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MEASURES THE VALUES OF WHICH ARE CLASSES OF EQUIVALENT MEASURABLE FUNCTIONS

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In this paper we consider properties of measures, the values of which are classes of equivalent measurable functions; such classes are called random variables.

0. Introduction and summary

The concept of a measure the values of which are random variables is a simultaneous generalization of the concepts of the real-valued measure and of the conditional probability. It is possible sometimes (but not always) to treat the conditional probability as a system of real-valued measures; we say in this case that the conditional probability is regular. It is, however, of interest to study the analogy between conditional probability and real-valued measure without the assumption of regularity and this is to what the following pages are essentially devoted.

The most important fact we systematically use is that the system of all finite random variables on a measurable space is a regular K-space and that the space of all random variables (not necessarily finite) on a measurable space, although being not a regular K-space, has certain important properties of a regular K-space. These properties are studied in sec. 3.

In sec. 4 three lemmas useful for further considerations are stated.

In sec. 5 a theorem on extension of a measure defined on a ring to a measure defined on a σ-ring is proved.

In sec. 6 the weak integral of a real-valued measurable function is defined and a theorem on a representation of a functional (the values of which are random variables) by a weak integral is proved.

In sec. 7 we study the problem of integration of functions the values of which are again measurable functions. The $\mathbf{W}$-integral is defined, for functions the values of which are $(\mathbf{W})$ measurable functions.

Both the **W**-integral and the weak integral have the usual properties of nonnegativity, linearity and continuity from below, the later implying the usual continuity $f_n \to f$, $0 \leqq f_n \leqq g$, Jg finite $\Rightarrow Jf_n \to Jf$, where J denotes the weak or the **W**-integral and $\to$ denotes the convergence induced by the partial ordering of measurable functions and random variables respectively.

The concept of a strong measure is introduced; a measure μ is strong if, roughly speaking, the **W**-integral exists for sufficiently ample σ-algebra **W**. Three theorems show conditions under which a measure μ is strong. In the third of them the concept of the degenerate functional is used; these functionals are used in the mathematical theory of the dynamic of turbulence (Blanc-Lapierre, Fortet [2], p. 613).

In sec. 8 further properties of the **W**-integral are proved. First the domain of definition of the **W**-integral is extended in a way analogous to the extension of a real-valued measure to its completion. The relation with the integral with respect to a system of real-valued measures is stated and theorems analogous to those of Fubini and Radon-Nikodym are proved.

In sec. 9 the conditional probability is studied. The assertion of Theorem 9.4 is near to the results of Shu-Teh Chen Moy [8], whose method we have used in the proof of Lemma 7.14. Theorem 9.5 says that every conditional probability is (as a measure) strong; on the other hand every strong measure is closely related to a conditional probability (Theorem 9.6).

In sec. 10 a further property of conditional probability is studied and results are obtained generalizing the author's results in [3].

There are essentially two ways in defining the integral. The first supposes essentially the elementary integral is first defined for characteristic functions of sets in a ring or in a σ-ring. This method is commonly used in the theory of measure and probability. The other method supposes the elementary integral is defined on a linear space of arbitrary real-valued functions, or, in a more general case, on a lattice (see e. g. McShane [9], Stone [11]).

Thus extending the domain of the elementary integral to a σ-complete lattice we obtain in the first case the system of all measurable characteristic functions, in the second case the system of all measurable functions. Thus in the first case further considerations are necessary to obtain the usual domain of the integral.

In this paper we use essentially the measure-theoretic consideration, but we attempt to unify the two aspects. For example we consider the outer measure μ^* $*$-induced by a functional J and the measure μ induced by μ^* and study the relation between J and μ. We suppose that J (which may be infinite) is defined on a system $\mathscr{D}J$ of non negative finite functions on a set X. Concerning $\mathscr{D}J$ we suppose only that with two functions f and g the system $\mathscr{D}J$ contains the functions $\max(f, g)$, $\min(f, g)$ and $f - \min(f, g)$. (See Theorems

5.8 and 5.13.) Thus $\mathscr{D}J$ may be for example the system of characteristic functions of sets in a ring, or the system of all non negative continuous funtions on a topological space. Both cases are of great importance but so far as I know, they are commonly treated by two different manners.

On the other hand, the two aspects differ more in our case than in the simpler case of a real-valued integrand, since the majority of difficulties does not consist in extending the elementary integral (the functional J in Lemma 7.9) but in proving that the elementary integral has the necessary properties, in particular that it is continuous from below.

We note that, under the restriction to σ-finite measures, Theorem 5.15 can be easily proved by means of Theorem 4.21, Chapt. IX of KANTOROVIČ, VULICH, PINSKER [5].

1. Basic definitions and notations

1.1. The symbol E denotes the space of all real numbers, $E^* = E \cup \{-\infty\} \cup \cup \{+\infty\}$ with usual conventions about ordering, multiplication and addition; in particular $0 . (\pm \infty) = 0$. Further we denote $E_+ = \{c; c \in E, c \geqq 0\}$ and $E_+^* = \{c; c \in E^*, c \geqq 0\}$.

Let $\{b_i\}$ be a finite or infinite sequence, let A be a set and let B be the set of all b_i. Then we write $\{b_i\} \subset\cdot A$ for $B \subset A$, $\{b_i\} \cdot\supset A$ for $B \supset A$ and $\{b_i\} \doteq A$ for $B = A$.

1.2. If $\mathbf{S}$ is a system of sets, then $\mathbf{S}_\cup$ ($\mathbf{S}_{\sigma\cup}$) is the system of all finite (countable) unions of sets in $\mathbf{S}$; similarly $\mathbf{S}_\cap$ is the system of all finite intersections of sets in $\mathbf{S}$; $\mathbf{S}_-$ denotes the system of all differences $A - B$, where $A \in \mathbf{S}$, $B \in \mathbf{S}$. $\mathbf{S}$ is called a lattice, if $\emptyset \in \mathbf{S}$, $\mathbf{S}_\cup \subset \mathbf{S}$, $\mathbf{S}_\cap \subset \mathbf{S}$; a pseudolattice[1]), if the system of all finite unions of disjoint sets in $\mathbf{S}$ is a lattice; a ring, if $\emptyset \in \mathbf{S}$, $\mathbf{S}_\cup \subset \mathbf{S}$, $\mathbf{S}_- \subset \mathbf{S}$; a σ-ring, if $\emptyset \in \mathbf{S}$, $\mathbf{S}_{\sigma\cup} \subset \mathbf{S}$, $\mathbf{S}_- \subset \mathbf{S}$. A ring ($\sigma$-ring) $\mathbf{S}$ is an algebra (σ-algebra), if $\bigcup \mathbf{S} \in \mathbf{S}$. If $\mathbf{C}$ is a system of sets, then $\mathbf{rC}$ resp. $\mathbf{sC}$ denotes the smallest ring resp. σ-ring which contains $\mathbf{C}$. We denote by $\mathfrak{B}$ the smallest σ-algebra containing all intervals $I \subset E$ and the sets $\{-\infty\}$, $\{+\infty\}$.

1.3. If T is a transformation, then $\mathscr{D}T$ is the set on which T is defined and $\mathscr{R}T = T(\mathscr{D}T)$. The meaning of symbols $T(A)$, $T^{-1}(B)$, $T(x) = Tx$ for $A \subset \subset \mathscr{D}T$, $B \subset \mathscr{R}T$, $x \in \mathscr{D}T$ is obvious. If V is also a transformation, $\mathscr{D}V \supset \mathscr{R}T$, then the symbol VT denotes the composed transformation. If $A \subset \mathscr{D}T$, then T_A is the transformation of A into $\mathscr{R}T$ defined by the relation $T_A x = Tx$ for every $x \in A$. If V and T are two transformations and $\mathscr{D}V \subset \mathscr{D}T$, $T_{\mathscr{D}V} = V$, then T is an extension of V, in symbols $T \succ V$. A transformation T is called measurable ($\mathbf{V}$, $\mathbf{S}$), if $\mathbf{S}$ and $\mathbf{V}$ are σ-rings, $\bigcup \mathbf{S} = \mathscr{D}T$, $\bigcup \mathbf{V} \supset \mathscr{R}T$ and $A \in \mathbf{V} \Rightarrow \Rightarrow T^{-1}(A) \in \mathbf{S}$.

[1]) It is easy to see that every semiring (see [4]) is a pseudolattice.

1.4. A (finite) real-valued function is a transformation f with $\mathscr{R}f \subset E^*$ ($\mathscr{R}f \subset E$). If A is a set, then $\mathbf{f}^*A$ ($\mathbf{f}A$) is the system of all (finite) real-valued functions defined on A. By the symbol $\mathbf{f}^*_+A$ resp. $\mathbf{f}_+A$ we denote the system of all functions belonging to $\mathbf{f}^*A$ resp. $\mathbf{f}A$, which are non negative. If $f_i \in \mathbf{f}^*A$ ($i = 1, 2, \ldots, k$), we denote $\bigwedge_{i=1}^{k} f_i = f_1 \wedge f_2 \wedge \ldots \wedge f_k = \inf(f_1, \ldots, f_k)$, $\bigvee_{i=1}^{k} f_i = f_1 \vee f_2 \vee \ldots \vee f_k = \sup(f_1, \ldots, f_k)$, $f_+ = f \vee 0$, $f_- = (-f)_+ = -(f \wedge 0)$. The symbols $\bigvee_{i=1}^{\infty} f_i$ and $\bigwedge_{i=1}^{\infty} f_i$ have the analogous meaning. Further $f_1 \leqq f_2$ means $f_1(x) \leqq f_2(x)$ in E^* for every $x \in A = \mathscr{D}f_i$; $f_1 < f_2$ means $f_1 \leqq f_2$ and $f_1 \neq f_2$; $f_i \to f$ or $\lim f_i = f$ means $f_i(x) \to f(x)$ in E^* for every $x \in A$; $f_i \nearrow f$ ($f_i \searrow f$) means $f_i \to f$ and $f_i \leqq f_{i+1}$ ($f_i \geqq f_{i+1}$) for $i = 1, 2, \ldots$

1.5. If A is a set, we denote by c_A the characteristic function of the set A (the meaning of the complement of A will be always clear from the context). Let $\mathbf{S}$ be a system of sets. Then we denote by $\mathbf{cS}$ the system of all functions c_A with $A \in \mathbf{S}$. If $\mathbf{S}$ is a system of sets, then a real-valued function f is called $\mathbf{S}$-simple, if $\mathscr{D}f = \bigcup \mathbf{S}$ and $f = \sum_{i=1}^{n} a_i \,.\, c_{A_i}$, where $a_i \in E_+$, $A_i \in \mathbf{S}$. If $\mathbf{S}$ is a σ-ring, then a real-valued function f is called ($\mathbf{S}$) measurable if $\mathscr{D}f = \bigcup \mathbf{S}$ and $f^{-1}(A) \in \mathbf{S}$ as soon as $0 \text{ non} \in A \in \mathfrak{B}$. If $\mathscr{A} \subset \mathbf{f}^*A$ then we denote by $\mathbf{k}\mathscr{A}$ the smallest σ-ring such that every $f \in A$ is ($\mathbf{k}\mathscr{A}$) measurable. We denote by $\mathbf{m}^*\mathbf{S}$ ($\mathbf{mS}$) the system of all (finite) real-valued ($\mathbf{S}$) measurable functions, by $\mathbf{m}^*_+\mathbf{S}$ ($\mathbf{m}_+\mathbf{S}$) the system of all $f \in \mathbf{m}^*\mathbf{S}$ ($f \in \mathbf{mS}$) which are non negative.

If A is a set, $\mathscr{A} \subset \mathbf{f}^*A$, then $\mathscr{A}_\vee$ resp. $\mathscr{A}_\wedge$ resp. $\mathscr{A}_{\sigma\vee}$ denotes the set of all $f \vee g$ resp. $f \wedge g$ resp. $\bigvee_{i=1}^{\infty} f_i$, where $f \in \mathscr{A}$, $g \in \mathscr{A}$, $\{f_i\}_{i=1}^{\infty} \subset\cdot \mathscr{A}$. If $\mathscr{A} \subset \mathbf{f}^*_+A$, $\mathscr{B} \subset \mathbf{f}_+A$, then we define

$$\mathscr{A}_-(\mathscr{B}) = \{f_2 - f_1;\ 0 \leqq f_1 \leqq f_2 \leqq f \in \mathscr{B},\quad f_1 \in \mathscr{A},\ f_2 \in \mathscr{A}\}\,.$$

If $\mathscr{A} = \mathscr{B} \in \mathbf{f}_+A$ we write $\mathscr{A}_- = \mathscr{A}_-(\mathscr{A})$.

Let $\mathscr{A} \subset \mathbf{f}A$. Then $\mathscr{A}$ is called an f-lattice, if $\mathscr{A} \subset \mathbf{f}_+A$, $0 \in \mathscr{A}$, $\mathscr{A}_\vee \subset \mathscr{A}$, $\mathscr{A}_\wedge \subset \mathscr{A}$; an f-ring, if $\mathscr{A}$ is an f-lattice and $\mathscr{A}_- \subset \mathscr{A}$; a basic system, if $\mathscr{A}$ is an f-ring and $f \in \mathscr{A}, c \in E_+ \Rightarrow c \,.\, f \in \mathscr{A}$, $f \wedge 1 \in \mathscr{A}$.

1.6. A real measure is a real-valued non negative function μ such that $\mathscr{D}\mu$ is a σ-ring and $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ as soon as $A_i \in \mathscr{D}\mu$ and $A_i \cap A_j = \emptyset$ for every $i = 1, 2, \ldots$; $j \neq i$. A measure μ is said to be totally σ-finite, if there exists a sequence of sets $\{A_i\}_{i=1}^{\infty} \subset\cdot \mathscr{D}\mu$ such that $\bigcup A_i = \bigcup \mathscr{D}\mu$ and $\mu(A_i) < +\infty$ for every $i = 1, 2, \ldots$. A measurable space is such a couple of σ-rings ($\mathbf{S}, \mathbf{S}_0$) that there exists a totally σ-finite measure μ such that $\mathbf{S} = \mathscr{D}\mu$ and $\mathbf{S}_0 = \{A;\ A \in \mathbf{S},\ \mu(A) = 0\}$. In such a case we say that ($\mathbf{S}, \mathbf{S}_0$) is induced by μ, or that μ induces $(\mathbf{S}, \mathbf{S}_0)$. If $(\mathbf{S}, \mathbf{S}_0)$ is a measurable space, then two $(\mathbf{S})$ measurable functions f_1, f_2 are $(\mathbf{S}_0)$ equivalent if there exists a set $V_0 \in \mathbf{S}_0$ such that $f_1(x) = f_2(x)$ for all $x \in \bigcup \mathbf{S} - V_0$. Thus the system $\mathbf{m}^*\mathbf{S}$ can be divided into disjoint classes of $(\mathbf{S}_0)$ equivalent functions; such classes are called random variables. If $\mathscr{M} \subset \mathbf{m}^*\mathbf{S}$, then the system of all such random variables, which contain at least one element of $\mathscr{M}$, is denoted by $\mathbf{n}_{\mathscr{M}}(\mathbf{S}, \mathbf{S}_0)$. In particular we denote by $\mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$ resp. $\mathbf{n}^*_+(\mathbf{S}, \mathbf{S}_0)$ resp. $\mathbf{n}(\mathbf{S}, \mathbf{S}_0)$ resp. $\mathbf{n}_+(\mathbf{S}, \mathbf{S}_0)$ the set $\mathbf{n}_{\mathscr{M}}(\mathbf{S}, \mathbf{S}_0)$ where $\mathscr{M} = \mathbf{m}^*\mathbf{S}$ resp. $\mathscr{M} = \mathbf{m}^*_+\mathbf{S}$ resp. $\mathscr{M} = \mathbf{m}\mathbf{S}$ resp. $\mathscr{M} = \mathbf{m}_+\mathbf{S}$.

The addition, multiplication and ordering of random variables are defined as follows (α and β are supposed to be random variables beloging to $\mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$): First we define $\alpha + \beta$ if and only if there exist two functions $f \in \alpha$, $g \in \beta$ such that $f + g$ is defined. In this case we define

$$\alpha + \beta = \{f + g; f \in \alpha, g \in \beta, f + g \quad \text{has a meaning}\} .$$

Further we put $\alpha \,.\, \beta = \{f \,.\, g; f \in \alpha, g \in \beta\}$. Finally we write $\alpha \leqq \beta$ if and only if there exist $f \in \alpha$, $g \in \beta$ such that $f \leqq g$. Obviously $\alpha + \beta$ and $\alpha \,.\, \beta$ are random variables and belong to $\mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$.

If $f \in \alpha \in \mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$, let us write for a moment $\alpha = n(f)$.

If $A \in \mathbf{S}$ then we denote $\chi_A = n(c_A)$. If $c \in E^*$, $f \in \mathbf{m}^*\mathbf{S}$, $fx = c$ for every $x \in \mathscr{D}f$, we denote both f and $n(f)$ by the same symbol c. Every totally σ-finite real-valued measure ξ induces a measurable space $(\mathbf{S}, \mathbf{S}_0)$; in such a case we write $\mathbf{n}^*\xi = \mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$ etc. We denote also the $\mathbf{S}_0$-equivalence of $f, g \in \mathbf{m}^*\mathbf{S}$ by $f = g\ [\xi]$. The elements of $\mathbf{n}^*$ resp. $\mathbf{n}$ resp. $\mathbf{n}^*_+$ are called random resp. finite random resp. non negative random variables. If $f \in \varphi \in \mathbf{n}^*_+\xi$, we define $\int \varphi \, \mathrm{d}\xi = \int f \, \mathrm{d}\xi$.

1.7. If a binary transitive relation $>$ is given in a set Y, we write $a \geqq b$ if and only if $a > b$ or $a = b$. Then a subset $B \subset Y$ is said to be bounded from below in Y, if there exists a $y \in Y$ such that $y \leqq b$ for every $b \in B$; we write in this case $y\ (\leqq)\ B$. By the symbol $\inf_A B$, if $A \subset Y$, we denote such an element of A that

$$\inf_A B\ (\leqq)\ B$$

and $h\ (\leqq)\ B$, $h \in A \Rightarrow h \leqq \inf_A B$. If $\inf_A B$ exists and if the relation $\geqq$ is antisymmetric, then $\inf_A B$ is uniquely determined.

In an analogous way the boundedness from above and $\sup_A B$ are defined. If $B \doteq \{b_i\}_{i=1}^k$, we write also

$$\sup_Y B = b_1 \vee b_2 \vee \ldots \vee b_k = \bigvee_{i=1}^{k} b_i , \quad \inf_Y B = b_1 \wedge b_2 \wedge \ldots \wedge b_k = \bigwedge_{i=1}^{k} b_i .$$

The convergence in a partially ordered set Y is defined in the following way: $b_i \to b_0$ if and only if $b_i \in Y$, $\bigvee_{m=1}^{\infty} \bigwedge_{i=m}^{\infty} b_i$ and $\bigwedge_{m=1}^{\infty} \bigvee_{i=m}^{\infty} b_i$ exist, $b_0 = \bigvee_{m=1}^{\infty} \bigwedge_{i=m}^{\infty} b_i = \bigwedge_{m=1}^{\infty} \bigvee_{i=m}^{\infty} b_i$.

It is easy to see that for real-valued functions this convergence coincides with the convergence everywhere. For random variables this convergence means $\alpha_i \to \alpha_0$ if and only if there exists a sequence $a_i \in \alpha_i$ such hat $a_i \to a_0$ or, what is the same, if for every sequence $a_i \in \alpha_i$ there exists a set $N \in \mathbf{S}_0$ (if $\alpha_i \in \mathbf{n}^*(\mathbf{S}, \mathbf{S}_0)$) such that $\lim_{i\to\infty} a_i(x) = a_0(x)$ for every $x \in \bigcup \mathbf{S} - N$.

1.8. If $(\mathbf{S}, \mathbf{S}_0)$ and $(\mathbf{V}, \mathbf{V}_0)$ are two measurable spaces, we write $(\mathbf{S}, \mathbf{S}_0) \leqq \leqq (\mathbf{V}, \mathbf{V}_0)$ if and only if $\mathbf{S} \subset \mathbf{V}$, $\bigcup \mathbf{V} = \bigcup \mathbf{S}$, $\mathbf{S}_0 = \mathbf{V}_0$; i. e., if $\mathbf{n}^*(\mathbf{S}, \mathbf{S}_0) \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, where $\subset$ denotes the usual set inclusion. If $0 \in \mathscr{M} \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, where $(\mathbf{V}, \mathbf{V}_0)$ is a measurable space, then there exists a smallest measurable space $(\mathbf{q}\mathscr{M}, \mathbf{q}_0\mathscr{M})$ such that $\mathbf{n}^*$ $(\mathbf{q}\mathscr{M}, \mathbf{q}_0\mathscr{M})$ contains $\mathscr{M}$. It is easy to see that

$$\mathbf{q}_0\mathscr{M} = \{A; c_A \in 0 \in \mathscr{M}\},$$
$$\mathbf{q}\mathscr{M} = \mathbf{s}\{A; A = g^{-1}(B); 0 \text{ non} \in B \in \mathfrak{B}, g \in \gamma \in \mathscr{M}\}.$$

2. The Radon-Nikodym derivatives

In this section we remind of certain properties of the Radon-Nikodym derivatives.

2.1. Definition. Let μ and ν be two real-valued measures. We say that ν is *absolutely continuous* with respect to μ ($\nu << \mu$) if $\mathscr{D}\mu = \mathscr{D}\nu$ and if $A \in \mathscr{D}\mu$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

2.2 Lemma. *Let μ and ν be two real measures, let μ be totally σ-finite and let $\nu << \mu$. Then there exists one and only one α such that*

$$\alpha \in \mathbf{n}^*_+\mu, \tag{2.2.1}$$
$$\beta \in \mathbf{n}^*_+\mu \Rightarrow \int \beta \, d\nu = \int \alpha \, . \, \beta \, d\mu. \tag{2.2.2}$$

2.3. Definition. Let μ and ν satisfy the conditions of the preceding Lemma, let α be the (unique) random variable satisfying (2.2.1) and (2.2.2). Then α is called the *Radon-Nikodym derivative*; it is denoted by the symbol $\frac{d\nu}{d\mu}$.

2.4. Lemma. *Let μ and ν_i be real measures, let μ be σ-finite and $\nu_i << \mu$ for every $i = 1, 2, \ldots$. Then*

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}; \tag{2.4.1}$$

$$\nu_1 \leqq \nu_2 \Rightarrow \frac{d\nu_1}{d\mu} \leqq \frac{d\nu_2}{d\mu}; \tag{2.4.2}$$

$$\nu_1 \leqq \nu_2 \leqq \ldots \Rightarrow \lim_{i\to\infty} \nu_i \textit{ is a real-valued measure, } \frac{d \lim_{i\to\infty} \nu_i}{d\mu} = \lim_{i\to\infty} \frac{d\nu_i}{d\mu}. \tag{2.4.3}$$

For proofs of (2.2) and (2.4) see for example Halmos [4], § 31, Theorem B and Exercises 7, § 32, Theorems A and B.

3. The spaces of random variables and the K-spaces

3.1. Lemma. *Let* $(\mathbf{V}, \mathbf{V}_0)$ *be a measurable space. Then there exists a pseudo-probability* ξ *(i. e. a real-valued measure* ξ *such that* $\xi(\bigcup \mathbf{V})$ *is equal to* 0 *or to* 1) *inducing* $(\mathbf{V}, \mathbf{V}_0)$.

