpha_k>0}^{d}\frac{d^{\alpha_k}\mathrm{P}_{\alpha_k}\left[f_{i_k}^{(\beta_k)}\right]}{d\xi_k^{\alpha_k}}\prod_{k=1,\alpha_k=0}^{d} f_{i_k}^{(\beta_k)}(0).\end{aligned}\tag{19}$$

Now, we characterize the non-zero multiindices σ,τ satisfying condition (7) for the polynomials $q_{i_1\dots i_d}$ related to the indices $i_1\dots i_d$ of the form (11). If $\deg(f_{i_k}^{(\beta_k)}) = i_k-\beta_k \le \alpha_k$ then $\mathrm{P}_{\alpha_k}\left[f_{i_k}^{(\beta_k)}\right] = f_{i_k}^{(\beta_k)}$ and

$$\frac{d^{\alpha_k}\mathrm{P}_{\alpha_k}\left[f_{i_k}^{(\beta_k)}\right]}{d\xi_k^{\alpha_k}} = f_{i_k}^{(\alpha_k+\beta_k)}.$$

Both this value for $\alpha_k>0$ and $f_{i_k}^{(\beta_k)}(0)$ for $\alpha_k=0$ is zero in the case $i_k-\beta_k<\alpha_k$ and non-zero when $i_k-\beta_k=\alpha_k$ for $k=1,\dots,d$. Hence, whenever there exists k such that $i_k-\beta_k<\alpha_k$, the product (19) is zero. Let us analyse the remaining case

$$i_k-\beta_k\ge\alpha_k \quad\text{for}\quad k=1,\dots,d. \tag{20}$$

By adding up these inequalities and using (11), we obtain $r+1-t \geq s$ or, equivalently, $s+1 \geq s$. Hence all inequalities from (20) except one are equalities and the exception is of the form $i_k - \beta_k = \alpha_k + 1$. As the factors in the product (19) related to the equalities are non-zero, (19) is equal to zero for all sequences of indices from (11) if and only if

$$\frac{d^{\alpha_k} \mathrm{P}_{\alpha_k}\left[f^{(\beta_k)}_{\alpha_k+\beta_k+1}\right]}{d\xi_k^{\alpha_k}} = 0 \quad \text{when} \quad \alpha_k > 0 \quad \text{and} \quad f^{(\beta_k)}_{\beta_k+1}(0) = 0 \quad \text{when} \quad \alpha_k = 0 \tag{21}$$

for $k = 1, \ldots, d$. In the case $\alpha_k > 0$, condition (21) is equivalent to

$$\frac{d\mathrm{P}_1\left[f^{(\alpha_k+\beta_k-1)}_{\alpha_k+\beta_k+1}\right]}{d\xi_k} = 0$$

due to Theorem 3. As $f_{\alpha_k+\beta_k+1}(\xi_k) = \prod_{\iota=0}^{\alpha_k+\beta_k}(\xi_k - \iota h) =$

$$\begin{aligned} &= \xi_k^{\alpha_k+\beta_k+1} - \frac{h}{2}(\alpha_k+\beta_k+1)(\alpha_k+\beta_k)\xi_k^{\alpha_k+\beta_k} \\ &+ \frac{h^2}{24}(\alpha_k+\beta_k+1)(\alpha_k+\beta_k)(\alpha_k+\beta_k-1)(3\alpha_k+3\beta_k+2)\xi_k^{\alpha_k+\beta_k-1} + p(\xi_k) \end{aligned}$$

for some polynomial p with $\deg(p) \leq \alpha_k + \beta_k - 2$, we obtain $f^{(\alpha_k+\beta_k-1)}_{\alpha_k+\beta_k+1}(\xi_k) =$

$$= \frac{(\alpha_k+\beta_k+1)!}{2}\left[\xi_k^2 - h(\alpha_k+\beta_k)\xi_k + \frac{h^2}{12}(\alpha_k+\beta_k-1)(3\alpha_k+3\beta_k+2)\right].$$