Proof. From the definition it follows that there exists a totally σ-finite measure η inducing $(\mathbf{V}, \mathbf{V}_0)$. If $\eta(\bigcup \mathbf{V}) = 0$, we put $\xi = \eta$. If $\eta(\bigcup \mathbf{V}) > 0$, then there exists a sequence $\{A_i\}_{i=1}^{\infty} \subset \mathbf{V}$ such that $\bigcup_{i=1}^{\infty} A_i = \bigcup \mathbf{V}$, $0 < \mu(A_i) < +\infty$; put $\xi(A) = \sum_{i=1}^{\infty} \frac{\eta(A \cap A_i)}{2^i \eta(A_i)}$.

3.2. Definition.[2]) Y is a *K-space*, if Y is a linear space with a *binary relation* $>$, satisfying

$$y > z \Leftrightarrow y - z > 0\,, \tag{3.2.1}$$

$$y > 0 \Rightarrow y \neq 0\,, \tag{3.2.2}$$

$$y > 0, z > 0 \Rightarrow y + z > 0\,, \tag{3.2.3}$$

$$\text{if } y \in Y, \text{ then there exists a } z \in Y \text{ such that } z \geqq 0\,, \quad z \geqq y\,, \tag{3.2.4}$$

$$y \in Y,\ c \in E,\ y > 0,\ c > 0 \Rightarrow c \,.\, y > 0\,, \tag{3.2.5}$$

for every non empty set $B \subset Y$ bounded from below in Y there exists

$$\inf_Y B\,. \tag{3.2.6}$$

3.3. Notation. If α and β are random variables, we write $\alpha > \beta$ if and only if $\alpha \geqq \beta$ and $\alpha \neq \beta$.

3.4. Theorem. *Let* $(\mathbf{V}, \mathbf{V}_0)$ *be a measurable space, let* $B \subset Y^* = \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$. *Then both* $\inf_{Y^*} B$ *and* $\sup_{Y^*} B$ *exist. Moreover a countable subset* $B' \subset B$ *exists such that* $\inf_{Y^*} B' = \inf_{Y^*} B$ *and* $\sup_{Y^*} B' = \sup_{Y^*} B$.

Proof. If $B = \emptyset$, then $\inf_{Y^*} B = +\infty$, $\sup_{Y^*} B = -\infty$. If $B \doteq \{\beta_i\}_{i=1}^{\infty}$, then obviously $\sup_{Y^*} B$ is the random variable containing the element $\bigvee_{i=1}^{\infty} b_i$, where $b_i \in \beta_i$. If B is uncountable, we proceed as follows.

Put, for every $\alpha \in Y^*$, $\varrho(\alpha) = \int \frac{\alpha}{1 + |\alpha|}\, d\xi$, where $\frac{+\infty}{1+\infty}$ and $\frac{-\infty}{1+\infty}$ mean 1 and -1 respectively and where ξ is a pseudoprobability inducing $(\mathbf{V}, \mathbf{V}_0)$. Clearly

$$\alpha < \beta \Rightarrow \varrho(\alpha) < \varrho(\beta)\,. \tag{3.4.1}$$

Let C be the set of all random variables of the form $\bigvee_{i=1}^{\infty} \alpha_i$, $\alpha_i \in B$. It is evident that if $\sup_{Y^*} C$ exists, so does $\sup_{Y^*} B$ and $\sup_{Y^*} C = \sup_{Y^*} B$.

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[2]) See [5].

Now let $\{\gamma_i\}_{i=1}^{\infty}$ be such a sequence that $\gamma_i \in C$, $\varrho(\gamma_i) \to \sup_{\gamma\in C} \varrho(\gamma)$. Since $\bigvee_{i=1}^{\infty} \gamma_i \in C$ and $\varrho(\gamma_i) \leqq \varrho(\bigvee_{i=1}^{\infty} \gamma_i)$ it follows that

$$\varrho(\bigvee_{i=1}^{\infty} \gamma_i) = \sup_{\gamma\in C} \varrho(\gamma) \tag{3.4.2}$$

and $\bigvee_{i=1}^{\infty} \gamma_i = \sup_{Y^*} C = \sup_{Y^*} B$. Indeed, if for a $\gamma \in C$ the inequality $\gamma \leqq \bigvee_{i=1}^{\infty} \gamma_i$ does not hold, then $\bigvee_{i=1}^{\infty} \gamma_i < (\bigvee_{i=1}^{\infty} \gamma_i) \vee \gamma \in C$, which is impossible according to (3.4.1) and (3.4.2). However, $\bigvee_{i=1}^{\infty} \gamma_i$ is supremum of a countable subset $B_1 \subset B$. By a similar argument we obtain a countable set $B_2 \subset B$ such that $\inf_{Y^*} B_2 = \inf_{Y^*} B$ and it suffices to put $B' = B_1 \cup B_2$.

3.5. Lemma. *Let* $Y = \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$, $Y^* = \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, $\emptyset \neq B \subset Y$. *Then the following four conditions are mutually equivalent*:

$$B \text{ is bounded from below in } Y\,, \tag{3.5.1}$$

$$\inf_{Y^*} B \in Y\,, \tag{3.5.2}$$

$$\inf_Y B \text{ exists and } \inf_{Y^*} B = \inf_Y B\,, \tag{3.5.3}$$

$$\inf_Y B \text{ exists}\,. \tag{3.5.4}$$

Proof. If (3.5.1) holds, then there exists an $\alpha \in Y$ such that $B\ (\geqq)\ \alpha$. Hence $\inf_{Y^*} B \geqq \alpha$. As $B \neq \emptyset$, there exists a $\beta \in B \subset Y$ and $\alpha \leqq \inf_{Y^*} B \leqq \beta$. Thus (3.5.2) holds. Clearly (3.5.2) $\Rightarrow$ (3.5.3) $\Rightarrow$ (3.5.4) $\Rightarrow$ (3.5.1).

3.6. Notation. In the next, if $A \subset Y^* = \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$, the symbol $\inf A$ denotes $\inf_{Y^*} A$.

3.7. Definition. Let Y be a K-space. Denote by $\tilde{Y}$ the space $Y \cup \{+\tilde{\infty}\} \cup \cup \{-\tilde{\infty}\}$, where $-\tilde{\infty} < y < +\tilde{\infty}$ for every $y \in Y$. [$y_n \to y$ in $\tilde{Y}$] means of course the convergence induced by the ordering in $\tilde{Y}$.

3.8. Lemma. *Let* $Y = \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$, $\{\alpha_i\}_{i=1}^{\infty} \subset Y$, $a_i \in \alpha_i$. *Then* $\alpha_i \to +\tilde{\infty}$ *in* $\tilde{Y}$ *if and only if the following condition is satisfied*: $\bigwedge_{i=1}^{\infty} \alpha_i \in Y$ *and there exists a set* $V \in \mathbf{V} - \mathbf{V}_0$ *such that* $\lim_{i\to\infty} a_i(x) = +\infty$ *for every* $x \in V$.

Proof. Let $\beta_i = \inf_{\tilde{Y}}\{\alpha_j;\ j = i, i+1, \ldots\}$. Then $\alpha_i \to +\tilde{\infty}$ in $\tilde{Y}$ if and only if $\beta_i \to +\tilde{\infty}$ in $\tilde{Y}$.

First let $\alpha_i \to +\tilde{\infty}$ in $\tilde{Y}$. Then $\beta_{i_0} \in Y$ for some i_0, and thus also $\bigwedge_{i=1}^{\infty} \alpha_i = \bigwedge_{i=1}^{i_0-1} \alpha_i \wedge \beta_{i_0} \in Y$. Further, if $b_i = \bigwedge_{j=i}^{\infty} a_j$, then $b_i \in \beta_i$ and $b = \lim_{i\to\infty} b_i$ exists. Obviously b is not in $\mathbf{m}\mathsf{V}$ and thus there exists a set $V \in \mathsf{V} - \mathsf{V}_0$ such that $b(x) = + \infty$ for $x \in V$; hence it follows that $a_i(x) \to + \infty$ for every $x \in V$ and the "only if" is proved.

On the other hand, if $a_i \in \alpha_i$, $V \in \mathsf{V} - \mathsf{V}_0$, $a_i (x) \to + \infty$ for every $x \in V$ and $\bigwedge_{i=1}^{\infty} \alpha_i \in Y$, then $\sup_{\tilde{Y}\, i} \beta_i = + \tilde{\infty}$ and thus $\alpha_i \to + \tilde{\infty}$ in $\tilde{Y}$.

The following lemma is a slight generalization of a theorem due to Fréchet (see [5], p. 177).

3.9. Lemma. *Let $Y = \mathbf{n}(\mathsf{V}, \mathsf{V}_0)$, let $\alpha_{ij} \in Y$, $\alpha_i \in Y$ for every i, j, let*

$$\lim_{j \to \infty} \alpha_{ij} = \alpha_i, \ i = 1, 2, \dots \text{ and } \alpha_i \to \alpha \quad \text{in } \tilde{Y} .$$

Then there exists a sequence of integers $n_1 < n_2 < n_3 < \dots$ such that $\alpha_{in_i} \to \alpha$ in $\tilde{Y}$.

Proof. Let ξ be a pseudoprobability (see Lemma 3.1) inducing $(\mathsf{V}, \mathsf{V}_0)$, let $a_{ij} \in \alpha_{ij}$, $a_i \in \alpha_i$, $a \in \alpha$. For every n there exists (Jegorov's Theorem) a set $W_n \in \mathsf{V}$ such that $\xi(W_n) < \frac{1}{n}$ and $a_{ij} \to a_i$ uniformly on $\bigcup \mathsf{V} - W_n$ for every $i = 1, 2, \dots$. But, for every $i = 1, 2, \dots, a_{ij} \to a_i$ uniformly on $\bigcup \mathsf{V} - V_n$, where $V_n = \bigcap_{j=1}^{n} W_j$. Clearly $V_1 \supset V_2 \supset \dots$ and $\xi(\bigcap_{n=1}^{\infty} V_n) = 0$. Accordingly, we may choose a sequence of integers $0 < n_1 < n_2 < \dots$ such that

$$|a_{in_i}(x) - a_i(x)| < \frac{1}{i} \quad \text{for every} \quad x \in \bigcup \mathsf{V} - V_i .$$

Suppose $\alpha_i \to \alpha \in Y$, $a \in \alpha$, $a_i \to a$. If $x \in \bigcup \mathsf{V} - \bigcap_{i=1}^{\infty} V_i$, then there exists an index i_0 such that $x \in \bigcup \mathsf{V} - V_i$ for all $i > i_0$; thus

$$|a_{in_i}(x) - a_i(x)| < \frac{1}{i} \text{ for } i > i_0,$$

which implies $a_{in_i}(x) \to a(x)$. Thus $a_{in_i} \to a$ on $\bigcup \mathsf{V} - \bigcap_{i=1}^{\infty} V_i$, i. e., $\alpha_{in_i} \to \alpha$.

Suppose $\alpha_i \to + \tilde{\infty}$. This is equivalent (see the preceding Lemma) to the existence of a set $M \in \mathsf{V} - \mathsf{V}_0$ such that $a_i(x) \to + \infty$ for every $x \in M$ and $\bigwedge_{i=1}^{\infty} \alpha_i \in Y$. But obviously $\bigwedge_{i=1}^{\infty} \alpha_{in_i} \in Y$ and $a_{in_i}(x) \to \infty$ for every $x \in M - \bigcap_{i=1}^{\infty} V_i$. Thus also $\alpha_{in_i} \to + \tilde{\infty}$.

Suppose $\alpha_i \to - \tilde{\infty}$. Then $(- \alpha_i) \to + \tilde{\infty}$, $(- \alpha_{in_i}) \to + \tilde{\infty}$ and $\alpha_{in_i} \to + \tilde{\infty}$. Thus the Theorem is proved.

3.10. Lemma. *Let* $M \subset \mathbf{n}^*_+(\mathbf{V}, \mathbf{V}_0)$ *and let*

$$\alpha \in M, \ \beta \in M, \ \alpha \neq \beta \Rightarrow \alpha \wedge \beta = 0 \, .$$

Then M is countable.

Proof. For every sequence $\{\alpha_i\} \subset\cdot M$ we have $\sum_{i=1}^{\infty} \int \alpha_i \wedge 1 \, \mathrm{d}\xi \leqq 1$, if ξ is a pseudoprobability inducing $(\mathbf{V}, \mathbf{V}_0)$. Thus the set of all $\alpha \wedge 1$, where $\alpha \in M$, is at most countable. Further $\alpha \neq \beta, \alpha \wedge \beta = 0 \Rightarrow \alpha \wedge 1 \neq \beta \wedge 1$, which finishes the proof.

3.11. Theorem. *Let* $(\mathbf{V}, \mathbf{V}_0)$ *be a measurable space. Then* $\mathbf{n}(\mathbf{V}, \mathbf{V}_0)$ *is a regular K-space.*

Proof. $\mathbf{n}(\mathbf{V}, \mathbf{V}_0)$ is a K-space according Theorem 3.4 and Lemma 3.5 and is regular according Lemmas 3.9 and 3.10 (see [5], Chapt. V.).

3.12. Definition. A subset B of a partially ordered set A is called *down oriented,* if $a \in B$, $b \in B$ implies the existence of such a $d \in B$ that $d \leqq a$, $d \leqq b$.

3.13. Theorem. *Let* $B \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$ *and let B be down oriented. Then there exists a sequence* $\{\beta_i\}_{i=1}^{\infty} \subset\cdot B$ *such that* $\beta_i \searrow \inf B$.

Proof. From Theorem 3.4 it follows that there exists a sequence $\{\alpha_i\}_{i=1}^{\infty} \subset\cdot B$ such that $\bigwedge_{i=1}^{\infty} \alpha_i = \inf B$. Now it suffices to choose $\beta_n \in B, \beta_n \leqq \beta_{n-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_n$ for every n (this is possible for B is down oriented).

3.14. Definition. If $(\mathbf{V}, \mathbf{V}_0)$ is a measurable space, $A \in \mathbf{V}$, then we denote by $\mathbf{P}_A$ the transformation from $\mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$ onto $\mathbf{n}^*({}_A\mathbf{V}, {}_A\mathbf{V}_0)$, where

$${}_A\mathbf{V} = \{B; A \supset B \in \mathbf{V}\} \, , \quad {}_A\mathbf{V}_0 = \{B; A \supset B \in \mathbf{V}_0\} \, ,$$

which satisfies the condition $f \in \varphi \in \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0) \Rightarrow f_A \in \mathbf{P}_A\varphi$.

Further, if $\alpha \in \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, $B \in \mathfrak{B}$, then we denote by $\alpha^{-1}(B)$ the system $\{A; A = a^{-1}(B), a \in \alpha\}$.

3.15. Lemma. *Let* $Y^* = \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, $M \in \mathbf{V}$. *Let* $\alpha \in Y^*, \beta \in Y^*$. *Then*

$$\alpha \geqq \beta \Rightarrow \mathbf{P}_M\alpha \geqq \mathbf{P}_M\beta \, , \tag{3.15.1}$$

$$\mathbf{P}_M\alpha > \mathbf{P}_M\beta \Leftrightarrow \alpha \, . \, \chi_M > \beta \, . \, \chi_M \, , \tag{3.15.2}$$

$$\mathbf{P}_M\alpha \geqq \mathbf{P}_M\beta \, , \quad \mathbf{P}_{\bigcup\mathbf{V}-M}\alpha \geqq \mathbf{P}_{\bigcup\mathbf{V}-M}\beta \Rightarrow \alpha \geqq \beta \, ; \tag{3.15.3}$$

if $\alpha + \beta$ *is defined, then*

$$\mathbf{P}_M(\alpha + \beta) = \mathbf{P}_M\alpha + \mathbf{P}_M\beta \, . \tag{3.15.4}$$

Proof. The Lemma follows from the definition of $\mathbf{P}_M$ immediately.

3.16. Theorem. *Let* $A \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, $M \in \mathbf{V}$. *Then*

$$\mathbf{P}_M \inf A = \inf \mathbf{P}_M(A) \tag{3.16.1}$$

and

$$\mathbf{P}_M \sup A = \sup \mathbf{P}_M(A) \, . \tag{3.16.2}$$

Hence, in particular, if $\{\alpha_i\}_{i=1}^{\infty} \subset \mathbf{n}_{+}^{*}(\mathbf{V}, \mathbf{V}_0)$, *then*

$$\mathbf{P}_M \sum_{i=1}^{\infty} \alpha_i = \sum_{i=1}^{\infty} \mathbf{P}_M \alpha_i \,. \tag{3.16.3}$$

Proof. Both members of (3.16.1) have a meaning. Let $\gamma \in \mathbf{P}_M(A)$. Then there exists a $\alpha \in A$ such that $\gamma = \mathbf{P}_M\alpha$. It is $\alpha \geqq \inf A$ and thus according to the preceding Lemma, $\gamma \geqq \mathbf{P}_M \inf A$. Hence $\mathbf{P}_M \inf A \leqq \inf \mathbf{P}_M(A)$. On the other hand, if

$$\gamma = \inf \mathbf{P}_M(A) > \mathbf{P}_M \inf A \quad \text{and} \quad \gamma = \mathbf{P}_M\beta \,, \quad \alpha = \inf A \,,$$

then $\beta \,.\, \chi_M > \alpha \,.\, \chi_M$ according to (3.15.2); thus

$$\beta \,.\, \chi_M + \alpha(1 - \chi_M) > \alpha = \inf A \quad \text{and} \quad \beta \,.\, \chi_M + \alpha(1 - \chi_M) \,(\leqq)\, A \,,$$

as it follows from (3.15.3). But this is impossible. Thus $\mathbf{P}_M \inf A = \inf \mathbf{P}_M(A)$. By duality $\mathbf{P}_M \sup A = \sup \mathbf{P}_M(A)$. We have proved (3.16.1) and (3.16.2). Since (3.16.3) follows from (3.15.4) and (3.16.2), the proof is complete.

4

In this paragraph three lemmas, which are more or less known, are stated for the convenience of the reader.

4.1. Notation. If A is a set, $\mathscr{A} \subset \mathbf{f}_{+}^{*}A$, then $\mathscr{A}_{+} = \{f; f = g + h,\, g \in \mathscr{A},\, h \in \mathscr{A}\}$, $\mathscr{A}_{\sigma+} = \{\sum_{i=1}^{\infty} f_i; \{f_i\}_{i=1}^{\infty} \subset \mathscr{A}\}$, $\mathscr{A}_{\times} = \{c \,.\, f; f \in \mathscr{A},\, c \in E_{+}\}$.

4.2. Lemma. *Let* $\mathbf{C}$ *be a pseudolattice,* $\mathscr{A} \subset \mathbf{f}_{+}^{*}(\bigcup \mathbf{C})$,

$$\mathscr{A}_{\sigma+} \subset \mathscr{A} \,, \quad \mathscr{A}_{-}(\mathbf{cC}) \subset \mathscr{A} \,, \quad \mathbf{cC} \subset \mathscr{A} \,. \tag{4.2.1}$$

Then

$$\mathscr{A} \supset \mathbf{csC} \,. \tag{4.2.2}$$

If even

$$\mathscr{A}_{\times} \subset \mathscr{A} \,,$$

then

$$\mathscr{A} \supset \mathbf{m}_{+}^{*}\mathbf{sC} \,. \tag{4.2.3}$$

Proof. Let us denote by $\mathbf{A}$ the system of all sets the characteristic functions of which are in $\mathscr{A}$. For $B \in \mathbf{C}$ let us denote by ${}_B\mathbf{C}$ the pseudolattice of all A, which satisfy $B \supset A \in \mathbf{C}$. Clearly $\mathbf{A} \supset {}_B\mathbf{C}$. Then from [4], § 5, Ex. (2), (3e) and (5) it follows that $\mathbf{A} \supset \mathbf{s}_B\mathbf{C}$. Thus

$$\mathbf{A} \supset \mathbf{B} = \bigcup \{\mathbf{s}_B\mathbf{C}; B \in \mathbf{C}\} \,,$$

where $\mathbf{B}$ is defined by the context. Now, if we define

$$\mathbf{D} = \{\sum_{i=1}^{\infty} A_i; A_i \in \mathbf{B},\, A_i \cap A_j = \emptyset \quad \text{for} \quad i \neq j\} \,,$$

then, since $\mathbf{B} \subset \mathbf{A}$ and $\mathscr{A}_{\sigma+} \subset \mathscr{A}$, we obtain $\mathbf{D} \subset \mathbf{A}$, or equivalently $\mathbf{cD} \subset \mathscr{A}$. But $\mathbf{D}$ is a σ-ring, $\mathbf{D} \supset \mathbf{C}$, and thus $\mathbf{D} \supset \mathbf{sC}$. (In fact $\mathbf{D} = \mathbf{sC}$.) We obtain $\mathscr{A} \supset \mathbf{cD} \supset \mathbf{csC}$ and (4.2.2) is proved.

Now suppose that $\mathscr{A}_\times \subset \mathscr{A}$. Let $f \in \mathbf{m}_+^*\mathbf{sC}$. Then there exists a sequence of $\mathbf{sC}$-simple functions f_n such that $\sum_{n=1}^{\infty} f_n = f$. But f_n are linear combinations of elements of $\mathbf{csC}$ and thus, since $\mathscr{A}_\times \subset \mathscr{A}$ and $\mathscr{A}_{\sigma+} \subset \mathscr{A}$, we obtain $\{f_n\} \subset\cdot \mathscr{A}$ and $\sum_{n=1}^{\infty} f_n = f \in \mathscr{A}$. Thus $\mathscr{A} \supset \mathbf{m}_+^*\mathbf{sC}$.

4.3. Lemma. *Let $\mathscr{F}$ be a basic system; denote*

$$\mathbf{F} = \{A;\, g_n \nearrow c_A,\, \{g_n\}_{n=1}^{\infty} \subset\cdot \mathscr{F}\}\,. \tag{4.3.1}$$

Then for every $f \in \mathscr{F}$, $c \in E_+$ we have

$$\{x;\, f(x) > c\} \in \mathbf{F} \tag{4.3.2}$$

and thus

$$\mathbf{k}\mathscr{F} = \mathbf{sF}\,. \tag{4.3.3}$$

Proof. The following proof is due to MAŘÍK [6]:

Let $f \in \mathscr{F}$, $c \in E_+$. Put

$$g_n = n\left[f \wedge \left(c + \frac{1}{n}\right) - f \wedge c\right].$$

Then $g_n \nearrow c_{\{x;\, f(x) > c\}}$, which proves (4.3.2). Hence it follows that $\mathbf{k}\mathscr{F} \subset \mathbf{sF}$. On the other hand $\mathbf{k}\mathscr{F} \supset \mathbf{F}$ and thus $\mathbf{k}\mathscr{F} = \mathbf{sF}$.

4.4. Lemma. *Let $\mathscr{B}$ be a basic system,*

$$\mathscr{A} \supset \mathscr{B}\,, \quad \mathscr{A}_{\sigma+} \subset \mathscr{A}\,, \quad \mathscr{A}_\times \subset \mathscr{A}\,, \quad \mathscr{A}_-(\mathscr{B}) \subset \mathscr{A}\,. \tag{4.4.1}$$

Then $\mathscr{A} \supset \mathbf{m}_+^*\mathbf{k}\mathscr{B}$.

Proof. Let $f \in \mathscr{B}$, $\mathbf{C}_f = \{A;\, g_n \nearrow c_A \leqq f,\, \{g_n\}_{n=1}^{\infty} \subset\cdot \mathscr{B}\}$. We have $\mathbf{cC}_f \subset \mathscr{A}$. Indeed, if $\{g_n\} \subset\cdot \mathscr{B}$, $g_n \nearrow c_A \leqq f$ and $h_1 = g_1$, $h_n = g_n - g_{n-1}$ for $n = 2, 3, \ldots$, then $h_n \in \mathscr{A}_-(\mathscr{B}) \subset \mathscr{A}$ and $c_A = \sum_{n=1}^{\infty} h_n \in \mathscr{A}_{\sigma+} \subset \mathscr{A}$. Clearly every $\mathbf{C}_f$ is a lattice; since $\mathscr{B}$ is a basic system, the union $\mathbf{C} = \bigcup\{\mathbf{C}_f;\, f \in \mathscr{B}\}$ is a lattice, too. We have $\mathbf{cC} \subset \mathscr{A}$ and we deduce easily that $\mathscr{A}_-(\mathbf{cC}) \subset \mathscr{A}$. Thus all the assumptions of Lemma 4.2 are satisfied, we get $\mathscr{A} \supset \mathbf{m}_+^*\mathbf{sC}$ and it remains to prove that $\mathbf{sC} \supset \mathbf{k}\mathscr{B}$.

Let $f \in \mathscr{B}$, $c \in E_+$. Then there exists a sequence $\{g_n\} \subset\cdot \mathscr{B}$ such that $g_n \nearrow c_{\{x;\, f(x) > c\}}$ (see Lemma 4.3). Since $g_n \leqq \frac{1}{c} \cdot f \in \mathscr{B}$, we have $\{x;\, f(x) > c\} \in \mathbf{C}_{\frac{1}{c}\cdot f} \subset \mathbf{C}$ and $\mathbf{k}\mathscr{B} \subset \mathbf{sC}$.

5. Functional, measure and outer measure
Definitions and principal properties

5.1. Definition. J is a *functional*, if J is a transformation, $\mathscr{D}J \subset \mathbf{f}^*X$, where X is a set (we shall write $X = \mathscr{D}^2 J$), $\mathscr{R}J \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, where $(\mathbf{V}, \mathbf{V}_0)$ is a measurable space. J is called *finite*, if $\mathscr{R}J \subset \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$; *additive*, if $\{f, g, f + g\} \subset\cdot \mathscr{D}J \Rightarrow J(f + g) = Jf + Jg$; *homogeneous*, if $\{f, c \,.\, f\} \subset\cdot \mathscr{D}J, c \in E \Rightarrow J(c \,.\, f) = c \,.\, Jf$; *linear*, if it is additive and homogeneous; continuous from below, if $\{f_n\}_{n=0}^{\infty} \subset\cdot \mathscr{D}J,\ f_n \nearrow f_0 \Rightarrow Jf_n \nearrow Jf_0$; *non negative*, if $f \in \mathscr{D}J,\ f \geqq 0 \Rightarrow Jf \geqq 0$; *subadditive*, if $\{f_1, f_2, f_1 \vee f_2\} \subset\cdot \mathscr{D}J \Rightarrow J(f_1 \vee f_2) \leqq Jf_1 + Jf_2$; *monotone*, if $\{f_1, f_2\} \subset\cdot \mathscr{D}J, f_1 \leqq f_2 \Rightarrow Jf_1 \leqq Jf_2$.

5.2. Definition. μ is called a *set function*, if $\mathscr{D}\mu$ is a system of sets, $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, where $(\mathbf{V}, \mathbf{V}_0)$ is a measurable space. We say that μ is *non negative*, if $\mathscr{R}\mu \subset \mathbf{n}_+^*(\mathbf{V}, \mathbf{V}_0)$; *monotone*, if $\{A, B\} \subset\cdot \mathscr{D}\mu,\ A \subset B \Rightarrow \mu(A) \leqq \mu(B)$; *σ-subadditive*, if $\{A_i\}_{i=1}^{\infty} \subset\cdot \mathscr{D}\mu,\ \bigcup_{i=1}^{\infty} A_i \in \mathscr{D}\mu \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) \leqq \sum_{i=1}^{\infty} \mu(A_i)$; *σ-additive*, if $\{A_i\}_{i=1}^{\infty} \subset\cdot \mathscr{D}\mu,\ \bigcup_{i=1}^{\infty} A_i \in \mathscr{D}\mu,\ A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$; *σ-finite*, if for every $A \in \mathscr{D}\mu$ there exists a sequence $\{A_i\}_{i=1}^{\infty} \subset\cdot \mathscr{D}\mu$ such that $\mu(A_i) \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$ for every i and $\bigcup_{i=1}^{\infty} A_i = A$. μ is called a *measure* if μ is a non negative σ-additive set function and if $\mathscr{D}\mu$ is a σ-ring.

5.3. Definition. $\boldsymbol{H}$ is a hereditary σ-ring, if $\boldsymbol{H}$ is a σ-ring and $A \subset B \in \boldsymbol{H} \Rightarrow A \in \boldsymbol{H}$.

5.4. Definition. μ^* is an outer measure, if μ^* is a non negative, monotone and σ-subadditive set function, if $\mathscr{D}\mu^*$ is a hereditary σ-ring and $\mu^*(\emptyset) = 0$.

5.5. Lemma. *Let J be a functional continuous from below, let $\mathscr{D}J$ be an f-lattice. Then there exists a unique functional $\bar{J}$ continuous from below such that $\bar{J} \succ J$ and $\mathscr{D}\bar{J} = \{h;\ f_n \nearrow h,\ f_n \in \mathscr{D}J\}$.*

Proof. Put

$$\bar{J}f = \lim_{n\to\infty} Jf_n\,, \tag{5.5.1}$$

if $f \in \mathscr{D}\bar{J}$, $\{f_n\}_{n=1}^{\infty} \subset\cdot \mathscr{D}J$, $f_n \nearrow f$. We shall show that this definition is independent of the choice of the particular sequence $\{f_n\}$. First we remark that continuity from below implies monotony. Now let $f_n \in \mathscr{D}J$, $g_n \in \mathscr{D}J$, $g_n \nearrow f$, $f_n \nearrow f$. We have $f_n \geqq f_n \wedge g_{n_0} \nearrow g_{n_0}$ and thus $\lim_{n\to\infty} Jf_n \geqq Jg_{n_0}$. Making $n_0 \to \infty$ we get $\lim_{n\to\infty} Jf_n \geqq \lim_{n\to\infty} Jg_n$ and from symmetry $\lim_{n\to\infty} Jf_n = \lim_{n\to\infty} Jg_n$. It remains to prove the continuity from below of $\bar{J}$. Let $f_{ni} \nearrow_i f_n$,[3] $f_{ni} \in \mathscr{D}J$, $f_n \nearrow f$. Put $g_n = f_{1n} \vee$

[3]) The index i in $\nearrow_i$ is used with the obvious meaning for preventing misunderstandings.

$\vee f_{2n} \vee \ldots \vee f_{nn}$. Then $g_n \in \mathscr{D}J$, $g_n \nearrow f$ and $g_n \leqq f_n$. Thus $Jg_n \leqq \bar{J}f_n \leqq \bar{J}f$ and $Jg_n \nearrow \bar{J}f$. Consequently $\bar{J}f_n \nearrow \bar{J}f$.

5.6. Notation. If J is a functional satisfying the conditions of Lemma 5.5, then we denote by $\bar{J}$ the functional defined by (5.5.1).

5.7. Lemma. *Let J be a functional continuous from below, let $\mathscr{D}J$ be an f-lattice. Then*

(5.7.1) *$\mathscr{D}\bar{J}$ is an f-lattice, $(\mathscr{D}\bar{J})_{\sigma\vee} \subset \mathscr{D}\bar{J}$,*

(5.7.2) *non negativity of J implies non negativity of $\bar{J}$,*

(5.7.3) *subadditivity of J implies subadditivity of $\bar{J}$,*

(5.7.4) *if J is additive, $(\mathscr{D}J)_+ \subset \mathscr{D}J$, then $\bar{J}$ is additive, $(\mathscr{D}\bar{J})_+ \subset \mathscr{D}\bar{J}$,*

(5.7.5) *if J is homogeneous, $(\mathscr{D}J)_\times \subset \mathscr{D}J$, then $\bar{J}$ is homogeneous and $(\mathscr{D}\bar{J})_\times \subset \mathscr{D}\bar{J}$.*

Proof obvious.

5.8. Theorem. *Let J be a non negative, homogeneous, subadditive and from below continuous functional, let $\mathscr{D}J$ be an f-lattice, let E_A, $\mathbf{H}$ and μ^* be defined as follows:*

$$A \subset \mathscr{D}^2J \Rightarrow E_A = \{\bar{J}g;\, c_A \leqq g \in \mathscr{D}\bar{J}\}\,, \tag{5.8.1}$$

$$\mathbf{H} = \{A;\, A \subset \mathscr{D}^2J,\ E_A \neq \emptyset\} \tag{5.8.2}$$

and

$$A \in \mathbf{H} \Rightarrow \mu^*(A) = \inf E_A\,. \tag{5.8.3}$$

Then μ^* is an outer measure.

Proof. $\mathbf{H}$ is obviously a hereditary σ-ring, $\mu^*(\emptyset) = 0$, $\mu^*(A) \leqq \mu^*(B)$ whenever $A \subset B$, $B \in \mathbf{H}$. It remains to prove the σ-subadditivity. We observe that every E_A is down oriented. Let $\{A_i\}_{i=1}^\infty \subset\cdot\, \mathbf{H}$ and let ξ be a pseudoprobability inducing $(\mathbf{V}, \mathbf{V}_0) = (\mathbf{q}\mathscr{R}J, \mathbf{q}_0\mathscr{R}J)$. Then there exist sequences (see Theorem 3.13)

$$\{\alpha_{ij}^{(m)}\}_{j=1}^\infty \subset\cdot\, E_{A_i}\ (i = 1, 2, \ldots;\ m = 1, 2, \ldots)$$

such that

$$\alpha_{ij}^{(m)}\ {}_j\!\searrow \mu^*(A_i)$$

and such that there exist sets M_{im}, M_i, satisfying the following relations:

$$M_{im} \in [\alpha_{i1}^{(m)}]^{-1}(\{+\infty\}]\,,\quad M_i \in [\mu^*(A_i)]^{-1}(\{+\infty\})\,,$$

$$M_{im} \supset M_{i,m+1}\,,\quad \xi(M_{im} - M_i) < \frac{1}{2^i m}\,.$$

We obtain $\xi(\bigcup_{i=1}^\infty M_{im} - \bigcup_{i=1}^\infty M_i) < \frac{1}{m}$ and $\xi(\bigcap_{m=1}^\infty \bigcup_{i=1}^\infty M_{im} - \bigcup_{i=1}^\infty M_i) = 0$. Accordingly, we may suppose, after modifying the sets M_{ij}, M_i by substraction of a set in $\mathbf{V}_0$, that $\bigcap_{m=1}^\infty \bigcup_{i=1}^\infty M_{im} = \bigcup_{i=1}^\infty M_i$. Defining $N_m = \bigcup \mathbf{V} - \bigcup_{i=1}^\infty M_{im}$ we obtain

$$\bigcup \mathbf{V} - \bigcup_{m=1}^\infty N_m = \bigcup_{i=1}^\infty M_i\,. \tag{5.8.4}$$

Now, fix an index m and denote $\alpha_{ij}^{(m)} = \alpha_{ij}$. Then $\mathbf{P}_{N_m}\alpha_{ij}$ are elements of the regular K-space

$$Y_m = \mathbf{n}({}_{N_m}\mathbf{V},\ {}_{N_m}\mathbf{V}_0)$$

and $\mathbf{P}_{N_m}\alpha_{ij}\ {}_j\!\searrow \mathbf{P}_{N_m}\mu^*(A_i)$ for every $i = 1, 2, \ldots$ (see 3.15, 3.16). Now the regularity of Y_m implies (see [5], Chapt. V, Theorem 1,25) the existence of a $\varrho \in Y_m$ satisfying the following condition: for every $\varepsilon > 0$ there exists a sequence of integers $n_1, n_2, \ldots$ such that

$$\mathbf{P}_{N_m}\alpha_{in_i} \leqq \mathbf{P}_{N_m}\mu^*(A_i) + \frac{\varepsilon}{2^i}\,\varrho\,, \quad i = 1, 2, \ldots. \tag{5.8.5}$$

However, for every i there exists a g_i such that

$$\alpha_{in_i} = \bar{J}g_i\,, \quad g_i \geqq c_{A_i}\,;$$

thus $\bar{J}(\bigvee_{i=1}^{\infty} g_i) \in E_{\bigcup_{i=1}^{\infty} A_i}$. Therefore, since $\bar{J}$ is subadditive and continuous from below,

$$\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \bar{J}(\bigvee_{i=1}^{\infty} g_i) \leqq \sum_{i=1}^{\infty} \bar{J}g_i = \sum_{i=1}^{\infty} \alpha_{in_i}\,.$$

We get (Lemma 3.15, Theorem 3.16)

$$\mathbf{P}_{N_m}\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \mathbf{P}_{N_m}\sum_{i=1}^{\infty} \alpha_{in_i} = \sum_{i=1}^{\infty} \mathbf{P}_{N_m}\alpha_{in_i}\,. \tag{5.8.6}$$

Thus, using (5.8.5),

$$\mathbf{P}_{N_m}\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \sum_{i=1}^{\infty} \mathbf{P}_{N_m}\mu^*(A_i) + \varepsilon\,.\,\varrho\,.$$

As $\varepsilon > 0$ was arbitrary, we get, using (3.15) and (3.16),

$$\mathbf{P}_{N_m}\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \mathbf{P}_{N_m}\sum_{i=1}^{\infty} \mu^*(A_i) \tag{5.8.7}$$

for every $m = 1, 2, \ldots$

Hence it is easy to see that

$$\mathbf{P}_{N}\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \mathbf{P}_{N}\sum_{i=1}^{\infty} \mu^*(A_i) \tag{5.8.8}$$

where $N = \bigcup_{m=1}^{\infty} N_m$. If $\bigcup \mathbf{V} - N \in \mathbf{V}_0$, then (5.8.8) holds for $N = \bigcup \mathbf{V}$ and the Theorem is proved. In the contrary case it follows from (5.8.4) that $\mathbf{P}_{\cup \mathbf{V} - N}\sum_{i=1}^{\infty} \mu^*(A_i) = +\infty$. This together with (5.8.8) gives $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leqq \sum_{i=1}^{\infty} \mu^*(A_i)$ and again the Theorem is proved.

5.9. Definition. We say that μ^* is $*$-*induced* by J, if μ^* and J satisfy the conditions of Theorem 5.8. We say that μ^* is $*$-induced by a non negative σ-additive set function μ, if μ^* is $*$-*induced* by J and $Jc_A = \mu(A)$ for every $A \in \mathscr{D}\mu$, $\mathscr{D}J = \mathbf{c}\mathscr{D}\mu$.

5.10. Definition. If μ^* is an outer measure, then any set $A \in \mathscr{D}\mu^*$, for which

$$B \in \mathscr{D}\mu^* \Rightarrow \mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A)\,, \qquad (5.11.1)$$

is called μ^*-measurable.

5.11. Definition. μ is a *complete measure,* if it is a measure and if $B \subset A \in \mathscr{D}\mu$, $\mu(A) = 0 \Rightarrow B \in \mathscr{D}\mu$.

5.12. Theorem. *If μ^* is an outer measure, $\mathbf{S}$ the system of all μ^*-measurable sets, then $\mathbf{S}$ is a σ-ring and $\mu^*_{\mathbf{S}}$ is a complete measure.*

Proof. The Theorem can be proved in a way formally identical with that of [4], § 11.

5.13. Theorem. *Let μ^* be an outer measure $*$-induced by an additive functional J, let $\mathscr{D}J$ be an f-ring. Then $c_A \in \mathscr{D}\bar{J}$ implies the μ^*-measurability of A and*

$$\mu^*(A) = \bar{J}c_A\,. \qquad (5.13.1)$$

Proof. (5.13.1) follows immediately from (5.8.3) and it remains to prove the μ^*-measurability of A. Let $B \in \mathscr{D}\mu^*$. Theorem 3.13 implies the existence of $\{\beta_i\}_{i=1}^{\infty} \subset\cdot E_B$ such that $\beta_i \searrow \mu^*(B)$. Choose a sequence $M_1 \supset M_2 \supset \ldots$ such that $M_i \in \beta_i^{-1}(\{+\infty\})$. Fix an integer m. Denote $(\mathbf{V}, \mathbf{V}_0) = (\mathbf{q}\mathscr{R}J, \mathbf{q}_0\mathscr{R}J)$, put $N_m = \bigcup \mathbf{V} - M_m$ and denote $Y_m = \mathbf{n}({}_{N_m}\mathbf{V}, {}_{N_m}\mathbf{V}_0)$. Then $\{\mathbf{P}_{N_m}\beta_i\}_{i=m}^{\infty} \subset\cdot Y_m$ and Y_m is a regular K-space. Thus, according to [5], Chapt. V, Theorem 1.24, there exists a $\varrho_1 \in Y_m$ satisfying the following condition: for every $\varepsilon > 0$ there exists an index n_0 such that $\mathbf{P}_{N_m}\beta_{n_0} \leqq \mathbf{P}_{N_m}\mu^*(B) + \varepsilon\,.\,\varrho_1$.

We have $\beta_{n_0} = \bar{J}g$, where $g \geqq c_B$, $g \in \mathscr{D}\bar{J}$. Since c_A belongs to $\mathscr{D}\bar{J}$ too, there exist $\{g_n\}_{n=1}^{\infty} \subset\cdot \mathscr{D}J$, $\{h_n\}_{n=1}^{\infty} \subset\cdot \mathscr{D}J$ such that

$$g_n \nearrow g \geqq c_B\,, \quad h_n \nearrow c_A\,.$$

Thus also $g_n \wedge h_n \nearrow g \wedge c_A$, $J(g_n \wedge h_n) \nearrow \bar{J}(g \wedge c_A)$ and

$$\mathbf{P}_{N_m}J(g_n \wedge h_n) \nearrow_n \mathbf{P}_{N_m}\bar{J}(g \wedge c_A)\,.$$

This is again the convergence in Y_m and thus there exists a $\varrho_2 \in Y_m$ satisfying the following condition: for every $\delta > 0$ there exists an integer k such that

$$\mathbf{P}_{N_m}\bar{J}(g \wedge c_A) \leqq \mathbf{P}_{N_m}J(g_k \wedge h_k) + \delta\,.\,\varrho_2\,.$$

Thus $\mathbf{P}_{N_m}\mu^*(A \cap B) \leqq \mathbf{P}_{N_m}J(g_k \wedge h_k) + \delta\,.\,\varrho_2$ for $c_{A\cap B} \leqq g \wedge c_A$. But $c_{B-A} \leqq$ $\leqq g - g \wedge c_A \leqq g - g_k \wedge h_k \in \mathscr{D}\bar{J}$. Thus

$$\begin{aligned}\mathbf{P}_{N_m}[\mu^*(B \cap A) + \mu^*(B - A)] &= \mathbf{P}_{N_m}\mu^*(B \cap A) + \mathbf{P}_{N_m}\mu^*(B - A) \leqq \\ &\leqq \mathbf{P}_{N_m}J(g_k \wedge h_k) + \delta\,.\,\varrho_2 + \mathbf{P}_{N_m}\bar{J}(g - g_k \wedge h_k) = \\ &= \mathbf{P}_{N_m}\bar{J}g + \delta\,.\,\varrho_2 \leqq \mathbf{P}_{N_m}\mu^*(B) + \delta\,.\,\varrho_2 + \varepsilon\,.\,\varrho_1\,.\end{aligned}$$

Making first $\delta \to 0$ and then $\varepsilon \to 0$, we obtain $\mathbf{P}_{N_m}(\mu^*(B \cap A) + \mu^*(B - A)) \leqq$ $\leqq \mathbf{P}_{N_m}\mu^*(B)$ for every m and thus

$$\mathbf{P}_{\bigcup_{m=1}^{\infty} N_m}[\mu^*(B \cap A) + \mu^*(B - A)] \leqq \mathbf{P}_{\bigcup_{m=1}^{\infty} N_m}\mu^*(B)\,.$$

Finally

$$\mathbf{P}_{\cup \mathbf{V} - \bigcup_{m=1}^{\infty} N_m} \mu^*(B) = +\infty \quad \text{or} \quad \bigcup \mathbf{V} - \bigcup_{m=1}^{\infty} N_m \in \mathbf{V}_0$$

and thus $\mu^*(B \cap A) + \mu^*(B - A) \leqq \mu^*(B)$. Since μ^* is subadditive, (5.11.1) holds and the proof is finished.

5.14. Definition. Let μ be a measure. Then $\bar{\mu}$ is called the *completion* of μ, if $\bar{\mu}$ is a complete measure, $\bar{\mu} \succ \mu$, and if to every $A \in \mathscr{D}\bar{\mu}$ there exist $M \subset M_1 \in \mathscr{D}\mu$, $N \subset N_1 \in \mathscr{D}\mu$, $A_1 \in \mathscr{D}\mu$ such that

$$A = (A_1 - M) \cup N\,, \quad \mu(M_1) = \mu(N_1) = 0\,.$$

5.15. Theorem. *Let μ be a non negative σ-additive set function, let $\mathscr{D}\mu$ be a ring, μ^* the outer measure $*$-induced by μ, $\mathbf{S}$ the σ-ring of all μ^*-measurable sets, $\nu = \mu^*_{\mathbf{S}}$.*

Then ν is a measure and $\nu \succ \mu$.

If, in addition, μ is σ-finite and ν_1 is a measure defined on $\mathbf{s}\mathscr{D}\mu$, $\nu_1 \succ \mu$, then ν is the completion of ν_1 and ν is σ-finite.

Proof. The first assertion of the Theorem follows from Theorem 5.8, if we put $Jc_A = \mu(A)$ for every $A \in \mathscr{D}\mu$, from Theorem 5.12, which shows that ν is a measure, and from Theorem 5.13, which shows $\nu \succ \mu$. The other assertions are easy to prove in a way commonly used for the case of real measure ([4], Sec. 13).

5.16. The following two Theorems are easy consequences of Lemmas 4.2 and 4.4.

Theorem. *Let $\mathscr{B}$ be a basic system or let $\mathscr{B} = \mathbf{cC}$, where $\mathbf{C}$ is a pseudolattice. Let J_1, J_2 be two non negative, linear and from below continuous functionals defined on $\mathbf{m}^*_+\mathbf{k}\mathscr{B}$ and finite on $\mathscr{B}$. Let J_1 and J_2 agree on $\mathscr{B}$.*

Then $J_1 = J_2$.

Proof. Let $\mathscr{A} = \{f;\, J_1 f = J_2 f\}$. Then $\mathscr{A} \supset \mathscr{B}$, $\mathscr{A}_{\sigma+} \subset \mathscr{A}$, $\mathscr{A}_\times \subset \mathscr{A}$. If $0 \leqq f_1 \leqq \leqq f_2 \leqq f \in \mathscr{B}$, $f_1 \in \mathscr{A}$, $f_2 \in \mathscr{A}$, then $J_1 f_1 = J_2 f_1$, $J_1 f_2 = J_2 f_2$ and both these random variables are finite, since $J_i f$ is so. Thus $J_1(f_2 - f_1) = J_1(f_2 - f_1) + + J_1 f_1 - J_1 f_1 = J_1 f_2 - J_1 f_1 = J_2 f_2 - J_2 f_1 = J_2(f_2 - f_1)$; we obtain $\mathscr{A}_-(\mathscr{B}) \subset \mathscr{A}$.