Then

$$\begin{aligned} \frac{d\mathrm{P}_1\left[f^{(\alpha_k+\beta_k-1)}_{\alpha_k+\beta_k+1}(\xi_k)\right]}{d\xi_k} &= \frac{f^{(\alpha_k+\beta_k-1)}_{\alpha_k+\beta_k+1}(\alpha_k H) - f^{(\alpha_k+\beta_k-1)}_{\alpha_k+\beta_k+1}(0)}{\alpha_k H} \\ &= (\alpha_k+\beta_k+1)!\frac{\alpha_k H}{2}\left[\alpha_k H - (\alpha_k+\beta_k)h\right]. \end{aligned}$$

By putting $h = 1/(s+t)$ and $H = 1/s$, we can see that condition (21) is equivalent to the condition

$$\frac{(\alpha_k+\beta_k+1)!\alpha_k}{2s^2(s+t)}\left(\alpha_k t - \beta_k s\right) = 0$$

and this one is equivalent to $\alpha_k = \beta_k \cdot s/t$. In the case $\alpha_k = 0$, an evaluation of $f^{(\beta_k)}_{\beta_k+1}(0)$ tells us that the condition $f^{(\beta_k)}_{\beta_k+1}(0) = 0$ means $\beta_k = 0$.

The results obtained in both cases lead to $\sigma = \tau \cdot s/t$.

4 Conclusions

We formulate a corollary of Theorem 4 characterizing multiindices ϱ such that our Problem has a solution on a unit simplex, illustrate the influence of the solutions of the Problem on the complexity of the method of averaging by an example and discuss some open problems.

Definition 3. Let $\varrho = (\gamma_1, \ldots, \gamma_d)$ be a multiindex of length r and l_ϱ the largest common divisor of $\gamma_1, \ldots, \gamma_d$. We call the multiindex ϱ *reduced* when $l_\varrho = 1$. If ϱ is non-reduced then we set $\overline{\gamma}_k = \gamma_k / l_\varrho$ for $k = 1, \ldots, d$ and say that the multiindex $\overline{\varrho} = (\overline{\gamma}_1, \ldots, \overline{\gamma}_d)$ is a *reduction* of ϱ of length $\overline{r} = r/l_\varrho$.

Corollary 1. There exists a solution σ, τ of the Problem related to a d-dimensional unit simplex $\hat{T}$ and a non-zero multiindex $\varrho = (\gamma_1, \ldots, \gamma_d)$ if and only if ϱ is non-reduced.

Proof. According to Theorem 4, non-zero multiindices σ, τ solve the Problem whenever $\sigma = \tau \cdot s/t$. As $s/t > 0$, we have $\varrho = \tau \cdot r/t$ and $r/t > 1$. Let us write $r/t = \overline{r}/\overline{t}$ so that the integers $\overline{r}, \overline{t}$ are relatively prime. Then, as $\overline{r} > \overline{t} \geq 1$ and

$$\gamma_k = \beta_k \cdot \frac{\overline{r}}{\overline{t}} \quad \text{for} \quad k = 1, \ldots, d,$$

the fractions $\beta_k / \overline{t}$ are integers for $k = 1, \ldots, d$. This and $\overline{r} > 1$ tell us that ϱ is non-reduced. On the other hand, if ϱ is non-reduced and $\varrho = l_\varrho \overline{\varrho}$ then the non-zero multiindices $\sigma = \overline{\varrho}, \tau = (l_\varrho - 1)\overline{\varrho}$ create a solution of the Problem.

It is an open question whether Corollary 1 can be generalized to arbitrary simplices. The statement of the following Lemma 2, see [4], Lemma 8, provides a partial positive answer to this question.