We may apply Lemma 4.2 (if $\mathscr{B} = \mathbf{cC}$) or Lemma 4.4 (if $\mathscr{B}$ is a basic system). We get $\mathscr{A} \supset \mathbf{m}^*_+\mathbf{k}\mathscr{B}$, i. e. $J_1 = J_2$.

5.17. Theorem. *Let $\mathscr{B}_i$ be a basic system or let $\mathscr{B}_i = \mathbf{cC}_i$, where $\mathbf{C}_i$ is a pseudolattice ($i = 1, 2$). Let $\overline{\mathscr{B}}_1 \supset \mathscr{B}_1$ and $[\overline{\mathscr{B}}_1]_{\sigma+} = \mathbf{m}^*_+\mathbf{k}\mathscr{B}_1$.*

Let F_j (for $j = 1, 2$) be a transformation defined on $\mathbf{m}^*_+\mathbf{k}\mathscr{B}_1 \times \mathbf{m}^*_+\mathbf{k}\mathscr{B}_2$. For every $[f, g] \in \mathbf{m}^*_+\mathbf{k}\mathscr{B}_1 \times \mathbf{m}^*_+\mathbf{k}\mathscr{B}_2$ let both $F_j(f, .)$ and $F_j(., g)$ are non negative, linear and from below continuous functionals. Let F_1 be finite on $\overline{\mathscr{B}}_1 \times \mathscr{B}_2$.

Let F_1 and F_2 agree on $\mathscr{B}_1 \times \mathscr{B}_2$.
Then $F_1 = F_2$.

Proof. The proof consists in a repeated application of the preceding Theorem.

Let $g \in \mathscr{B}_2$ be fixed. Then $F_1(.\,, g)$, $F_2(.\,, g)$ are two non negative, linear and from below continuous functionals which agree and are finite on $\mathscr{B}_1$. Thus according to the preceding Theorem, $F_1(.\,, g) = F_2(.\,, g)$; in particular F_1 and F_2 agree and are finite on $\overline{\mathscr{B}}_1 \times \mathscr{B}_2$.

Now let $f_1 \in \overline{\mathscr{B}}_1$ be fixed. A new application of the preceding Theorem shows that $F_1(f, .) = F_2(f, .)$ and thus F_1, F_2 agree on $\overline{\mathscr{B}}_1 \times \mathbf{m}_+^*\mathbf{k}\mathscr{B}_2$.

Finally let $[f, g] \in \mathbf{m}_+^*\mathbf{k}\mathscr{B}_1 \times \mathbf{m}_+^*\mathbf{k}\mathscr{B}_2$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset\cdot$ $\subset\cdot \overline{\mathscr{B}}_1$ such that $f = \sum_{n=1}^{\infty} f_n$. Thus from the additivity and continuity from below it follows that $F_1(f, g) = \sum_{n=1}^{\infty} F_1(f_n, g) = \sum_{n=1}^{\infty} F_2(f_n, g) = F_2(f, g)$ and the Theorem is proved.

6. The weak integral

6.1. In the whole section let μ be a measure.

6.2. Lemma. *There exists a unique linear functional J such that $\mathscr{D}J$ consists of all $\mathscr{D}\mu$-simple functions and such that $Jc_A = \mu(A)$ for every $A \in \mathscr{D}\mu$. The functional J is non negative, linear and continuous from below; if $a_i \in E_+$, $A_i \in \mathscr{D}\mu$, then*

$$J \sum_{i=1}^{n} a_i \,.\, c_{A_i} = \sum_{i=1}^{n} a_i \,.\, \mu(A_i)\,. \tag{6.2.1}$$

Proof. From additivity of μ it follows that J can be unambiguously defined by (6.2.1). Then J is non negative and linear. Conversely, if J is linear and $Jc_A = \mu(A)$ for every $A \in \mathscr{D}\mu$, then J must be of the form (6.2.1). It remains to prove that J is continuous from below.

Let f_i be $\mathscr{D}\mu$-simple, $f_i \nearrow h = \sum_{i=1}^{k} a_i \,.\, c_{A_i}$, $a_i \in E_+$, $A_i \in \mathscr{D}\mu$. We may suppose that $0 < a_1 < a_2 < \ldots < a_k$. Fix an $m > \dfrac{1}{a_1}$ and put $h_m = \sum_{i=1}^{k} \left(a_i - \dfrac{1}{m}\right) . \, c_{A_i}$. Clearly h_m is simple too. If $Q_n = \{x; x \in \bigcup_{i=1}^{k} A_i, f_n(x) > h_m(x)\}$, then

$$Q_n \in \mathscr{D}\mu\,, \quad Q_n \subset Q_{n+1} \to \bigcup_{i=1}^{k} A_i\,.$$

Thus

$$Jf_n \geqq Jc_{Q_n} \, . \, f_n \geqq J\left(\sum_{i=1}^{k}\left(a_i - \frac{1}{m}\right) . \, c_{Q_n \cap A_i}\right) =$$

$$= \sum_{i=1}^{k}\left(a_i - \frac{1}{m}\right) . \, \mu(Q_n \cap A_i) \nearrow_n \sum_{i=1}^{k}\left(a_i - \frac{1}{m}\right) . \, \mu(A_i) = Jh_m \, .$$

It follows that $\lim_{n\to\infty} Jf_n \geqq \lim_{n\to\infty} Jh_n$. On the other hand

$$\lim_{m\to\infty} Jh_m = \lim \sum_{i=1}^{k}\left(a_i - \frac{1}{m}\right) . \, \mu(A_i) = \sum_{i=1}^{k} a_i \, . \, \mu(A_i) = Jh \, .$$

Thus $\lim Jf_n \geqq Jh$ and, since $Jf_n \leqq Jh$ for every n, we get $Jf_n \nearrow Jh$.

6.3. Definition. Let J be defined by (6.2.1). We define, for every $f \in \mathscr{D}\bar{J}$, the weak integral of f with respect to μ by the relation

$$\int f \, \mathrm{d}\mu = \bar{J} f \, . \tag{6.3.1}$$

6.4. Theorem. *The weak integral $\int . \, \mathrm{d}\mu$ is a non negative, linear, continuous from below functional defined on $\mathbf{m}_+^* \mathscr{D}\mu$.*

Proof. The Theorem is a consequence of Lemmas 6.2, 5.7 and 5.5.

6.5. Theorem. *Let J be a non negative, linear and from below continuous functional defined on a basic system $\mathscr{D}J$. Let μ^* be $*$-induced by J, let $\mu = \mu^*_{\mathbf{k}\mathscr{D}J}$.*

Then μ is a measure and

$$\int . \, \mathrm{d}\mu \succ J \, . \tag{6.5.1}$$

If J is finite, then μ is the unique measure on $\mathbf{k}\mathscr{D}J$ satisfying (6.5.1).

Proof. Putting $\mathscr{D}J = \mathscr{F}$ and using the notation of Lemma 4.3 we obtain $\mathbf{k}\mathscr{D}J = \mathbf{sF}$. Let $\mathbf{S}$ be the system of all μ^*-measurable sets. From Theorem 5.13 it follows that $\mathbf{F} \subset \mathbf{S}$; as $\mathbf{S}$ is a σ-ring, we have $\mathbf{k}\mathscr{D}J = \mathbf{sF} \subset \mathbf{S}$. Hence and from Theorem 5.12 it follows that μ is a measure. Using (5.13.1) we obtain

$$A \in \mathbf{F} \Rightarrow \mu(A) = \mu^*(A) = \bar{J} c_A \, . \tag{6.5.2}$$

Now let $f \in \mathscr{D}J$. For every $a \in E_+$ the set $M_a = \{x; f(x) > a\}$ belongs to $\mathbf{F}$, $c_{M_a} \in \mathscr{D}\bar{J}$ (Lemma 4.3) and $\bar{J} c_{M_a} = \mu(M_a)$ ((6.5.2)).

Put

$$g_{nm} = \frac{m}{n} \, . \, c_{M_{\frac{m}{n}}} \, , \qquad g_n = \bigvee_{m=1}^{\infty} g_{nm} \, .$$

We have $c_{M_{\frac{m}{n}}} \in \mathscr{D}\bar{J}$ and, for $\mathscr{D}J$ is a basic system, $g_{nm} \in \mathscr{D}\bar{J}$, $g_n \in \mathscr{D}\bar{J}$ ((5.7.1), (5.7.5)). Further $g_n = \sum_{m=1}^{\infty} \frac{1}{n} \, . \, c_{M_{\frac{m}{n}}}$ and, from the linearity and continuity from below of $\bar{J}$ and of $\int . \, \mathrm{d}\mu$ (Lemmas 5.5 and 5.7),

$$\bar{J} g_n = \frac{1}{n} \sum_{m=1}^{\infty} \bar{J} c_{M_{\frac{m}{n}}} = \frac{1}{n} \sum_{m=1}^{\infty} \mu(M_{\frac{m}{n}}) = \int g_n \, \mathrm{d}\mu \, .$$

Again, since $g_{2^k} \nearrow f$, $Jf = \lim_{k\to\infty} Jg_{2^k} = \lim_{k\to\infty} \int g_{2^k}\, d\mu = \int f\, d\mu$. The unicity of the measure μ in the case J is finite follows easily from Theorem 5.16. Indeed, if μ and ν are measures defined on $\mathbf{k}\mathscr{D}J$ and $\int f\, d\mu = \int f\, d\nu = Jf$ for every $f \in \mathscr{D}J$, then $\int .\, d\mu = \int .\, d\nu$; hence it follows that $\mu = \nu$.

6.6. Definition. We say that a measure μ is *induced by a functional* J, if J is non negative, linear and continuous from below, if $\mathscr{D}J$ is a basic system, $\mathscr{D}\mu = \mathbf{k}\mathscr{D}J$ and $\int .\, d\mu \succ J$.

6.7. Theorem. *Let μ be a measure, let $\{f_n\}_{n=0}^{\infty} \subset \mathbf{m}_+^*\mathscr{D}\mu$, $A \in \mathscr{D}\mu$, $\mu(A)$ finite, $g \in \mathbf{m}_+^*\mathscr{D}\mu$, $\int g\, d\mu$ finite. Then*

$$f_n \,.\, c_A \to f_0 \,.\, c_A \quad \textit{uniformly} \Rightarrow \int f_n \,.\, c_A\, d\mu \to \int f_0 \,.\, c_A\, d\mu\,; \tag{6.7.1}$$

$$f_n \leqq g\,,\ n = 1, 2, \ldots, f_n \to f_0 \Rightarrow \int f_n\, d\mu \to \int f_0\, d\mu\,. \tag{6.7.2}$$

Proof. The Theorem is a consequence of the linearity and continuity from below of $\int .\, d\mu$.

6.8. Definition. A functional J is called *σ-finite*, if for every $f \in \mathscr{D}J$ there exists such a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{D}J$ that Jf_n are finite and $f_n \nearrow f$.

6.9. Theorem. *Let μ be a measure. Then the following three propositions are mutually equivalent:*

$$\mu \textit{ is } \sigma\textit{-finite}\,, \tag{6.9.1}$$

$$\int .\, d\mu \textit{ is } \sigma\textit{-finite}\,, \tag{6.9.2}$$

$$\mu \textit{ is induced by a finite functional}\,. \tag{6.9.3}$$

Proof. Let J be a finite functional inducing μ, let $A \in \mathscr{D}\mu$. Then $\mu(A) = \inf E_A$, $E_A \neq \emptyset$, where E_A is defined by (5.8.1) (see Theorems 5.8 and 6.5). Thus there exists a non-decreasing sequence $\{g_n\}_{n=1}^{\infty} \subset \mathscr{D}J$ such that $\bigvee_{n=1}^{\infty} g_n \geqq c_A$. If we put $B_n = A \cap \{x; g_n(x) > \frac{1}{2}\}$, then $\bigcup_{n=1}^{\infty} B_n = A$. For, if $x \in A$, then there exists an index n_0 such that $g_{n_0}(x) > \frac{1}{2}$; thus $x \in B_{n_0}$. Now $\mu(B_n)$ are finite. Indeed,

$$\mu(B_n) \leqq \mu(\{x; g_n(x) > \tfrac{1}{2}\}) \leqq \int 2 \,.\, g_n\, d\mu = 2 \,.\, Jg_n\,.$$

We have proved (6.9.3) ⇒ (6.9.1).

Let μ be σ-finite, let $f \in \mathbf{m}_+^*\mathscr{D}\mu$; then there exists a sequence of sets B_n in $\mathscr{D}\mu$ the measure of which is finite and the union of which is equal to $\{x; f(x) > 0\}$. Then the sequence $f_n = n \wedge f \,.\, c_{B_n}$ has the properties required in Definition 6.8. Thus $\int .\, d\mu$ is σ-finite and (6.9.1) ⇒ (6.9.2).

If $\int .\, d\mu$ is σ-finite and if $\mathscr{L}$ is the system of all such $f \in \mathbf{m}_+\mathscr{D}\mu$ that $\int f\, d\mu$ is finite, then $\mathscr{L}$ is a basic system and $[\int .\, d\mu]_{\mathscr{L}}$ induces μ. Thus (6.9.2) ⇒ (6.9.3).

7. The strong measure

7.1. Notation. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are σ-rings, then we denote

$$\mathbf{S}_1 \circ \mathbf{S}_2 = \{A \times B; A \in \mathbf{S}_1, B \in \mathbf{S}_2\}\,, \quad \mathbf{S}_1 \mathbf{x}\, \mathbf{S}_2 = \mathbf{s}(\mathbf{S}_1 \circ \mathbf{S}_2)\,.$$

If μ is a measure, $\mathbf{V}$ a σ-ring, we denote $\mathbf{V}_\mu = \mathscr{D}\mu \,\mathbf{x}\, \mathbf{V}$.

7.2. Notation. If f is a function, Ω is a set, then by f^Ω we denote the function defined on $\mathscr{D}f \times \Omega$ by the relation $f^\Omega(x, \omega) = f(x)$ for every $x \in \mathscr{D}f$, $\omega \in \Omega$.

Similarly by ${}^\Omega f$ we denote the function defined on $\Omega \times \mathscr{D}f$ and such that ${}^\Omega f(\omega, x) = f(x)$ for every $\omega \in \Omega$, $x \in \mathscr{D}f$.

7.3. Remark. The purpose of the remark is to motivate the definitions of this section.

In the preceding section we have defined the weak integral for functions, the values of which are real numbers. But since the values of our measure are random variables, it seems to be natural to define an integral for functions, the values of which are random variables, too. Unfortunately this way leads to an integral with little useful properties, as we shall see in (7.17). Thus we shall proceed in a somewhat different way, which will be shown to be more successful.

To fix the ideas, let μ be a measure, $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$; we shall try to extend the domain of the weak integral to the class $\mathfrak{M}$ with the following properties: 1. The elements of $\mathfrak{M}$ are functions defined on $\bigcup \mathscr{D}\mu$ with values in $\mathbf{m}_+^*\mathbf{V}$. (If $f \in \mathfrak{M}$, $x \in \bigcup \mathscr{D}\mu$, $v \in \bigcup \mathbf{V}$, then $fx \in \mathbf{m}_+^*\mathbf{V}$, $fxv \in E_+^*$.) 2. $\mathfrak{M}$ contains all non negative $(\mathscr{D}\mu)$ measurable real-valued functions, the values of which are regarded as constant functions. (We note that constant functions are $(\mathbf{V})$ measurable for $\mathbf{V}$ is a σ-algebra.) 3. $\mathfrak{M}_{\sigma+} \subset \mathfrak{M}$; if $f \in \mathfrak{M}$, $g \in \mathfrak{M}$, $f - g$ has *a* meaning, then $(f - g)_+ \in \mathfrak{M}$. 4. $g \in \mathbf{m}_+^*\mathbf{V}$, $f \in \mathfrak{M}$, h is a function, $\mathscr{D}h = \bigcup \mathscr{D}\mu$, $\mathscr{R}h \subset \mathbf{m}_+^*\mathbf{V}$, $hxv = gv\,.\,fxv \Rightarrow h \in \mathfrak{M}$. 5. $\mathfrak{M}$ is the smallest class satisfying the conditions already listed.

The conditions 1., 2., 3. and 5. have an obvious meaning. The condition 4. corresponds to the fact that, if $\mathbf{S}$ is a σ-ring, then $f \in \mathbf{m}_+^*\mathbf{S}$, $g \in E_+^* \Rightarrow g\,.\,f \in \mathbf{m}_+^*\mathbf{S}$.

Now it is easy to see that instead of considering a function f such that $\mathscr{D}f = \bigcup \mathscr{D}\mu$ and $\mathscr{R}f \subset \mathbf{m}_+^*\mathbf{V}$ it is possible and less complicated to consider the real valued function $\tilde{f}$ defined on $\bigcup \mathscr{D}\mu \times \bigcup \mathbf{V}$ and satisfying the relation $\tilde{f}(x, v) = fxv$. In this language it is easy to see that the class $\mathfrak{M}$ (or, more precisely, the image of $\mathfrak{M}$) is equal to the class $\mathbf{m}_+^*(\mathscr{D}\mu \,\mathbf{x}\, \mathbf{V})$.

Now in the extension of the domain of the weak integral (which may be supposed to be $\mathbf{m}_+^*(\mathscr{D}\mu \,\mathbf{x}\, \{\bigcup \mathbf{V}, \emptyset\})$) to $\mathbf{m}_+^*(\mathscr{D}\mu \,\mathbf{x}\, \mathbf{V})$, the following homogeneity condition will be essential. If $g \in \mathbf{m}_+^*\mathbf{V}$ and $f \in \mathbf{m}_+^*(\mathscr{D}\mu \,\mathbf{x}\, \mathbf{V})$, $X = \bigcup \mathscr{D}\mu$, $\Omega = \bigcup \mathbf{V}$, then $J({}^X g\,.\,f) = \bar{g}\,.\,Jf$, where J denotes the integral and $\bar{g}$ is the random variable containing g. Thus starting with the definition $Jc_{A \times \Omega} = \mu(A)$

for $A \in \mathscr{D}\mu$, we have by the homogeneity condition $Jc_{A\times B} = J^{x}c_{B} \,.\, c_{A\times\Omega} = \chi_{B} \,.\, \mu(A)$ and the further extension from $\mathbf{c}(\mathscr{D}\mu \circ \mathbf{V})$, if it is possible, is determined by additivity and continuity.

Perhaps it is convenient to say something else about the meaning of the homogeneity condition. If we notice that in the analogy between our measure and the real measure the random variables and $(\mathbf{V})$ measurable functions play the rôle of the real numbers, we may regard our homogeneity condition as analogical to the usual homogeneity.

Now we are not able to prove in general the existence of an integral with the properties mentioned above. Moreover, the σ-algebra $\mathbf{V}$ in the above considerations is not uniquely determined by the measure μ (and it would be unreasonable to put $\mathbf{V} = \mathbf{q}\mathscr{R}\mu$). Thus if μ is given, we shall consider the extension of the weak integral to the system $\mathbf{m}_{+}^{*}(\mathscr{D}\mu \times \mathbf{W})$, where $\mathbf{W}$ is a σ-algebra of subsets of $\Omega = \bigcup \mathbf{q}\mathscr{R}\mu$. Let us rewrite the homogeneity condition: if $f \in \mathbf{m}_{+}^{*}(\mathscr{D}\mu \times \mathbf{W})$, $g \in \mathbf{m}_{+}^{*}\mathbf{W}$, then $J({}^{x}g \,.\, f) = \bar{g} \,.\, h$, where $\bar{g}$ is the random variable containing g. However, we must define what it means "the random variable containing g" and what is the meaning of the multiplication $\bar{g} \,.\, Jf$. This can be made, if we require that there exists a measurable space $(\mathbf{V}, \mathbf{V}_0)$ such that $\mathscr{R}J \subset \mathbf{n}_{+}^{*}(\mathbf{V}, \mathbf{V}_0)$ and such that $\mathbf{W} \subset \mathbf{V}$. Then we can define $\bar{g}$ by the relation $g \in \bar{g} \in \mathbf{n}_{+}^{*}(\mathbf{V}, \mathbf{V}_0)$, which has a meaning, since $g \in \mathbf{W} \subset \mathbf{V}$. Further we can define the multiplication $\bar{g} \,.\, Jf$ as the multiplication in $\mathbf{n}_{+}^{*}(\mathbf{V}, \mathbf{V}_0)$. We note that $\mathbf{V}_0 = \mathbf{q}_0\mathscr{R}\mu$, since the hereditary σ-ring $\mathbf{V}_0$ is determined by the random variable $J0 = \mu(\emptyset)$. Thus the relation $g \in \bar{g} \in \mathbf{n}^{*}(\mathbf{V}, \mathbf{V}_0)$ for a function $g \in \mathbf{W}$ does not depend on the particular choice of the measurable space $(\mathbf{V}, \mathbf{V}_0)$ (i. e., on the particular choice of the σ-algebra $\mathbf{V}$), since it holds if and only if $\bar{g}$ is the class of all functions measurable $\{\mathbf{s}(\mathbf{W} \cup \mathbf{V}_0)\}$, which are $(\mathbf{V}_0)$ equivalent with g. If g is finite then $\bar{g} = \{g + \Theta;\, \Theta \in \mu(\emptyset)\}$.

7.4. Definition. A functional J is called a $\mathbf{W}$-integral with respect to a measure μ, if $\mathbf{W}$ is a σ-algebra, $\bigcup \mathbf{W} = \bigcup \mathbf{q}\mathscr{R}\mu$, J is non negative, linear, continuous from below, defined on $\mathbf{m}_{+}^{*}\,\mathbf{W}_{\mu}$ and if the following conditions hold (we denote $X = \bigcup \mathscr{D}\mu$, $\Omega = \bigcup \mathbf{W}$):

There exists a measurable space $(\mathbf{V}, \mathbf{V}_0)$ such that $\mathscr{R}J \subset \mathbf{n}_{+}^{*}(\mathbf{V}, \mathbf{V}_0)$, $\mathbf{W} \subset \mathbf{V}$; (7.4.1)

$$A \in \mathscr{D}\mu \Rightarrow Jc_{A}^{\Omega} = \mu(A)\,; \tag{7.4.2}$$

$$f \in \mathbf{m}_{+}^{*}\,\mathbf{W}_{\mu}\,, \quad g \in \mathbf{m}_{+}^{*}\,\mathbf{W}\,, \quad g \in \bar{g} \in \mathbf{n}^{*}(\mathbf{V}, \mathbf{V}_0) \Rightarrow J({}^{x}g \,.\, f) = \bar{g} \,.\, Jf\,. \tag{7.4.3}$$

7.5. Remark. If $\mathbf{W} = \{\emptyset, \bigcup \mathbf{q}\mathscr{R}\mu\}$, $Jf^{\cup \mathbf{W}} = \int f \,\mathrm{d}\mu$ for $f \in \mathbf{m}_{+}^{*}\mathscr{D}\mu$, then J is the unique $\mathbf{W}$-integral with respect to μ.

Further we note that, if $\mathbf{W}$ is a σ-algebra, $\bigcup \mathbf{W} = \bigcup \mathbf{q}\mathscr{R}\mu$, $\mathbf{W} \subset \mathbf{Z}$ and a $\mathbf{Z}$-integral with respect to the measure μ exists, then a $\mathbf{W}$-integral with respect to μ exists, too. For, if J is a $\mathbf{Z}$-integral, then $J_{\mathbf{m}_{+}^{*}\mathbf{W}_{\mu}}$ is a $\mathbf{W}$-integral.

Finally we remark that if J is a $\mathbf{W}$-integral with respect to a σ-finite measure μ, if $(\mathbf{Z}, \mathbf{Z}_0)$ is a measurable space such that $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{Z}, \mathbf{Z}_0)$ and $\mathbf{W} \subset \mathbf{Z}$, then $\mathscr{R}J \subset \mathbf{n}^*_+(\mathbf{Z}, \mathbf{Z}_0)$. This can be proved as follows. Let $\mathscr{A}$ denote the system of all such functions f which belong to $\mathscr{D}J$ and for which $Jf \in \mathbf{n}^*(\mathbf{Z}, \mathbf{Z}_0)$. Then from (7.4.2) and (7.4.3) it follows that $\mathscr{A} \supset \mathbf{c}[\mathscr{D}\mu \circ \mathbf{W}] \supset \mathbf{c}[\mathbf{S} \circ \mathbf{W}]$, where $\mathbf{S}$ is the system of all such sets $A \in \mathscr{D}\mu$ for which $\mu(A)$ is finite. An application of Lemma 4.2 gives the desired result: $\mathscr{A} = \mathbf{m}^*_+\mathbf{s}[\mathbf{S} \circ \mathbf{W}] = \mathbf{m}^*_+[\mathscr{D}\mu \times \times \mathbf{W}] = \mathscr{D}J$.

7.6. Theorem. *If μ is a σ-finite measure, $\mathbf{W}$ is a σ-algebra, then there exists at most one $\mathbf{W}$-integral with respect to μ.*

Proof. Let J_1 and J_2 be two $\mathbf{W}$-integrals with respect to μ; denote $\Omega = = \bigcup \mathbf{W}$. Let $\mathbf{S}$ be the system of all such sets $A \in \mathscr{D}\mu$ for which $\mu(A)$ is finite. Let $A \in \mathbf{S}$, $B \in \mathbf{W}$. Then $J_1 c_{A \times B} = \chi_B \,.\, J_1 c_A^\Omega = \chi_B \,.\, \mu(A) = J_2 c_{A \times B}$, where $\chi_B = \{c_B + \Theta; \Theta \in \mu(\emptyset)\}$.

Thus J_1 and J_2 agree and are finite on $\mathbf{c}[\mathbf{S} \circ \mathbf{W}]$.

But $\mathbf{S} \circ \mathbf{W}$ is a pseudolattice, $\mathbf{m}^*\mathbf{kc}[\mathbf{S} \circ \mathbf{W}] = \mathscr{D}J_1 = \mathscr{D}J_2$. From Theorem 5.16 it follows that $J_1 = J_2$.

7.7 Definition. μ is a *strong* measure, if there exists a measurable space $(\mathbf{V}, \mathbf{V}_0)$ such that $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$ and such that there exists a $\mathbf{V}$-integral with respect to μ.

Remark. From Remark 7.5 it follows that μ is strong if and only if a $\mathbf{q}\mathscr{R}\mu$-integral with respect to μ exists.

Remark. The rest of this section is devoted to give sufficient conditions for a measure to be strong. We do not know if there exists a measure which is not strong.

7.8. Definition.[4]) A system $\mathscr{A} \subset \mathbf{f}(X)$, where X is a set, has the *property LK*, if

$$\{f_n\}_{n=0}^\infty \subset \cdot\, \mathscr{A}\ , \quad f_n \nearrow f_0 \Rightarrow \sup_{x \in X} \left|f_n(x) - f_0(x)\right| \to 0\ .$$

A σ-ring $\mathbf{S}$ is said to be an LK-σ-ring, if there exists a basic system $\mathscr{A}$ with the property LK such that $\mathscr{A} \subset \mathbf{f}_+(\bigcup \mathbf{S})$ and $\mathbf{k}\mathscr{A} = \mathbf{S}$.

A measure μ is an *LK-measure*, if there exists a basic system $\mathscr{A} \subset \mathbf{f}_+(\bigcup \mathscr{D}\mu)$ with the property LK such that $\mathbf{k}\mathscr{A} = \mathscr{D}\mu$ and $f \in \mathscr{A} \Rightarrow \int f \,\mathrm{d}\mu$ is finite.

7.9. Lemma. *Let μ be a measure, $\mathbf{W}$ a σ-algebra, $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$, $\mathbf{W} \subset \mathbf{V}$, $X = = \bigcup \mathscr{D}\mu$, $\Omega = \bigcup \mathbf{V} = \bigcup \mathbf{W}$. Suppose there exists a non negative, finite, linear and from below continuous functional J defined on such a basic system $\mathscr{D}J$ that $\mathbf{k}\mathscr{D}J = \mathscr{D}\mu \times \mathbf{W}$. Let $\mathscr{G}$ and $\mathscr{B}$ be basic systems,*

$$\mathbf{W} = \mathbf{k}\mathscr{G}\ , \quad \mathscr{D}\mu = \mathbf{k}\mathscr{B}\ . \tag{7.9.1}$$

[4]) See also Remark 7.16.

Let

$$g \in \mathscr{G}\,,\quad h \in \mathscr{B}\,,\quad g \in \bar{g} \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0) \Rightarrow {}^X g \,.\, h^\Omega \in \mathscr{D}J\,,\quad J[{}^X g \,.\, h^\Omega] = \bar{g} \,.\, \int h \,\mathrm{d}\mu\,. \quad (7.9.2)$$

Then μ is σ-finite and the (unique) **W**-*integral with respect to μ exists.*

Proof. From Theorems 6.5 and 6.9 it follows that there exists a σ-finite measure ν defined on $\mathscr{D}\mu \times \mathbf{W}$ such that $\int . \,\mathrm{d}\nu \succ J$. The weak integral $\int . \,\mathrm{d}\nu$ is defined on $\mathbf{m}_+^*(\mathscr{D}\mu \times \mathbf{W}) = \mathbf{m}_+^* \mathbf{W}_\mu$; we shall prove that it is the **W**-integral with respect to μ. The conditions to be verified are contained in Definition (7.4); the only nontrivial among them are (7.4.2) and (7.4.3).

For every $g \in \mathbf{m}_+^* \mathbf{W}$ let $\bar{g}$ denote the random variable in $\mathbf{n}_+^{*'}(\mathbf{V}, \mathbf{V}_0)$ containing g. Let us denote, for $g \in \mathbf{m}_+^* \mathbf{W}$, $h \in \mathbf{m}_+^* \mathscr{D}\mu$

$$F_1(g, h) = \bar{g} \,.\int h \,\mathrm{d}\mu\,, \quad F_2(g, h) = \int {}^X g \,.\, h^\Omega \,\mathrm{d}\nu\,. \quad (7.9.3)$$

(We note that ${}^X g \,.\, h^\Omega$ is $(\mathscr{D}\mu \times \mathbf{W})$ measurable although ${}^X g$ may not be so.) Put $\mathscr{B}_1 = \mathscr{G}$, $\mathscr{B}_2 = \mathscr{B}$, $\overline{\mathscr{B}}_1 = \mathbf{m}_+ \mathbf{W}$. The functionals $F_i(g, .)$, $F_i(., h)$ are non negative, linear and from below continuous; F_1, F_2 are defined on $\mathbf{m}_+^* \mathbf{k}\mathscr{B}_1 \times \mathbf{m}_+^* \mathbf{k}\mathscr{B}_2$. We shall show that F_1 is finite on $\overline{\mathscr{B}}_1 \times \mathscr{B}_2$. Let $h \in \mathscr{B}_2 = \mathscr{B}$. Since **W** is a σ-algebra and $\mathbf{k}\mathscr{G} = \mathbf{W}$, we can choose a sequence $g_n \in \mathscr{G}$ such that $\Omega = \bigcup_{n=1}^{\infty} M_n$, where $M_n = \{\omega; g_n(\omega) > 0\}$. According to (7.9.2) the integrals $J[{}^X g_n \,.\, h^\Omega] = \bar{g}_n \int h \,\mathrm{d}\mu$ are finite, which yields the finiteness of $\int h \,\mathrm{d}\mu$. Hence F_1 is finite on $\overline{\mathscr{B}}_1 \times \mathscr{B}_2$. Now F_1 agree with F_2 on $\mathscr{B}_1 \times \mathscr{B}_2$ (see (7.9.2)). From Theorem 5.17 it follows that $F_1 = F_2$. In particular, we get, for every $A \in \mathscr{D}\mu$,

$$\mu(A) = F_1(1, c_A) = F_2(1, c_A) = \int c_A^\Omega \,\mathrm{d}\nu\,; \quad (7.9.3)$$

thus (7.4.2) is satisfied.

Now, if $g \in \mathbf{m}_+ \mathbf{W}$, $A \in \mathscr{D}\mu$, $B \in \mathbf{W}$, then

$$\bar{g} \,.\int c_{A \times B} \,\mathrm{d}\nu = \bar{g} \,.\int {}^X c_B \,.\, c_A^\Omega \,\mathrm{d}\nu = \bar{g} \,.\, F_2(c_B, c_A) = \bar{g} \,.\, F_1(c_B, c_A) = \bar{g} \,.\, \bar{c}_B \,.\int c_A \,\mathrm{d}\mu =$$
$$= F_1(g \,.\, c_B, c_A) = F_2(g \,.\, c_B, c_A) = \int {}^X(g \,.\, c_B) \,.\, c_A^\Omega \,\mathrm{d}\nu = \int {}^X g \,.\, c_{A \times B} \,\mathrm{d}\nu\,.$$

Now we beg the reader to forget the former definitions of $F_1, F_2, \mathscr{B}_1, \mathscr{B}_2, \overline{\mathscr{B}}_1$. We define, for every $g \in \mathbf{m}_+^* \mathbf{W}$, $f \in \mathbf{m}_+^* \mathbf{W}_\mu$

$$F_1(f, g) = \bar{g} \,.\int f \,\mathrm{d}\nu\,, \quad F_2(f, g) = \int {}^X g \,.\, f \,\mathrm{d}\nu\,.$$

We have proved that $F_1(g, f) = F_2(g, f)$ for $g \in \mathbf{m}_+ \mathbf{W}$, $f \in \mathbf{c}[\mathscr{D}\mu \circ \mathbf{W}]$. Denote by **C** the pseudolattice of all sets C in $\mathscr{D}\mu \circ \mathbf{W}$ for which $\nu(C)$ is finite. As ν is σ-finite, $\mathbf{sC} = \mathbf{V}_\mu$. Again the assumptions of Theorem 5.17 are satisfied, if we put $\mathscr{B}_1 = \overline{\mathscr{B}}_1 = \mathbf{m}_+ \mathbf{W}$, $\mathscr{B}_2 = \mathbf{cC}$. Thus $F_1 = F_2$, i. e., (7.4.3) is satisfied. Thus $\int . \,\mathrm{d}\nu$ is the **W**-integral with respect to μ.

The σ-finiteness of μ follows from the σ-finiteness of ν and from (7.9.3). The unicity of the **W**-integral follows from the σ-finiteness of μ and from Theorem 7.6.

7.10. Theorem. *Let μ be a σ-finite measure, let $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$ and let $\mathbf{W}$ be an LK-σ-algebra, $\mathbf{W} \subset \mathbf{V}$, $\bigcup \mathbf{V} = \bigcup \mathbf{W} = \Omega$.*

Then the $\mathbf{W}$-integral with respect to the measure μ exists.

Proof. Denote $X = \bigcup \mathscr{D}\mu$, let $\mathscr{G}$ be a basic system with the property LK, such that $\mathbf{k}\mathscr{G} = \mathbf{W}$. Let further $\mathbf{B} = \{A; A \in \mathscr{D}\mu, \mu(A) \text{ finite}\}$ and let $\mathscr{B}$ be the system of all $\mathbf{B}$-simple functions.

If $A_i \in \mathbf{B}$, $g_i \in \mathscr{G}$, $g_i \in \bar{g}_i \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$ $(i = 1, 2, \ldots, n)$, put

$$J \sum_{i=1}^{n} {}^{x}g_i \,.\, c^{\Omega}_{A_i} = \sum_{i=1}^{n} \bar{g}_i \,.\, \mu(A_i)\,. \tag{7.10.1}$$

From the additivity of μ it follows that the relations

$$\sum_{i=1}^{n} {}^{x}g_i \,.\, c^{\Omega}_{A_i} = \sum_{i=1}^{m} {}^{x}h_i \,.\, c^{\Omega}_{B_i}\,, \quad h_i \in \mathscr{G}\,, \quad B_i \in \mathbf{B}\,, \quad h_i \in \bar{h}_i \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$$

imply

$$\sum_{i=1}^{n} \bar{g}_i \,.\, \mu(A_i) = \sum_{i=1}^{m} \bar{h}_i \,.\, \mu(B_i)\,;$$

thus J is unambiguously defined. Since in (7.10.1) we may always suppose that the sets A_i are disjoints, it is easy to see, that $\mathscr{D}J$ is a basic system. Further obviously $\mathbf{k}\mathscr{D}J = \mathscr{D}\mu \times \mathbf{W}$; $\mathbf{k}\mathscr{B} = \mathscr{D}\mu$;

$$g \in \mathscr{G}, \quad g \in \bar{g} \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0), \quad f \in \mathscr{B} \Rightarrow {}^{x}g \,.\, f^{\Omega} \in \mathscr{D}\, J,\; J^{x}g \,.\, f^{\Omega} = \bar{g} \,.\, \textstyle\int f \,\mathrm{d}\mu;$$

J is non negative, finite, linear. Thus, if J is continuous from below, then all the assumptions of Lemma 7.9 are satisfied and the $\mathbf{W}\!\int . \,\mathrm{d}\mu$ exists.

We shall prove the continuity from below of J.

Let $f_i \nearrow f_0$, $\{f_i\}_{i=0}^{\infty} \subset\cdot\, \mathscr{D}J$. Put $h_m = \left[f_0 - \dfrac{1}{m}\right]_+$.

We have $\{f_i(x, .)\}_{i=0}^{\infty} \subset\cdot\, \mathscr{G}$ for every $x \in X$ and $\mathscr{G}$ has the property LK. It follows that for every $x \in X$ and $m > 0$ there exists an index n such that $f_n(x, .) \geqq f_0(x, .) - \dfrac{1}{m}$ and hence $f_n(x, .) \geqq \left(f_0(x, .) - \dfrac{1}{m}\right) \vee 0 = h_m(x, .)$.

Fix an m and denote $Q_n = \{x; x \in X, f_n(x, .) \geqq h_m(x, .)\}$. Clearly $Q_n \subset Q_{n+1}$, $\bigcup Q_n = \bigcup \mathscr{D}\mu$. Since $f_0 \in \mathscr{D}J$, we can write

$$f_0 = \sum_{j=1}^{k} {}^{x}g_j \,.\, c^{\Omega}_{A_j}\,, \quad A_j \in \mathbf{B}\,, \quad A_j \cap A_i = \emptyset \text{ for } i \neq j\,, \quad g_j \in \mathscr{G}\,,$$

and $h_m = \displaystyle\sum_{j=1}^{k} \left({}^{x}g_j - \frac{1}{m}\right)_+ c^{\Omega}_{A_j}$.

$\mathscr{G}$ is a basic system and thus $\left(g_j - \dfrac{1}{m}\right)_+ = g_j - \left(g_j \wedge \dfrac{1}{m}\right) \in \mathscr{G}$. Consequently

$$Jf_n \geqq Jh_m \,.\, c^{\Omega}_{Q_n} = J \sum_{j=1}^{k} {}^{x}\left(g_j - \frac{1}{m}\right)_+ . \, c^{\Omega}_{A_j \cap Q_n} \geqq \sum_{j=1}^{k} \left(\bar{g}_j - \frac{1}{m}\right) . \, \mu(A_j \cap Q_n)$$

and thus

$$\lim_{n\to\infty} Jf_n \geqq \sum_{j=1}^{k} \left(\bar{g}_j - \frac{1}{m}\right) . \mu(A_j) \nearrow_m \sum_{j=1}^{k} \bar{g}_j . \mu(A_j) = Jf_0 .$$

Since $Jf_n \leqq Jf_0$ we get $Jf_n \nearrow Jf_0$.

7.11. Theorem. *Let μ be an LK-measure, $\mathscr{R}\mu \subset \mathbf{n}_+^*(\mathbf{V}, \mathbf{V}_0)$. Then the* **V***-integral with respect to μ exists; thus μ is a strong measure.*

Proof. Let $\mathscr{B}$ be a basic system with the property LK, $\mathbf{k}\mathscr{B} = \mathscr{D}\mu$, $f \in \mathscr{B} \Rightarrow$ $\Rightarrow \int f \,\mathrm{d}\mu$ is finite. Denote $F = [\int . \,\mathrm{d}\mu]_{\mathscr{B}}$.

Put $J \sum_{i=1}^{n} {}^{X}c_{B_i} . f_i^{\Omega} = \sum_{i=1}^{n} \chi_{B_i} . Ff_i$ for $B_i \in \mathbf{V}$, $f_i \in \mathscr{B}$ $(i = 1, 2, \ldots, n)$.

Let $\mathscr{G}$ be the system of all **V**-simple functions. Clearly $\mathscr{D}J$ is a basic system, $\mathbf{k}\mathscr{D}J = \mathbf{V}_\mu$, (7.9.1) and (7.9.2) hold. J is non negative, finite and linear. We shall prove the continuity from below of J.

Let $\{f_i\}_{i=0}^{\infty} \subset \mathscr{D}J$, $f_i \nearrow f_0$.

Put $h_m = \left(f_0 - \frac{1}{m}\right)_+$, $Q_n = \{\omega; f_n(., \omega) \geqq h_m(., \omega)\}$. As in the preceding proof, we have $Q_n \subset Q_{n+1} \subset \ldots$, $\bigcup Q_n = \bigcup \mathbf{V}$.

If $f_0 = \sum_{j=1}^{q} {}^{X}c_{B_j} . g_j^{\Omega}$, where

$$B_j \in \mathbf{V} , \quad B_j \cap B_i = \emptyset \quad \text{for } i \neq j,\ g_j \in \mathscr{B} ,$$

then $h_m = \sum_{j=1}^{q} {}^{X}c_{B_j} \left(g_j - \frac{1}{m}\right)_+^{\Omega}$ and

$$Jf_n \geqq Jh_m . {}^{X}c_{Q_n} = J \sum_{j=1}^{q} {}^{X}c_{B_j \cap Q_n} . \left(g_j - \frac{1}{m}\right)_+^{\Omega} = \sum_{j=1}^{q} \chi_{B_j \cap Q_n} . F\left(g_j - \frac{1}{m}\right)_+ .$$

Thus $\lim_{n\to\infty} Jf_n \geqq \sum_{j=1}^{q} \chi_{B_j} . F\left(g_j - \frac{1}{m}\right)_+^{\Omega} \nearrow_m \sum_{j=1}^{q} \chi_{B_j} . Fg_j = Jf_0$. Finally $Jf_n \leqq Jf_0$ implies that $Jf_n \nearrow Jf_0$. Thus the conditions of Lemma 7.9 are satisfied for $\mathbf{W} = \mathbf{V}$ and the **V**-integral with respect to μ exists.

7.12. Definition. J is called a *degenerate functional* if:

(7.12.1) J is a non negative, linear and from below continuous functional,

(7.12.2) there exists a $c \in E_+$ such that $f \in \mathscr{D}J$, $f \leqq 1 \Rightarrow Jf \leqq c$,

(7.12.3) there exists a transformation Z from $\mathscr{D}^2 J$ into $\bigcup \mathbf{q}\mathscr{R}J$ such that

$$0 \leqq g \in \bar{g} \in \mathscr{R}J , \quad f \in \mathscr{D}J \Rightarrow f . gZ \in \mathscr{D}J , \quad J(f . gZ) = \bar{g} . Jf .$$

7.13. Lemma. *If μ is a measure induced by a degenerate functional J, then the weak integral $\int . \,\mathrm{d}\mu$ is degenerate and μ is finite.*

Proof. Without loss of generality we may suppose that the functional J is finite. (In the contrary case we may put $J_0 = J_{\mathscr{M}}$, where $\mathscr{M}$ is the system of all bounded functions in $\mathscr{D}J$. Clearly[5]) $\mathscr{M} = \mathscr{D}J_0$ is a basic system and J_0 induces μ, since, as it is easy to see, $\bar{J} = \bar{J}_0$. From (7.12.2) it follows that J_0 is finite.)

Let Z be the transformation satisfying (7.12.3). We shall prove that

$$0 \leqq g \in \bar{g} = \int h \, \mathrm{d}\mu \,, \quad h \in \mathbf{m}_+^* \mathscr{D}\mu \,, \quad f \in \mathbf{m}_+^* \mathscr{D}\mu \Rightarrow \\ \Rightarrow f \,.\, gZ \in \mathbf{m}_+^* \mathscr{D}\mu \,, \quad \bar{g} \,.\, \int f \, \mathrm{d}\mu = \int gZ \,.\, f \, \mathrm{d}\mu \,. \tag{7.13.1}$$

Let $h \in \mathscr{D}J$ and let $\mathscr{A}^h$ denote the system of all such $f \in \mathbf{m}_+^* \mathscr{D}\mu$, for which (7.13.1) holds (the function h being fixed). We have $\mathscr{A}^h \subset \mathscr{D}J$, $\mathscr{A}^h_{\sigma+} \subset \mathscr{A}^h$, $\mathscr{A}^h_{\times} \subset \mathscr{A}$, $\mathscr{A}^h_-(\mathscr{D}J) \subset \mathscr{A}$. Lemma 4.4 applied, we get $\mathscr{A} \supset \mathbf{m}_+^* \mathscr{D}\mu$ and thus (7.13.1) holds for every $h \in \mathscr{D}J$, $f \in \mathbf{m}_+^* \mathscr{D}\mu$.

Let $\mathscr{L}$ be the system of all $f \in \mathbf{m}_+^* \mathscr{D}\mu$ such that $\int f \, \mathrm{d}\mu$ is finite. Let $f \in \mathscr{L}$ and let $\mathscr{C}^f$ be the system of all such h that (7.13.1) holds for the fixed function f. We have $\mathscr{C}^f \supset \mathscr{D}J$; since $\bar{g} \,.\, \int f \, \mathrm{d}\mu$ is finite for every $\bar{g} \in \mathscr{R}J$, we see that $\mathscr{C}^f_-(\mathscr{D}J) \subset \mathscr{C}^f$; the inclusions $\mathscr{C}^f_{\sigma+} \subset \mathscr{C}^f$, $\mathscr{C}^f_{\times} \subset \mathscr{C}^f$ are obvious. A new application of Lemma 4.4 gives $\mathscr{C}^f \supset \mathbf{m}_+^* \mathscr{D}\mu$.

Thus (7.13.1) holds for every $h \in \mathbf{m}_+^* \mathscr{D}\mu$, $f \in \mathscr{L}$. From the continuity from below and from the σ-finiteness of $\int . \, \mathrm{d}\mu$ it follows that (7.13.1) holds for every f and h.