Lemma 2. If $r \in \{2, 3, \ldots\}$ and $k \in \{1, 2\}$ then

$$\frac{\partial^r \mathrm{P}_{T,r}(p)}{\partial x_k^r} = \frac{\partial \mathrm{P}_{T,1}(\partial^{r-1} p / \partial x_k^{r-1})}{\partial x_k}$$

for all 2-dimensional simplices T and polynomials $p = p(x_1, x_2)$ of degree $r + 1$.

Example 1. For $u(x, y) = \ln(x^2 + 0.2y^4 + 0.5) \cdot \exp(xy - \sin(x + 2y) - 3)$ and an inner vertex $a = [0, 0]$ with the neighbours $ha^1, \ldots, ha^7$ of certain triangulations $\mathcal{T}_h$ – Fig. 3 illustrates the neighbourhood $\Theta(a) = T_1 \cup \ldots \cup T_7$ of a in $\mathcal{T}_h$ – find the errors of the approximations of $\partial^3 u / \partial x^3(a)$ by means of the method of averaging with the parameters $m = 3 = r$ for such values of h that $h(a) = 2^{-1}, \ldots, 2^{-8}$.

In this example, the multiindex $\varrho = (3, 0)$ is non-reduced. Setting $\sigma = \overline{\varrho} = (1, 0)$ and $\tau = (2, 0)$, we can see that the reduced systems of 6 equations in 7 unknowns indicated in Table 1 are in fact reduced systems of 4 equations in 7 unknowns due to Lemma 2. These systems are exactly the reduced systems for the superapproximation

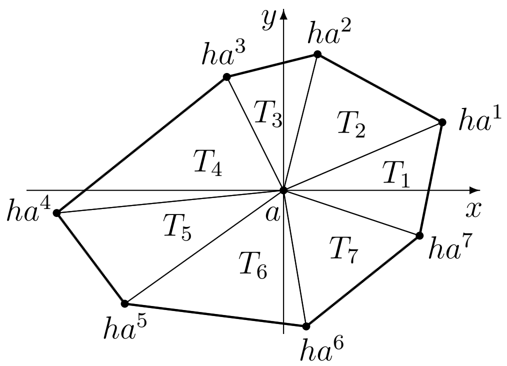

Fig. 3: *Neighbourhood* $\Theta(a) = T_1 \cup \ldots \cup T_7$ *of* a *in* $\mathcal{T}_h$.

i	h_i	e_i	$\log \frac{e_i}{e_{i-1}} / \log \frac{h_i}{h_{i-1}}$
1	5E−1	−1.57729E−1	
2	2.5E−1	−4.38036E−2	1.84833
3	1.25E−1	−1.13485E−2	1.94855
4	6.25E−2	−2.86352E−3	1.98664
5	3.125E−2	−7.17266E−4	1.99721
6	1.5625E−2	−1.79366E−4	1.99960
7	7.8125E−3	−4.48941E−5	1.99831
8	3.90625E−3	−1.13726E−5	1.98096

Tab. 2: *The errors* $e_i = \partial^3 u/\partial x^3(a) - G_{3,(3,0)}[u](a)$ *and the estimates of the order of accuracy.*

of the first derivative $\partial u/\partial x(a)$ by the method of averaging with the parameters $m = 1 = r$. We solve these underdetermined systems of 4 equations by the Householder QR-algorithm described in [5] and use their solutions $f_1, \ldots, f_7$ in the computation of the approximation $\mathrm{G}_{3,(3,0)}[u](a)$ according to (4).

Table 2 presents the values of errors $e_i = \partial^3 u/\partial x^3(a) - \mathrm{G}_{3,(3,0)}[u](a)$ related to the parameters $h(a) = h_i = 2^{-i}$ for $i = 1, \ldots, 8$. The last column indicates that $e_i = O(h_i^2)$.

The special method of averaging ($d = 2, m = 1 = r$) has been analysed in [3] completely. On the contrary, concerning the general method, answers to many open questions would increase its applicability. Among them, besides the generalization of Corollary 1, validity of the hypothesis (H) and applicability of the method in the boundary vertices should be studied.

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