Thus it remains to prove the existence of a $c \in E_+$ such that

$$f \in \mathbf{m}_+ \mathscr{D}\mu \,, \quad f \leqq 1 \Rightarrow \int f \, \mathrm{d}\mu \leqq c \,. \tag{7.13.2}$$

From the assumptions it follows that there exists a $c \in E_+$ such that (7.13.2) holds for every $f \in \mathscr{D}J$. Putting $\mathscr{D}J = \mathscr{F}$ and using the notation of Lemma 4.3, we see that $A \in \mathbf{F} \Rightarrow \mu(A) \leqq c$. For there exists a sequence $\{f_n\}_{n=1}^\infty \subset\cdot \mathscr{D}J$, $f_n \nearrow c_A$. Thus $f_n \leqq 1$, $Jf_n \leqq c$, $\mu(A) = \lim\limits_{n\to\infty} \int f_n \, \mathrm{d}\mu = \lim\limits_{n\to\infty} Jf_n \leqq c$. We have $\mathbf{sF} = \mathbf{k}\mathscr{D}J = \mathscr{D}\mu$ and $\mathbf{F}$ is a lattice. Let $B \in \mathscr{D}\mu$. Thus there exists a sequence $\{C_n\}_{n=1}^\infty \subset\cdot \mathbf{F}$ such that $\bigcup\limits_{n=1}^{\infty} C_n \supset B$ ([4], § 5, Theorem D). As $\mathbf{F}$ is a lattice, $\bigcup\limits_{n=1}^{m} C_n \in \mathbf{F}$ for every $m = 1, 2, \ldots$ and thus

$$\mu(B) \leqq \mu(\bigcup_{n=1}^{\infty} C_n) = \lim_{m\to\infty} \mu(\bigcup_{n=1}^{m} C_n) \leqq c \,.$$

Thus $B \in \mathscr{D}\mu \Rightarrow \mu(B) \leqq c$ and hence

$$f \in \mathbf{m}_+ \mathscr{D}\mu \,, \quad f \leqq 1 \Rightarrow \int f \, \mathrm{d}\mu \leqq c \,,$$

which accomplishes the proof.

7.14. Lemma. *Let μ be a measure induced by a degenerate functional,*

$$(\mathbf{V}, \mathbf{V}_0) = (\mathbf{q}\mathscr{R}\mu, \mathbf{q}_0\mathscr{R}\mu) \,. \tag{7.14.1}$$

---

[5]) From Definition 6.6 it follows that $\mathscr{D}J$ is a basic system.

Then there exists a transformation Z from $\bigcup \mathscr{D}\mu$ into $\bigcup \mathbf{V}$ such that

$$0 \leqq g \in \bar{g} \in \mathbf{n}_+^*(\mathbf{V}, \mathbf{V}_0)\,, \quad f \in \mathbf{m}_+^* \mathscr{D}\mu \Rightarrow \\ \Rightarrow f\,.\,gZ \in \mathbf{m}_+^* \mathscr{D}\mu\,, \quad \int f\,.\,gZ\,\mathrm{d}\mu = \bar{g}\,.\int f\,\mathrm{d}\mu\,. \tag{7.14.2}$$

Proof. From the preceding Lemma it follows that $\int\,.\,\mathrm{d}\mu$ is degenerate. Thus there exists a transformation Z from $\bigcup \mathscr{D}\mu$ into $\bigcup \mathbf{q}\mathscr{R}\mu = \bigcup \mathbf{V}$ such that

$$0 \leqq g \in \bar{g} \in \mathscr{R}[\int\,.\,\mathrm{d}\mu]\,, \quad f \in \mathbf{m}_+^* \mathscr{D}\mu \Rightarrow \\ \Rightarrow f\,.\,gZ \in \mathbf{m}_+^* \mathscr{D}\mu\,, \quad \int f\,.\,gZ\,\mathrm{d}\mu = \bar{g}\,.\int f\,\mathrm{d}\mu\,. \tag{7.14.3}$$

Now let $\mathscr{L}$ be the system of all bounded functions measurable $(\mathscr{D}\mu)$. Put $Jf = \int f_+\,\mathrm{d}\mu - \int f_-\,\mathrm{d}\mu$ for every $f \in \mathscr{L}$; this is possible because there exists (for $\int\,.\,\mathrm{d}\mu$ is degenerate) such a constant $c \in E_+$ that $f \in \mathbf{m}_+\mathscr{D}\mu, f \leqq 1 \Rightarrow \Rightarrow \int f\,\mathrm{d}\mu \leqq c$; obviously

$$f \in \mathscr{L}\,, \quad |f| \leqq 1 \Rightarrow |Jf| \leqq 2c\,.$$

From there we proceed by a method essentially due to Moy [8]. Let us denote by $\mathscr{L}_1$ the system of all bounded functions measurable ($\mathbf{V}$). Let $\mathscr{C}$ be the system of all such $g \in \mathscr{L}_1$ for which

$$h \in \mathscr{L}\,, \quad g \in \bar{g} \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0) \Rightarrow h\,.\,gZ \in \mathscr{L}\,, \quad J(h\,.\,gZ) = \bar{g}\,.\,Jh\,.$$

Denoting $\mathscr{P} = \{g;\, 0 \leqq g \in \bar{g} \in \mathscr{R}[\int\,.\,\mathrm{d}\mu]\}$, we obtain, according to (7.14.3),

$$\mathscr{C} \supset \mathscr{P} \cap \mathscr{L}_1\,. \tag{7.14.4}$$

Clearly $1 \in \mathscr{C}$ and

$$\{g_1, g_2\} \subset \mathscr{C} \Rightarrow g_1 + g_2 \in \mathscr{C}\,, \quad g_1 - g_2 \in \mathscr{C}\,, \quad g_1\,.\,g_2 \in \mathscr{C}\,, \tag{7.14.5}$$

the last inclusion being a consequence of

$$J(h\,.\,(g_1\,.\,g_2)\,Z) = J(h\,.\,g_1Z\,.\,g_2Z) = \bar{g}_1\,.\,J(h\,.\,g_1Z) = \bar{g}_1\,.\,\bar{g}_2\,.\,Jh\,.$$

Thus if Λ is a polynomial, $g \in \mathscr{C}$, then $\Lambda g \in \mathscr{C}$.

Let $\{g_n\}_{n=1}^\infty \subset \mathscr{C}$, $|g_n| \leqq a$ for $n = 1, 2, \ldots, a \in E_+$, $g_n \to g$. Then $g \in \mathscr{C}$.

For if $h \in \mathscr{L}$, then, by assumption, $h\,.\,g_nZ \in \mathscr{L}$; thus $h\,.\,gZ \in \mathbf{m}^*\mathscr{D}\mu$ and $h\,.\,gZ \in \mathscr{L}$ because $|h\,.\,gZ| \leqq a|h| \in \mathscr{L}$. Further from $|h\,.\,g_nZ| \leqq a\,.\,|h|$ and from Theorem 6.7 it follows that $J(h\,.\,gZ) = \lim_{n\to\infty} J(h\,.\,g_nZ) = \lim_{n\to\infty} \bar{g}_n\,.\,Jh = \bar{g}\,.\,Jh$, where $g_n \in \bar{g}_n$, $g \in \bar{g}$. Similarly we can prove, using (7.14.3), that if $\{g_n\}_{n=1}^\infty \subset \mathscr{C}$, $g_n \to g$ uniformly on $\bigcup \mathbf{V}$, then $g \in \mathscr{C}$.

Let Φ be a continuous real-valued function defined on E, let $g \in \mathscr{C}$; we shall prove $\Phi g \in \mathscr{C}$. Indeed, g is bounded and thus $\mathscr{R}g$ is contained in a finite interval. Thus there exists a sequence of polynomials Λ_n such that $\Lambda_n g \to \Phi g$ uniformly on $\bigcup \mathbf{V}$, what gives $\Phi g \in \mathscr{C}$; thus if $g \in \mathscr{C}$, then $|g| \in \mathscr{C}$. Hence and from (7.14.5) it follows that $\mathscr{B} = \{g;\, g \in \mathscr{C};\, g \geqq 0\}$ is a basic system. From (7.14.4) it follows that $\mathbf{k}\mathscr{B} = \mathbf{q}\mathscr{R}\mu = \mathbf{V}$.

Let $f \in \mathbf{m}_+\mathscr{D}\mu$ and let $\mathscr{A}$ denote the system of all such $g \in \mathbf{m}_+^*\mathbf{V}$ that $f \,.\, gZ \in \mathbf{m}^*\mathscr{D}\mu$. We have $\mathscr{A} \supset \mathscr{B}$, $\mathscr{A}_{\sigma+} \subset \mathscr{A}$, $\mathscr{A}_\times \subset \mathscr{A}$, $\mathscr{A}_-(\mathscr{B}) \subset \mathscr{A}$. From Lemma 4.4 it follows that $\mathscr{A} \supset \mathbf{m}_+^*\mathbf{V}$. Thus $f \,.\, gZ \in \mathbf{m}_+^*\mathscr{D}\mu$ for every $f \in \mathbf{m}_+\mathscr{D}\mu$, $g \in \mathbf{m}_+^*\mathbf{V}$ and, obviously, for every $f \in \mathbf{m}_+^*\mathscr{D}\mu$, $g \in \mathbf{m}_+^*\mathbf{V}$, too.

Now let $\mathscr{B}_2$ be the system of all bounded functions $f \in \mathbf{m}_+\mathscr{D}\mu$, let $\mathscr{B}_1 = \mathscr{B}$, $\overline{\mathscr{B}}_1 = \mathbf{m}_+\mathbf{V}$. Put $F_1(g, f) = \bar{g} \,.\, \int f \,\mathrm{d}\mu$, $F_2(g, f) = \int f \,.\, gZ \,\mathrm{d}\mu$ for every $f \in \mathbf{m}_+^*\mathscr{D}\mu$, $0 \leqq g \in \bar{g} \in \mathbf{n}_+^*(\mathbf{V}, \mathbf{V}_0)$. Then all the assumptions of Theorem 5.17 are satisfied and thus $F_1 = F_2$, which completes the proof of (7.14.2).

7.15. Theorem. *Every measure μ induced by a degenerate functional is strong.*

Proof. Let $(\mathbf{V}, \mathbf{V}_0)$ and Z satisfy (7.14.1) and (7.14.2). Let us denote $X = \bigcup \mathscr{D}\mu$, $\mathscr{G} = \mathbf{m}\mathbf{V}$, $\Omega = \bigcup \mathbf{V}$. Let $\mathscr{B}$ be the system of all $\mathscr{D}\mu$-simple functions. For

$$0 \leqq g_i \in \bar{g}_i \in \mathbf{n}_+(\mathbf{V}, \mathbf{V}_0)\,, \quad f_i \in \mathscr{B}$$

we put

$$J \sum_{i=1}^{n} {}^X g_i \,.\, f_i^\Omega = \sum_{i=1}^{n} \bar{g}_i \,.\, \int f_i \,\mathrm{d}\mu = \int \sum_{i=1}^{n} f_i \,.\, g_i Z \,\mathrm{d}\mu\,, \qquad (7.15.1)$$

where the last equality follows from the preceding Lemma.

Clearly (7.9.1) and (7.9.2) hold, J is a non negative, finite and linear functional, $\mathscr{D}J$ is a basic system, $\mathbf{k}\mathscr{D}J = \mathscr{D}\mu \times \mathbf{V}$. The only non trivial property is the continuity from below. Thus let

$$f_i = \sum_{j=1}^{n_i} {}^X g_{ji} \,.\, f_{ji}^\Omega\,, \quad f_{ji} \in \mathscr{B}\,, \quad 0 \leqq g_{ji} \in \bar{g}_{ji} \in \mathbf{n}_+(\mathbf{V}, \mathbf{V}_0)\,, \quad f_i \nearrow f_0$$

and put $\tilde{f}_i = \sum_{j=1}^{n_i} g_{ji}Z \,.\, f_{ji}$. If $x \in \bigcup \mathscr{D}\mu$, then $[x, Zx] \in \bigcup \mathscr{D}\mu \times \bigcup \mathbf{V}$ and thus

$$\tilde{f}_i(x) = f_i(x, Zx) \nearrow f_0(x, Zx) = \tilde{f}_0(x)\,.$$

Thus $\tilde{f}_i \nearrow \tilde{f}_0$. Further from (7.15.1) it follows that

$$Jf_i = \int \tilde{f}_i \,\mathrm{d}\mu \nearrow \int \tilde{f}_0 \,\mathrm{d}\mu = Jf_0\,.$$

J is continuous from below, and the application of Lemma 7.9 yields, if we put $\mathbf{W} = \mathbf{V}$, the desired result: μ is strong.

7.16. Remark. Although we do not know whether every measure is strong, Theorems 7.10, 7.11, and 7.15 give sufficient conditions for a measure to be strong, which are often satisfied. For example, it is easy to see that the σ-ring of all Borel (or Baire) sets in a locally compact Hausdorff space is a LK-σ-ring.

7.17. Remark. We keep the promise from (7.1) concerning the possibility of defining the integral for functions the values of which are random variables.

Let ξ be the Lebesgue measure on (0, 1), let $(\mathbf{V}, \mathbf{V}_0)$ be induced by ξ, let $\mu(A) = \chi_A \in \mathbf{n}(\mathbf{V}, \mathbf{V}_0)$ for every $A \in \mathscr{D}\mu = \mathbf{V}$. Let $\mathfrak{M}$ be a system of functions defined on (0, 1) the values of which are elements of $\mathbf{n}(\mathbf{V}, \mathbf{V}_0)$. Let $\mathfrak{M}$ contain all functions of the form $\sum_{i=1}^{n} \alpha_i \,.\, c_{A_i}$, where $\alpha_i \in \mathbf{n}_+(\mathbf{V}, \mathbf{V}_0)$, $A_i \in \mathscr{D}\mu$. Let J be a linear functional on $\mathfrak{M}$, let

$$A \in \mathscr{D}\mu\,, \quad \alpha \in \mathbf{n}_+(\mathbf{V}, \mathbf{V}_0) \Rightarrow J(\alpha \,.\, c_A) = \alpha \,.\, \mu(A)\,.$$

Let $g_n = \sum_{i=1}^{n} \chi\left(\frac{i-1}{n}, \frac{i}{n}\right) \,.\, c\left(\frac{i-1}{n}, \frac{i}{n}\right)$. Then $Jg_n = 1$ for every $n = 1, 2, \ldots$, although $g_{2^n} \searrow 0 \in \mathfrak{M}$ [6]) and the functional J is not continuous from below.

8. The W-and the WI-integral

8.1. Conventions. In this section μ denotes always a σ-finite measure, $(\mathbf{V}, \mathbf{V}_0)$ a measurable space such that $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$. We shall suppose that $\mathbf{W}$ is such a σ-algebra that $\mathbf{W} \subset \mathbf{V}$ and that the (unique, according to Theorem 7.6 and the assumed σ-finiteness of μ) $\mathbf{W}$-integral with respect to μ exists: we shall denote it by $\mathbf{W}\int \,.\, \mathrm{d}\mu$. If μ is a strong measure, then there exists a σ-algebra $\mathbf{Z}$ such that $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{Z}, \mathbf{V}_0)$ and that $\mathbf{Z}\int \,.\, \mathrm{d}\mu$ exists; in this case we shall assume the σ-algebra $\mathbf{V}$ has been chosen is such a way that $\mathbf{V}\int \,.\, \mathrm{d}\mu$ exists, too. Finally we denote $X = \bigcup \mathscr{D}\mu$ and $\Omega = \bigcup \mathbf{W} = \bigcup \mathbf{V}$.

8.2. Definition. Let us write (for $\{f_1, f_2\} \subset \mathbf{f}_+^*(\bigcup \mathbf{W}_\mu)$) $f_1 = f_2\,[\mu, \mathbf{W}]$, if and only if there exist functions $g_i \in \mathbf{m}_+^* \mathbf{W}_\mu$ $(i = 1, 2)$ such that $g_1 \leqq f_j \leqq g_2$ $(j = 1, 2)$ and $\mathbf{W}\int g_1 \,\mathrm{d}\mu = \mathbf{W}\int g_2 \,\mathrm{d}\mu$.

Let us define $\mathbf{i}(\mu, \mathbf{W}) = \{f;\, f = f\,[\mu, \mathbf{W}]\}$.

8.3. Theorem. *Let* $\mathscr{M} = \mathbf{i}(\mu, \mathbf{W})$. *Then* $\mathscr{M}_{\sigma+} \subset \mathscr{M}$, $\mathscr{M}_\times \subset \mathscr{M}$, $\mathscr{M}_-(\mathbf{f}_+(\bigcup \mathbf{W}_\mu)) \subset \subset \mathscr{M}$.

Proof. Obvious.

8.4. Definition. The **WI**-integral with respect to μ is defined on $\mathbf{i}(\mu, \mathbf{W})$ by means of the relation

$$f = g\,[\mu, \mathbf{W}]\,, \quad g \in \mathbf{m}_+^* \mathbf{W}_\mu \Rightarrow \mathbf{WI}\int f \,\mathrm{d}\mu = \mathbf{W}\int g \,\mathrm{d}\mu\,.$$

8.5. Theorem. *The* **WI**-*integral with respect to* μ *is non negative,* **W**-*linear and continuous from below.*

Proof. Obvious.

8.6. Theorem. *Let* ν *be a real-valued function,* $\mathscr{D}\nu = \mathscr{D}\mu \times \Omega$. *For every* $\omega \in \Omega$ *let* $\nu(.\,, \omega)$ *be a measure. For every* $A \in \mathscr{D}\mu$ *let*

$$\nu(A, .) \in \mu(A)\,. \tag{8.6.1}$$

Then: If $f \in \mathbf{m}_+^* \mathbf{W}_\mu$, *then*

$$h(\omega) = \int f(.\,, \omega) \,\mathrm{d}\nu(.\,, \omega) \tag{8.6.2}$$

[6]) I. e., for every $x \in (0, 1)$, $g_{2^n}(x) \searrow 0$ in $\mathbf{n}(\mathbf{V}, \mathbf{V}_0)$.

exists for every $\omega \in \Omega$ *and*

$$h \in \mathbf{W}\!\int f \,\mathrm{d}\mu \,. \tag{8.6.3}$$

If $f \in \mathbf{i}(\mu, \mathbf{W})$, *then there exists a set* $V \in \mathbf{V}_0$ *such that for every* $\omega \in \Omega - V$ *the integral*

$$h(\omega) = \int f(.\,, \omega) \,\mathrm{d}\overline{\nu(.\,, \omega)}\;^{7)} \tag{8.6.4}$$

exists and, defining h *on* V *in such a way that* $h \in \mathbf{m}^*\mathbf{V}$ *(this can be done, e. g., by putting* $h(x) = 0$ *for* $x \in V$*), we have*

$$h \in \mathbf{WI}\!\int f \,\mathrm{d}\mu \,. \tag{8.6.5}$$

Proof. We shall prove the first part of the Theorem. We note that, for every $f \in \mathbf{m}_+^* \mathbf{W}_\mu$, $f(.\,, \omega)$ is $(\mathscr{D}\mu)$ measurable for every $\omega \in \Omega$ and h is measurable $(\mathbf{V})$. Let us denote by $J_1 f$ the random variable in $\mathbf{n}_+^*(\mathbf{V}, \mathbf{V}_0)$ containing h and let us write $J_2 f = \mathbf{W}\!\int f \,\mathrm{d}\mu$.

Let $\mathbf{S}$ be the system of all sets $A \in \mathscr{D}\mu$, for which $\mu(A)$ is finite.

Let $A \in \mathbf{S}$, $B \in \mathbf{W}$, $f = c_{A \times B}$. Then $h(\omega) = c_B(\omega) \,.\, \nu(A, \omega)$ and, according to (8.6.1), $J_1 f = \chi_B \,.\, \mu(A) = J_2 f$. J_1 and J_2 agree and are finite on $\mathbf{c}(\mathbf{S} \circ \mathbf{V})$. From the σ-finiteness of μ it follows that $\mathbf{kc}(\mathbf{S} \circ \mathbf{V}) = \mathscr{D}\mu \times \mathbf{V}$. Lemma 5.16 gives $J_1 = J_2$; thus the first assertion of the Theorem is proved.

Now let $f \in \mathbf{i}(\mu, \mathbf{W})$. Thus there exist two functions $\{g_1, g_2\} \subset\cdot\, \mathbf{m}_+^* \mathbf{W}_\mu$, such that $g_1 \leqq f \leqq g_2$ and that $\mathbf{W}\int g_1 \,\mathrm{d}\mu = \mathbf{W}\int g_2 \,\mathrm{d}\mu$. We define

$$h_i(\omega) = \int g_i(.\,, \omega) \,\mathrm{d}\nu(.\,, \omega) \quad (i = 1, 2) \,.$$

We have $\{h_1, h_2\} \subset\cdot\, \mathbf{W}\int g_1 \,\mathrm{d}\mu$. Thus there exists a set $V \in \mathbf{V}_0$ such that $h_1(\omega) = h_2(\omega)$ for every $\omega \in \Omega - V$. For every $\omega \in \Omega$ we have $g_1(.\,, \omega) \leqq f(.\,, \omega) \leqq g_2(.\,, \omega)$, which with the preceding equality yields the $(\mathscr{D}\overline{\nu(.\,, \omega)})$ measurability of $f(.\,, \omega)$ for every $\omega \in \Omega - V$. But $h(\omega) = \int f(.\,, \omega) \,\mathrm{d}\overline{\nu(.\,, \omega)} = h_1(\omega)$ for every $\omega \in \Omega - V$; hence we deduce easily the second assertion of the Theorem.

8.7. Theorem. *Let* μ_0, μ_1, μ_2 *be* σ*-finite measures,* $\mathscr{D}\mu_0 = \mathscr{D}\mu_1 \times \mathscr{D}\mu_2$. *Let* $\bigcup \mathbf{q}\mathscr{R}\mu_i = \Omega$, $\bigcup \mathscr{D}\mu_i = X_i$ $(i = 0, 1, 2)$ *and*

$$\mathbf{q}_0\mathscr{R}\mu_2 \subset \mathbf{q}_0\mathscr{R}\mu_0 \,, \quad \mathbf{q}_0\mathscr{R}\mu_1 = \mathbf{q}_0\mathscr{R}\mu_0 \,.^{8)} \tag{8.7.1}$$

Let the $\mathbf{W}$*-integral with respect to* μ_i $(i = 0, 1, 2)$ *exist, let* $\mathbf{W} \supset \mathbf{q}\mathscr{R}\mu_2$. *Let*

$$A_i \in \mathscr{D}\mu_i \,, \quad g_i \in \mu(A_i) \;(i = 1, 2) \Rightarrow g_1 \,.\, g_2 \in \mu_0(A_1 \times A_2) \,. \tag{8.7.2}$$

---

[7]) Of course, $\overline{\nu(.\,, \omega)}$ is the completion of $\nu(.\,, \omega)$.

[8]) This asymmetry has the following reason. The weakness of the condition (8.7.4) (see Remark 8.8) is closely connected with the σ-ring $\mathbf{q}_0\mathscr{R}\mu_2$. If, for example, $\mathbf{q}_0\mathscr{R}\mu_2 = \{\emptyset\}$, then (8.7.4) determines in a unique way the function h_f. Therefore we require $\mathbf{q}_0\mathscr{R}\mu_2 \subset \mathbf{q}_0\mathscr{R}\mu_0$ instead of $\mathbf{q}_0\mathscr{R}\mu_2 = \mathbf{q}_0\mathscr{R}\mu_0$. On the other hand such a weakened condition for μ_1 leads to complications and seems to us to be superfluous.

Let $f \in \mathbf{m}_+^* \mathbf{W}_{\mu_0}$.
Then there exists a

$$h_f \in \mathbf{m}_+^* \mathbf{W}_{\mu_1} \tag{8.7.3}$$

such that

$$x_1 \in X_1 \Rightarrow h_f(x_1, .) \in \alpha_f(x_1) = \mathbf{W} \int f(x_1, *, .) \, \mathrm{d}\mu_2 \,, \tag{8.7.4}$$

$$\mathbf{W} \int h_f \, \mathrm{d}\mu_1 = \mathbf{W} \int f \, \mathrm{d}\mu_0 \tag{8.7.5}$$

and

$$h \in \mathbf{m}_+^* \mathbf{W}_{\mu_1} \,,$$
$$h^*(x_1, x_2, \omega) = h(x_1, \omega) \Rightarrow \mathbf{W} \int h \,.\, h_f \, \mathrm{d}\mu_1 = \mathbf{W} \int h^* \,.\, f \, \mathrm{d}\mu_0 \,. \tag{8.7.6}$$

Proof. Let us denote, for every $g \in \mathbf{m}_+^* \mathbf{q}\mathscr{R}\mu_i$, by $n_i(g)$ the random variable in $\mathbf{n}_+^*(\mathbf{q}\mathscr{R}\mu_i, \mathbf{q}_0\mathscr{R}\mu_i)$ containing g. Further denote

$$\mathbf{S}_i = \{A; A \in \mathscr{D}\mu_i, \mu_i(A) \text{ is finite}\} \,, \quad \mathscr{B} = \mathbf{c}[\mathbf{S}_1 \circ \mathbf{S}_2 \circ \mathbf{W}] \,.$$

Now let $\mathscr{A}$ be the system of all f for which the assertion of the Theorem holds. It is to prove that $\mathscr{A} = \mathbf{m}_+^* \mathbf{W}_{\mu_0}$; according to Lemma 4.2 it suffices to prove that

1. $\mathscr{A}_{\sigma+} \subset \mathscr{A}$, $\mathscr{A}_{\times} \subset \mathscr{A}$, 2. $\mathscr{A}_{-}(\mathscr{B}) \subset \mathscr{A}$, 3. $\mathscr{A} \supset \mathscr{B}$.

1. Let $\{f_i\}_{i=1}^{\infty} \subset \cdot\, \mathscr{A}$. Obviously $\sum_{i=1}^{\infty} f_i = f$ implies $\sum_{i=1}^{\infty} \alpha_{f_i}(x_1) = \alpha_f(x_1)$ for every $x_1 \in X_1$. Let, for every $i = 1, 2, \ldots$, the functions h_{f_i} satisfy the assertions of the Theorem. Then, if we put $h_f = \sum_{i=1}^{\infty} h_{f_i}$, we have $h_f(x_1, .) \in \alpha_f(x_1)$ for every $x_1 \in X_1$, and (8.7.4) holds. (8.7.3) is obvious. Let $h \in \mathbf{m}_+^* \mathbf{W}_{\mu_1}$ or $h = 1$; then, by assumption, $\mathbf{W} \int h \,.\, h_{f_i} \, \mathrm{d}\mu_1 = \mathbf{W} \int h^* \,.\, f_i \, \mathrm{d}\mu_0$ and thus

$$\mathbf{W} \int h \,.\, h_f \, \mathrm{d}\mu_1 = \sum_{i=1}^{\infty} \mathbf{W} \int h \,.\, h_{f_i} \mathrm{d}\mu_1 = \sum_{i=1}^{\infty} \mathbf{W} \int h^* \,.\, f_i \, \mathrm{d}\mu_0 = \mathbf{W} \int h^* \,.\, f \, \mathrm{d}\mu_0 \,.$$

Hence (8.7.5) and (8.7.6) are satisfied. Thus $f \in \mathscr{A}$, i. e., we have proved that $\mathscr{A}_{\sigma+} \subset \mathscr{A}$. The relation $\mathscr{A}_{\times} \subset \mathscr{A}$ can be proved in an analogous way.

2. Let $\{f_1, f_2\} \subset \cdot\, \mathscr{A}$, $f_0 \in \mathscr{B}$, $f_1 \leqq f_2 \leqq f_0$, let h_{f_1}, h_{f_2} satisfy (8.7.3) to (8.7.6). Let

$$M_1 = \{[x_1, \omega]; [x_1, \omega] \in X_1 \times \Omega \,, \quad h_{f_1}(x_1, \omega) = +\infty\} \,,$$
$$M_2 = \{[x_1, \omega]; [x_1, \omega] \in X_1 \times \Omega - M_1 \,, \quad h_{f_2}(x_1, \omega) - h_{f_1}(x_1, \omega) < 0\} \,.$$

We note that

$$\mathbf{W} \int c_{M_j} \,.\, h_{f_i} \, \mathrm{d}\mu_1 = \mathbf{W} \int c_{M_j}^* \,.\, f_i \, \mathrm{d}\mu_0 \leqq \mathbf{W} \int f_0 \, \mathrm{d}\mu_0$$

where the last integral is finite according to (8.7.2). Hence it follows, in the first place, that

$$\mathbf{W} \int c_{M_1} \, \mathrm{d}\mu_1 = 0 \,.$$

For in the contrary case the integral $n_1(+\infty) \,.\, \mathbf{W}\int c_{M_1}\,\mathrm{d}\mu_1 = \mathbf{W}\int h_{f_1} \,.\, c_{M_1}\,\mathrm{d}\mu_1$ is infinite, which is impossible.

In the second place, according to (8.7.6),

$$n_1(0) = n_0(0) \leqq \mathbf{W}\int c^*_{M_2} \,.\, [f_2 - f_1]\,\mathrm{d}\mu_0 = \mathbf{W}\int c^*_{M_2} \,.\, f_2\,\mathrm{d}\mu_0 - \mathbf{W}\int c^*_{M_2} \,.\, f_1\,\mathrm{d}\mu_0 =$$
$$= \mathbf{W}\int c_{M_2} \,.\, h_{f_2}\,\mathrm{d}\mu_1 - \mathbf{W}\int c_{M_2} \,.\, h_{f_1}\,\mathrm{d}\mu_1 = -\,\mathbf{W}\int c_{M_2} \,.\, [h_{f_1} - h_{f_2}]\,\mathrm{d}\mu_1 \leqq n_1(0)\,.$$

Hence

$$\mathbf{W}\int c_{M_2}\,\mathrm{d}\mu_1 \leqq \lim_{n\to\infty} \mathbf{W}\int n \,.\, c_{M_2} \,.\, [h_{f_1} - h_{f_2}]\,\mathrm{d}\mu_1 = 0$$

and thus again $\mathbf{W}\int c_{M_2}\,\mathrm{d}\mu_1 = 0$.

We conclude that, if two functions in $\mathbf{m}^*_+\mathbf{W}_{\mu_1}$ agree on $A = X_1 \times \Omega - (M_1 \cup M_2)$, then they have the same $\mathbf{W}$-integral with respect to μ_1. Put $h_{f_2-f_1} = c_A \,.\, h_{f_2} - c_A \,.\, h_{f_1}$. Clearly $h_{f_2-f_1} \in \mathbf{m}^*_+\mathbf{W}_{\mu_1}$. Further the set $P(x_1)$ of all such $\omega \in \Omega$, for which $[x_1, \omega] \in M_1 \cup M_2$, belongs to $\mathbf{q}_0\mathscr{R}\mu_2$ according to the relations

$$\mathbf{W}\int f_1(x_1, *, .)\,\mathrm{d}\mu_2 \leqq \mathbf{W}\int f_2(x_1, *, .)\,\mathrm{d}\mu_2 \leqq \mathbf{W}\int f_0(x_1, *, .)\,\mathrm{d}\mu_2 \in \mathbf{n}(\mathbf{W}, \mathbf{q}_0\mathscr{R}\mu_2)\,.$$

But hence it follows that $h_{f_2-f_1}(x_1, .) \in \alpha_{f_2-f_1}(x_1)$ for every $x_1 \in X_1$. Thus $h_{f_2-f_1}$ satisfies the conditions (8.7.3) and (8.7.4).

Finally let $h = 1$ or $h \in \mathbf{m}^*_+\mathscr{D}\mu_1$. Then

$$\mathbf{W}\int h \,.\, h_{f_2-f_1}\,\mathrm{d}\mu_1 = \lim_{n\to\infty} \mathbf{W}\int (n \wedge h) \,.\, h_{f_2-f_1}\,\mathrm{d}\mu_1 =$$
$$= \lim_{n\to\infty} [\mathbf{W}\int (n \wedge h) \,.\, c_A \,.\, h_{f_2}\,\mathrm{d}\mu_1 - \mathbf{W}\int (n \wedge h) \,.\, c_A \,.\, h_{f_1}\,\mathrm{d}\mu_1] =$$
$$= \lim_{n\to\infty} [\mathbf{W}\int (n \wedge h) \,.\, h_{f_2}\,\mathrm{d}\mu_1 - \mathbf{W}\int (n \wedge h) \,.\, h_{f_1}\,\mathrm{d}\mu_1] =$$
$$= \lim_{n\to\infty} [\mathbf{W}\int (n \wedge h)^* \,.\, f_2\,\mathrm{d}\mu_0 - \mathbf{W}\int (n \wedge h)^* \,.\, f_1\,\mathrm{d}\mu_0] =$$
$$= \lim_{n\to\infty} \mathbf{W}\int (n \wedge h)^* \,.\, [f_2 - f_1]\,\mathrm{d}\mu_0 = \mathbf{W}\int h^* \,.\, [f_2 - f_1]\,\mathrm{d}\mu_0$$

(the integrals $\mathbf{W}\int (n \wedge h) \,.\, c_A \,.\, h_{f_1}\,\mathrm{d}\mu_1$ are finite, since $\mathbf{W}\int h_{f_1}\,\mathrm{d}\mu_1$ is so). Thus $h_{f_2-f_1}$ satisfies (8.7.5) and (8.7.6) and $f_2 - f_1 \in A$. Thus $\mathscr{A}_-(\mathscr{B}) \subset \mathscr{A}$.

3. Let $f = c_{A_1\times A_2\times B}$, $A_1 \in \mathbf{S}_1$, $A_2 \in \mathbf{S}_2$, $B \in \mathbf{W}$. We shall show that the function h_f, defined by $h_f = c_{A_1\times B} \,.\, {}^{X_1}g$, $g \in \mu_2(A_2)$, $g \geqq 0$, satisfies the conditions (8.7.3) to (8.7.6). First, since $g \in \mathbf{q}\mathscr{R}\mu_2 \subset \mathbf{W}$, we have $h_f \in \mathbf{W}_{\mu_1}$ and (8.7.3) holds. Further

$$\alpha_f(x_1) = \mathbf{W}\int c_{A_1}(x_1) \,.\, c_{A_2\times B}\,\mathrm{d}\mu_2 = c_{A_1}(x_1) \,.\, n_2(c_B) \,.\, \mu_2(A_2)\,;$$

thus $h_f(x_1, .) \in \alpha_f(x_1)$ and (8.7.4) holds.

We shall prove (8.7.6). Let us define, for every $h \in \mathbf{m}^*_+\mathbf{W}_{\mu_1}$,

$$J_1 h = \mathbf{W}\int h \,.\, h_f\,\mathrm{d}\mu_1\,, \quad J_2 h = \mathbf{W}\int h^* \,.\, f\,\mathrm{d}\mu_0\,.$$

If $h = c_{M\times N}$, $M \in \mathbf{S}_1$, $N \in \mathbf{W}$, then

$$J_1 h = \mathbf{W}\int c_{(A_1\cap M)\times(B\cap N)} \,.\, {}^{X_1}g\,\mathrm{d}\mu_1 = n_1(g) \,.\, n_1(c_{B\cap N}) \,.\, \mu_1(A_1 \cap M)\,.$$

From (8.7.2) and from the assumption $\mathbf{q}_0\mathscr{R}\mu_0 = \mathbf{q}_0\mathscr{R}\mu_1$ it follows that $n_1(g) \cdot$ $\cdot\, \mu_1(A_1 \cap M) = \mu_0((A_1 \cap M) \times A_2)$. Since $n_1(c_{B \cap N}) = n_0(c_{B \cap N})$, we obtain

$$J_1 h = n_0(c_{B \cap N}) \cdot \mu_0((A_1 \cap M) \times A_2) = \mathbf{W} \int c_{(A_1 \cap M) \times A_2 \times (B \cap N)} \, \mathrm{d}\mu_0 = J_2 h \,.$$

Thus J_1 and J_2 agree and are finite on $\mathbf{c}[\mathbf{S}_1 \circ \mathbf{W}]$. From Theorem 5.16 it follows that $J_1 = J_2$ and (8.7.6) is satisfied. In particular, if $M = A_1$, $N = B$, we have $h \,.\, h_f = h_f$, $h^* \,.\, f = f$, and thus (8.7.5) holds, too.

We have $f \in \mathscr{A}$ and thus $\mathscr{A} \supset \mathscr{B}$.

8.8. Remark. Theorem 8.7 is much weaker then the usual Fubini Theorem, for the relation (8.7.4) does not generally determine the integral $\mathbf{W} \int h_f \, \mathrm{d}\mu_1$.

We shall illustrate the situation by an example.

Let ξ be the Lebesgue measure on $(0, 1)$, let $(\mathbf{W}, \mathbf{W}_0)$ be induced by ξ, let $n(g)$ denote, for every $g \in \mathbf{m}^*\mathbf{W}$, the random variable in $\mathbf{n}^*(\mathbf{W}, \mathbf{W}_0)$ which contains g.

Let $\mathscr{D}\mu_1 = \mathbf{W}$ and $A \in \mathscr{D}\mu_1 \Rightarrow \mu_1(A) = n(c_A)$, let μ_2 be defined on $\{\emptyset, \{a\}\}$, $\mu_2(\{a\}) = n(1)$, $\mu_2(\emptyset) = n(0)$. Then $\mathscr{D}\mu_1 \times \mathscr{D}\mu_2 = \{A \times \{a\}, A \in \mathscr{D}\mu_1\}$ and we may define μ_0 by the relation $\mu_0(A \times \{a\}) = \mu_1(A)$. The assumptions of Theorem 8.7 are satisfied.

Let $f = 0$. Then $\alpha_f(x_1) = n(0)$ for every $x_1 \in X_1 = (0, 1)$. Choose $h_f(x_1, \omega) = c_{\{\omega\}}(x_1)$ for every $x_1 \in X_1$, $\omega \in (0, 1)$. Then $h_f(x_1, .) \in \alpha_f(x_1)$ for every $x_1 \in X_1$ but[9]) $\mathbf{W} \int h_f \, \mathrm{d}\mu_1 = n(1) \neq \mathbf{W} \int f \, \mathrm{d}\mu_0 = n(0)$.

8.9. Theorem. *Let μ be a strong measure and let ξ be a pseudoprobability inducing $(\mathbf{V}, \mathbf{V}_0)$.*

Then there exists one and only one real measure ν defined on $\mathscr{D}\mu \times \mathbf{V}$ such that

$$A \times B \in \mathscr{D}\mu \circ \mathbf{V} \Rightarrow \nu(A \times B) = \int_B \mu(A) \, \mathrm{d}\xi \,. \tag{8.9.1}$$

The measure ν satisfies the following conditions:

$$\mathbf{i}(\mu, \mathbf{V}) = \mathbf{m}^*_+ \mathscr{D}\bar{\nu} \tag{8.9.2}$$

and

$$f \in \mathbf{m}^*_+ \mathscr{D}\bar{\nu} \,, \quad B \in \mathbf{V} \Rightarrow \int_B (\mathbf{VI} \int f \, \mathrm{d}\mu) \, \mathrm{d}\xi = \int_{X \times B} f \, \mathrm{d}\bar{\nu} \,, \tag{8.9.3}$$

i. e.,

$$f \in \mathbf{m}^*_+ \mathscr{D}\bar{\nu},\ \Phi_f(B) = \int_{X \times B} f \, \mathrm{d}\bar{\nu} \quad \text{for every} \quad B \in \mathbf{V} \Rightarrow \mathbf{VI} \int f \, \mathrm{d}\mu = \frac{\mathrm{d}\Phi_f}{\mathrm{d}\xi} \,. \tag{8.9.4}$$

Proof. If we define ν_0 on $\mathbf{r}(\mathscr{D}\mu \circ \mathbf{V})$ by means of the relation (8.9.1) and by the additivity, then ν_0 is a real-valued, non negative and additive set function. If

$$\{D_i\}_{i=1}^{\infty} = \{\bigcup_{j=1}^{n_i} A_{ij} \times B_{ij}\}_{i=1}^{\infty} \subset \mathbf{r}(\mathscr{D}\mu \circ \mathbf{V}) \,,$$

[9]) See Remark 7.17.

where $(A_{ij} \times B_{ij}) \cap (A_{ik} \times B_{ik}) = \emptyset$ for $j \neq k$, is a non decreasing sequence of sets, the union of which is equal to $D \in \mathbf{r}(\mathscr{D}\mu \circ \mathbf{V})$, then

$$\nu_0(D_i) = \sum_{j=1}^{n_i} \int_{B_{ij}} \mu(A_{ij})\, \mathrm{d}\xi = \int \sum_{j=1}^{n_i} \chi_{B_{ij}} \,.\, \mu(A_{ij})\, \mathrm{d}\xi =$$
$$= \int [\mathbf{V} \int c_{D_i}\, \mathrm{d}\mu]\, \mathrm{d}\xi \nearrow \int [\mathbf{V} \int c_D\, \mathrm{d}\mu]\, \mathrm{d}\xi = \nu_0(D)\,,$$

for $c_{D_i} \nearrow c_D$ implies $\mathbf{V} \int c_{D_i}\, \mathrm{d}\mu \nearrow \mathbf{V} \int c_D\, \mathrm{d}\mu$. Thus ν_0 is σ-additive; we shall prove that it is σ-finite, too. It suffices to prove that for every $A \in \mathscr{D}\mu$ there exists a sequence $\{D_n\} \subset \mathscr{D}\nu_0$ such that $\bigcup D_n = A \times \Omega$ and $\nu_0(D_n) < +\infty$ for every n. Let $A \in \mathscr{D}\mu$. Then, since μ is supposed to be σ-finite (see 8.1), there exists a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathscr{D}\mu$ such that $\bigcup A_n = A$ and $\mu(A_n)$ is finite. We can choose, for every n, a finite function $g_n \in \mu(A_n)$. Put $B_{n,m} = \{\omega;\, g_n(\omega) < m\}$. We have $\nu_0(A_n \times B_{nm}) < +\infty$ for every n, m and $\bigcup_n \bigcup_m (A_n \times B_{nm}) = A$. Thus ν_0 is σ-finite and there exists (Theorem 5.15) a unique measure ν defined on $\mathbf{V}_\mu$ such that $\nu \succ \nu_0$.

Now let us denote by $\mathbf{C}$ the system of all sets in $\mathscr{D}\mu \circ \mathbf{V}$ of finite ν-measure. $\mathbf{C}$ is clearly a pseudolattice and $\mathbf{sC} = \mathbf{V}_\mu$. Fix $B \in \mathbf{V}$ and put $J_1 f = \int_{X \times B} f\, \mathrm{d}\nu$, $J_2 f = \int_B (\mathbf{V} \int f\, \mathrm{d}\mu)\, \mathrm{d}\xi$ for every $f \in \mathbf{m}_+^* \mathscr{D}\nu = \mathbf{m}_+^* \mathbf{V}_\mu$. From (8.9.1) it follows that J_1 and J_2 agree and are finite on $\mathbf{cC} \subset \mathbf{c}(\mathscr{D}\mu \circ \mathbf{V})$. Theorem 5.16 gives $J_1 = J_2$; we have proved (8.9.3) for $f \in \mathbf{m}_+^* \mathscr{D}\nu = \mathbf{m}_+^* \mathbf{V}_\mu$.

Now let $g_1 \leqq g_2$, $\{g_1, g_2\} \subset \mathbf{m}_+^* \mathbf{V}_\mu$. Then

$$\mathbf{V} \int g_1\, \mathrm{d}\mu = \mathbf{V} \int g_2\, \mathrm{d}\mu \Leftrightarrow \int g_1\, \mathrm{d}\nu = \int g_2\, \mathrm{d}\nu\,;$$

hence it is easy to see that both (8.9.2) and (8.9.3) hold.

Remark. Theorem 8.9 shows that the $\mathbf{V} \int .\, \mathrm{d}\mu$ can be defined as a Radon-Nikodym derivative (see 8.9.4). It is easy to see that all properties of the $\mathbf{V}$-integral studied up to this time (if we suppose that μ is strong) are easy consequences of the relation (8.9.4) and of the properties of the Radon-Nikodym derivatives.

Unfortunately this method cannot be applied if μ is not strong, for in this case there does not exist the real measure ν satisfying (8.9.1) and (8.9.4).

8.10. Theorem. *Let μ_1 and μ_2 be two σ-finite measures defined on a σ-algebra $\mathbf{S}$,*

$$\mathscr{R}\mu_i \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)\,, \quad i = 1, 2$$

and let the $\mathbf{V}$-integral exist with respect to μ_i for both $i = 1, 2$.

Let

$$f \in \mathbf{m}_+ \mathbf{V}_{\mu_1}\,, \quad \mathbf{V} \int f\, \mathrm{d}\mu_1 = 0 \Rightarrow \mathbf{V} \int f\, \mathrm{d}\mu_2 = 0\,. \tag{8.10.1}$$

Then there exists a $g \in \mathbf{m}_+^* \mathbf{V}_{\mu_1}$ such that

$$f \in \mathbf{i}(\mu_1, \mathbf{V}) \Rightarrow \mathbf{VI} \int f\, \mathrm{d}\mu_2 = \mathbf{VI} \int f \,.\, g\, \mathrm{d}\mu_1\,. \tag{8.10.2}$$

Proof. Let ξ be a pseudoprobability inducing $(\mathbf{V}, \mathbf{V}_0)$. Let ν_i be a measure satisfying (8.9.1) for $\mu = \mu_i$ $(i = 1, 2)$. Let $f \in \mathbf{m}_+\mathscr{D}\nu_1$, $\int f \,\mathrm{d}\nu_1 = 0$. Then from (8.9.3) it follows that $\mathbf{V} \int f \,\mathrm{d}\mu_1 = 0$; from (8.10.1) and (8.9.4) it follows that $\int f \,\mathrm{d}\nu_2 = 0$. Thus $\nu_2 << \nu_1$.

Now μ and ξ are σ-finite; this implies that ν_1 is σ-finite, too. Finally both ν_1, ν_2 are defined on the σ-algebra $\mathbf{S} \times \mathbf{V}$. Thus from Lemma 2.2 it follows that there exists a $g \in \mathbf{m}_+^*(\mathbf{S} \times \mathbf{V}) = \mathbf{m}_+^*\mathbf{V}_{\mu_1}$ such that

$$f \in \mathbf{m}_+^*\mathscr{D}\bar{\nu}_1 \Rightarrow \int f \,\mathrm{d}\bar{\nu}_2 = \int f \,.\, g \,\mathrm{d}\bar{\nu}_1 \,.$$

However, this implies (according to Theorem 8.9) the relation (8.10.2).[10])

8.11. Notation. If ν is a measure, T a transformation measurable $(\mathbf{T}, \mathscr{D}\nu)$, then we denote by νT^{-1} the measure defined on $\mathbf{T}$ by the relation

$$A \in \mathbf{T} \Rightarrow \nu T^{-1}(A) = \nu(T^{-1}(A)) \,.$$

We call the attention to the vagueness of the notation just introduced. Indeed, the measure νT^{-1} is not determined by ν, T but by ν, T and $\mathbf{T}$. However the σ-ring $\mathbf{T}$ will be always marked before using the symbol νT^{-1}.

8.12. Theorem. *Let T be a transformation measurable $(\mathbf{T}, \mathscr{D}\mu)$, let μT^{-1} be σ-finite. Let us denote, for every $f \in \mathbf{f}(\bigcup \mathbf{T} \times \Omega)$, by f_T the function defined on $X \times \Omega$ by the relation $x \in X$, $\omega \in \Omega \Rightarrow f_T(x, \omega) = f(T(x), \omega)$. Then the $\mathbf{W}$-integral with respect to μT^{-1} exists and*

$$f \in \mathbf{i}(\mu T^{-1}, \mathbf{W}) \Rightarrow f_T \in \mathbf{i}(\mu, \mathbf{W}), \quad \mathbf{WI} \int f \,\mathrm{d}\mu T^{-1} = \mathbf{WI} \int f_T \,\mathrm{d}\mu \,. \qquad (8.12.1)$$

Proof. Let $f \in \mathbf{m}_+^*\mathbf{W}_{\mu T^{-1}}$. Then clearly the $(\mathbf{T}, \mathscr{D}\mu)$ measurability of T implies $f_T \in \mathbf{m}_+^*\mathbf{W}_\mu$. Let us denote $Jf = \mathbf{W} \int f_T \,\mathrm{d}\mu$ for every $f \in \mathbf{m}_+^*\mathbf{W}_{\mu T^{-1}}$. Then J is a non negative, linear and from below continuous functional. Let

$$g \in \mathbf{m}_+^*\mathbf{W} \,, \quad f \in \mathbf{m}_+^*\mathbf{W}_{\mu T^{-1}} \,, \quad g \in \bar{g} \in \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0) \,.$$

Then $(g^\Omega \,.\, f)_T = g^\Omega \,.\, f_T$ and thus

$$J(g^\Omega \,.\, f) = \mathbf{W} \int g^\Omega \,.\, f_T \,\mathrm{d}\mu = \bar{g} \,.\, \mathbf{W} \int f_T \,\mathrm{d}\mu = \bar{g} \,.\, Jf \,.$$

Thus J satisfies (7.4.3). Further, if $A \in \mathscr{D}\mu T^{-1}$, we have

$$Jc_A^\Omega = \mathbf{W} \int [c_A^\Omega]_T \,\mathrm{d}\mu = \mathbf{W} \int c_{T^{-1}(A)}^\Omega \,\mathrm{d}\mu = \mu T^{-1}(A) \,.$$

Thus all the conditions of Definition 7.4 are satisfied and J is the (unique, according to the σ-finiteness of μT^{-1}) $\mathbf{W}$-integral with respect to μT^{-1}.

Now, if

$$g^1 \leqq f \leqq g^2 \,, \quad \{g^1, g^2\} \subset \mathbf{W}_{\mu T^{-1}} \,, \quad \mathbf{W} \int g^1 \,\mathrm{d}\mu T^{-1} = \mathbf{W} \int g^2 \,\mathrm{d}\mu T^{-1} \,,$$

then

$$g_T^1 \leqq f_T \leqq g_T^2 \,, \quad \mathbf{W} \int g_T^1 \,\mathrm{d}\mu = \mathbf{W} \int g_T^2 \,\mathrm{d}\mu \,;$$
$$\mathbf{WI} \int f \,\mathrm{d}\mu T^{-1} = \mathbf{W} \int g^1 \,\mathrm{d}\mu T^{-1} = \mathbf{W} \int g_T^1 \,\mathrm{d}\mu = \mathbf{WI} \int f_T \,\mathrm{d}\mu$$

and (8.12.1) holds.

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[10]) The Theorem is a generalization of a result of A. Špaček [10].

9. Conditional probability and expectation

9.1. Definition. α is a *probability*, if α is a real measure, $\mathscr{D}\alpha$ is a σ-algebra and $\alpha(\bigcup \mathscr{D}\alpha) = 1$.

9.2. Definition. A measure μ is called the *conditional probability* $(\alpha, V, \mathbf{V})$ (and denoted by $p_{\alpha,V,\mathbf{V}}$), if

$$\alpha \text{ is a probability, } \mathbf{V} \text{ is a } \sigma\text{-algebra,} \tag{9.2.1}$$

$$V \text{ is a transformation measurable } (\mathbf{V}, \mathscr{D}\mu)\,, \tag{9.2.2}$$

$$\mathscr{D}\mu = \mathscr{D}\alpha, \quad \mathscr{R}\mu \subset \mathbf{n}^*(\alpha V^{-1}) \qquad (\mathbf{V} = \mathscr{D}\alpha V^{-1})\,, \tag{9.2.3}$$

$$B \in \mathbf{V}\,, \quad A \in \mathscr{D}\alpha \Rightarrow \alpha(A \cap V^{-1}(B)) = \int_B \mu(A)\, \mathrm{d}\alpha V^{-1}\,. \tag{9.2.4}$$

9.3. Definition.[11]) If μ is the conditional probability $(\alpha, V, \mathbf{V})$ then the weak integral with respect to μ is called the *conditional expectation* $(\alpha, V, \mathbf{V})$ and denoted by $e_{\alpha,V,\mathbf{V}}$.

9.4. Theorem. *The necessary and sufficient condition for a measure μ to be a conditional probability $(\alpha, V, \mathbf{V})$ for some $\alpha, V, \mathbf{V}$ is that μ is induced by a degenerate functional and that $\bigcup \mathscr{D}\mu \in \mathscr{D}\mu$, $\mu(\bigcup \mathscr{D}\mu) = 1$.*

Proof. The necessity follows from the known properties of $e_{\alpha,V,\mathbf{V}}$ which is a degenerate functional inducing μ.

Conversely, let μ be induced by a degenerate functional and let $(\mathbf{V}, \mathbf{V}_0) = (\mathbf{q}\mathscr{R}\mu, \mathbf{q}_0\mathscr{R}\mu)$. Then according to Lemma 7.14 there exists a transformation V from $\bigcup\mathscr{D}\mu$ into $\bigcup\mathbf{V}$ such that

$$f \in \mathbf{m}^*_+\mathscr{D}\mu\,, \quad 0 \leq g \in \bar{g} \in \mathbf{n}^*_+(\mathbf{V}, \mathbf{V}_0) \Rightarrow$$
$$\Rightarrow f \,.\, g\, V \in \mathbf{m}^*_+\mathscr{D}\mu\,, \quad \int f \,.\, gV\, \mathrm{d}\mu = \bar{g} \,.\, \int f\, \mathrm{d}\mu\,.$$

Putting $f = 1$, $g = c_B$ we get $B \in \mathbf{V} \Rightarrow V^{-1}(B) \in \mathscr{D}\mu$, i. e., V is measurable $(\mathbf{V}, \mathscr{D}\mu)$. Putting $f = c_A$, $g = c_B$, we get $A \times B \in \mathscr{D}\mu \circ \mathbf{V} \Rightarrow \mu(A \cap V^{-1}(B)) = \chi_B \,.\, \mu(A)$.

Thus let us define α on $\mathscr{D}\mu$ by the relation $A \in \mathscr{D}\mu \Rightarrow \alpha(A) = \int \mu(A)\, \mathrm{d}\xi$, where ξ is a probability inducing $(\mathbf{V}, \mathbf{V}_0)$ (such a probability exists for $\mu(\bigcup \mathscr{D}\mu) = 1$ and thus $\mathbf{V} \neq \mathbf{V}_0$; in addition, $\mu(\bigcup \mathscr{D}\mu) = 1$ implies that α is a probability, too). We have

$$B \in \mathbf{V}\,, \quad A \in \mathscr{D}\mu \Rightarrow \int_B \mu(A)\, \mathrm{d}\xi = \int \chi_B \,.\, \mu(A)\, \mathrm{d}\xi = \int \mu(A \cap V^{-1}(B))\, \mathrm{d}\xi =$$
$$= \alpha(A \cap V^{-1}(B))$$

and thus (9.2.4) holds if only $\xi = \alpha V^{-1}$. But

$$B \in \mathbf{V} \Rightarrow \alpha V^{-1}(B) = \int \mu(V^{-1}(B))\, \mathrm{d}\xi = \int_B \mu(\bigcup \mathscr{D}\mu)\, \mathrm{d}\xi = \int_B \mathrm{d}\xi = \xi(B)\,.$$

Thus $\mu = p_{\alpha,V,\mathbf{V}}$.

Remark. Theorem 9.4 is closely related to the results of Moy [8], who, however, assumes the measure α to be given in advance.

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[11]) Obviously our definition coincides with the usual one.

9.5. Theorem. *For every conditional probability $p = p_{\alpha,V,\mathbf{V}}$ the $\mathbf{V}$-integral exists; therefore p is strong.*

Proof. p is induced by a degenerate functional (Theorem 9.4), is therefore strong and the $\mathbf{q}\mathscr{R}p$-integral with respect to p exists. Thus it suffices to prove $\mathbf{q}\mathscr{R}p = \mathbf{V}$. If $A \in \mathbf{V}$, then $V^{-1}(A) \in \mathscr{D}\mu$ and $p(V^{-1}(A)) = \chi_A$. Hence $\mathbf{V} \subset \mathbf{q}\mathscr{R}p$; the contrary inclusion follows from (9.2.3). Thus $\mathbf{V} = \mathbf{q}\mathscr{R}p$.

9.6. Theorem. *Let μ be a strong measure, $\bigcup \mathscr{D}\mu \in \mathscr{D}\mu$, $\mu(\bigcup \mathscr{D}\mu) = 1$. Then there exists a conditional probability $p = p_{\alpha,V,\mathbf{V}}$ (for some α, V, $\mathbf{V}$) and a transformation T measurable $(\mathscr{D}\mu, \mathscr{D}p)$ such that $\mu = pT^{-1}$.*

Proof. Let ξ be such a probability that $\mathscr{R}\mu \subset \mathbf{n}^*_+\xi$ and such that the $\mathscr{D}\xi$-integral with respect to μ exists. Then according to Theorem 8.9 there exists a measure ν satisfying (8.9.1), if we denote $\mathscr{D}\xi = \mathbf{V}$. Let us define

$$p(D) = \mathbf{V}\int c_D \, \mathrm{d}\mu \quad \text{for every} \quad D \in \mathscr{D}\mu \times \mathbf{V}\,,$$
$$T(x, \omega) = x\,, \quad V(x, \omega) = \omega \quad \text{for every} \quad x \in \bigcup \mathscr{D}\mu\,, \quad \omega \in \bigcup \mathbf{V}\,.$$

Then T is $(\mathscr{D}\mu, \mathscr{D}p)$ measurable, V is $(\mathbf{V}, \mathscr{D}p)$ measurable and, for every $A \in \mathscr{D}\mu$, $B \in \mathbf{V}$, we have

$$pT^{-1}(A) = \mathbf{V}\int c_{A\times\Omega} \, \mathrm{d}\mu = \mu(A)\,,$$
$$\nu V^{-1}(B) = \nu(\bigcup \mathscr{D}\mu \times B) = \int_B \mu(\bigcup \mathscr{D}\mu) \, \mathrm{d}\xi = \xi(B)\,.$$

Finally, if $D \in \mathscr{D}p$, $B \in \mathbf{V}$, $X = \bigcup \mathscr{D}\mu$, we have (see (8.9.3))

$$\nu(D \cap V^{-1}(B)) = \nu(D \cap (X \times B)) = \int_{X\times B} c_D \, \mathrm{d}\nu = \int_B (\mathbf{V}\int c_D \, \mathrm{d}\mu) \, \mathrm{d}\xi =$$
$$= \int_B p(D) \, \mathrm{d}\nu V^{-1}\,.$$

Thus p is the conditional probability $(\nu, V, \mathscr{D}\xi)$, $\mu = pT^{-1}$, q. e. d.

9.7. Theorem. *Let μ be a measure, $\bigcup \mathscr{D}\mu \in \mathscr{D}\mu$, $\mu(\bigcup \mathscr{D}\mu) = 1$, $\mathscr{R}\mu \subset \mathbf{n}^*(\mathbf{V}, \mathbf{V}_0)$; let the $\mathbf{V}$-integral with respect to μ exist. Then there exist α and V such that $\mathbf{V}_\mu = \mathscr{D}\alpha$ and the $\mathbf{V}$-integral is the conditional expectation $(\alpha, V, \mathbf{V})$, i. e.,*

$$f \in \mathbf{m}^*_+\mathbf{V}_\mu \Rightarrow \mathbf{V}\int f \, \mathrm{d}\mu = e_{\alpha,V,\mathbf{V}}f\,.$$

Proof. Regarding the proof of the preceding Theorem, we see that $\mathbf{V}\int . \, \mathrm{d}\mu$ is the weak integral with respect to the conditional probability p. Thus it is the conditional expectation.

10. Further properties of conditional expectation

10.1. Remark. In this section we shall give some generalizations of the results of [3].

We recapitulate the problem. Suppose that α is a probability, T and V are two $(\mathscr{D}\alpha)$ measurable functions and that h is a non negative real-valued function defined on $E \times E$ such that $h(T, V)$ and $h(T, v)$ (for every $v \in E$) are $(\mathscr{D}\alpha)$ measurable functions. Now let $g \in e_{\alpha,V,\mathfrak{B}}h(T, V)$, $g_v \in e_{\alpha,V,\mathfrak{B}}h(T, v)$ (for every $v \in E$). If the set $\{v\}$ has a positive αV^{-1}-measure, then it is easy to see and well known that $g_v(v) = g(v)$; roughly speaking, if $\alpha V^{-1}(\{v\}) > 0$, then the conditional expectation given $V = v$ of the function $h(T, V)$ equals to that of the function $h(T, v)$. The paper [3] and the following section are devoted to similar considerations in the more general case without the assumption $\alpha V^{-1}(\{v\}) > 0$.

10.2. Assumptions. We assume that α is a probability, T and V are two transformations measurable $(\mathbf{T}, \mathscr{D}\alpha)$ and $(\mathbf{V}, \mathscr{D}\alpha)$ respectively, $(\mathbf{V}, \mathbf{V}_0)$ is the measurable space induced by αV^{-1}. Let $[T, V]$ denote the transformation measurable $(\mathbf{T} \times \mathbf{V}, \mathscr{D}\alpha)$ defined by the relation $[T, V](x) = [T(x), V(x)] \in \in \bigcup \mathbf{T} \times \bigcup \mathbf{V}$ for every $x \in \bigcup \mathscr{D}\alpha$. We denote by p the conditional probability $p_{\alpha,V,\mathbf{V}}$, by $\mathbf{e}$ the conditional expectation $e_{\alpha,V,\mathbf{V}}$; we denote $\bigcup \mathscr{D}\alpha = X$, $\bigcup \mathbf{V} = \Omega$; if $f \in \mathbf{f}^*(\bigcup \mathbf{T} \times \Omega)$ then by f_T we denote (as in Theorem 8.11) the function defined on $X \times \Omega$ by the relation $f_T(x, \omega) = f(Tx, \omega)$ for every $[x, \omega] \in X \times \Omega$. By ν_T we denote the probability satisfying (8.9.1) with $\mu = pT^{-1}$ and $\xi = = \alpha V^{-1}$. The probability ν_T, which is closely related with the $\mathbf{V}$-integral with respect to pT^{-1}, has now a self-reliant meaning. For, if $A \in \mathbf{T}$ and $B \in \mathbf{V}$, then

$$\nu_T(A \times B) = \int_B pT^{-1}(A)\, \mathrm{d}\alpha V^{-1} = \alpha(T^{-1}(A) \cap V^{-1}(B)) .$$

Thus $\nu_T = \alpha[T, V]^{-1}$ (for preventing misunderstandings we recapitulate that $\mathscr{D}\alpha[T, V]^{-1} = \mathbf{T} \times \mathbf{V}$).

If, in particular, T is the identical transformation, then we write $\nu_T = \nu$. Finally let us denote

$$D = \{[x, \omega];\ x \in X,\ \omega \in \Omega,\ Vx = \omega\}, \quad D_1 = \{[\omega, \omega];\ \omega \in \Omega\} .$$

10.3. Lemma. *Let ν^* be the outer measure $*$-induced by the measure ν. Then $\nu^*(D) = 1$.*

Proof. $\{A_i\}_{i=1}^{\infty} \subset\cdot\, \mathscr{D}\alpha$, $\{B_i\}_{i=1}^{\infty} \subset\cdot\, \mathbf{V}$, $\bigcup_{i=1}^{\infty} A_i \times B_i \supset D \Rightarrow \sum_{i=1}^{\infty} \nu(A_i \times B_i) = $

$$= \sum_{i=1}^{\infty} \alpha(A_i \cap V^{-1}(B_i)) \geqq \alpha\Big(\bigcup_{i=1}^{\infty} (A_i \cap V^{-1}(B_i))\Big) = \alpha(X) = 1.$$

10.4. Lemma. *If $D_1 \in \mathbf{V} \times \mathbf{V}$, then $D \in \mathscr{D}\alpha \times \mathbf{V}$.*

Proof. If we define $U(x, \omega) = [Vx, \omega]$ for every $[x, \omega] \in X \times \Omega$, then U is $(\mathbf{V} \times \mathbf{V}, \mathscr{D}\alpha \times \mathbf{V})$ measurable and thus $D = U^{-1}(D_1) \in \mathscr{D}\alpha \times \mathbf{V}$.

10.5. Theorem. *Let $f \in \mathbf{m}_+^*\mathscr{D}\alpha$, $h \in \mathbf{i}(pT^{-1}, \mathbf{V})$, $f(x) = h(Tx, Vx)$ for every $x \in X$. Then $\mathbf{e}f = \mathbf{VI} \int h \,\mathrm{d}pT^{-1}$.*

Proof. From $h \in \mathbf{i}(pT^{-1}, \mathbf{V})$ and from Theorem 8.11 it follows that $h_T \in \mathbf{i}(p, \mathbf{V})$. Hence and from Theorem 8.9 we obtain that $h_T \in \mathbf{m}_+^*\mathscr{D}\nu$. Further h_T and f^{Ω} agree on D, $\nu^*(D) = 1$ and $\{h_T, f^\Omega\} \subset \mathbf{m}^*_+\mathscr{D}\bar{\nu}$; it follows that $h_T = f^\Omega\,[\bar{\nu}]$. Thus $h_T = f^\Omega\,[p, \mathbf{V}]$ and

$$\mathbf{e}f = \int f\,\mathrm{d}p = \mathbf{V}\int f^\Omega\,\mathrm{d}p = \mathbf{VI}\int h_T\,\mathrm{d}p = \mathbf{VI}\int h\,\mathrm{d}pT^{-1}\,,$$

which is the desired result.

10.6. Corollary. *Let $D \in \mathscr{D}\alpha \times \mathbf{V}$ or $D_1 \in \mathbf{V} \times \mathbf{V}$. Let $f \in \mathbf{m}^*_+\mathscr{D}\alpha$, $h \in \mathbf{f}^*_+(X \times \Omega)$, $f(x) = h(x, Vx)$ for every $x \in X$.*

Then $\mathbf{e}f = \mathbf{VI}\int h\,\mathrm{d}p$.

Proof. The measurability of D implies $\nu(D) = 1$ (Lemmas 10.3 and 10.4); thus $f^\Omega = h[\bar{\nu}]$ and $h \in \mathbf{i}(p, \mathbf{V})$. The assumptions of the preceding Theorem are satisfied for the identical transformation T and thus $\mathbf{e}f = \mathbf{VI}\int h\,\mathrm{d}p$.

10.7. Theorem. *Let the assumptions of Theorem* 10.5 *hold. Let P be a function defined on $\mathbf{T} \times \Omega$ such that $P(., \omega)$ is a probability for every $\omega \in \Omega$ and $P(A, .) \in \overline{pT^{-1}(A)}$ for every $A \in \mathbf{T}$. Then, if $g \in \mathbf{f}^*\Omega$, $g(\omega) = \int h(., \omega)\,\mathrm{d}P(., \omega)\ [\overline{\alpha V^{-1}}]$, then $g \in \mathbf{e}f$.*

Proof. From Theorem 8.6 it follows that $g \in \mathbf{VI}\int h\,\mathrm{d}pT^{-1}$ and $\mathbf{VI}\int h\,\mathrm{d}pT^{-1} = \mathbf{e}f$ according to Theorem 10.5.

10.8. Definition. T and V are *α-independent,* if

$$A \in \mathbf{T},\ B \in \mathbf{V} \Rightarrow \alpha(T^{-1}(A) \cap V^{-1}(B)) = \alpha T^{-1}(A)\,.\,\alpha V^{-1}(B).$$

10.9. Theorem. *Let the assumptions of Theorem* 10.5 *hold, let T and V be α-independent.*

Then if $g \in \mathbf{f}^*\Omega$, $g(\omega) = \int h(., \omega)\,\mathrm{d}\alpha T^{-1}\ [\overline{\alpha V^{-1}}]$, then $g \in \mathbf{e}f$.

Proof. If we define $P(A, \omega) = \alpha T^{-1}(A)$, then the conditions of the preceding Theorem hold, since the relation $\alpha T^{-1}(A) \in \overline{pT^{-1}(A)}$ is an easy consequence of the independence of T and V. (We note that the Theorem can be also proved by a direct verification of the relation $B \in \mathbf{V} \Rightarrow \int_B g\,\mathrm{d}\alpha V^{-1} = \int_{V^{-1}(B)} f\,\mathrm{d}\alpha$ by the use of the Fubini Theorem.)

10.10. Remark. In all Theorems in this section we have assumed that $h \in \mathbf{i}(pT^{-1}, \mathbf{V})$. This condition is necessary for the integral $\mathbf{VI}\int h\,\mathrm{d}pT^{-1}$ to have a meaning and thus it is necessary in Theorem 10.5. However in Theorems 10.7 and 10.9 the integrals $\int h(., \omega)\,\mathrm{d}P(., \omega)$ and $\int h(., \omega)\,\mathrm{d}\alpha T^{-1}$ are defined for an ampler class of functions that $\mathbf{i}(pT^{-1}, \mathbf{V})$. Nevertheless we shall show that the condition $h \in \mathbf{i}(pT^{-1}, \mathbf{V})$ is essential. For simplicity we shall assume that T and V are α-independent and that h is a characteristic function. We remark that $\mathbf{i}(pT^{-1}, \mathbf{V}) = \mathscr{D}\bar{\nu}_T = \mathscr{D}\overline{\alpha[T, V]^{-1}}$.

10.11. Theorem. *Let $\mathbf{T}$ be a σ-algebra; let $A \subset \bigcup \mathbf{T} \times \Omega$; $h = c_A$; $h(., \omega) \in \mathbf{m}\mathscr{D}\overline{\alpha T^{-1}}$ for every $\omega \in \Omega - V_0$, $V_0 \in \mathbf{V}_0$; $f = h(T(.), V(.)) \in \mathbf{m}\,\mathscr{D}\alpha$; h non $\in \mathscr{D}\overline{\alpha[T, V]^{-1}}$. Finally, let T and V be α-independent.*

Put $g(\omega) = \int h(., \omega)\, \mathrm{d}\alpha T^{-1}$ *for* $\omega \in \Omega - V_0$ *and define* $g(\omega)$ *for* $\omega \in V_0$ *in an arbitrary way.*

Then there exists a probability β *with the following properties*: T *and* V *are* β*-independent and measurable* $(\mathbf{T}, \mathscr{D}\beta)$ *and* $(\mathbf{V}, \mathscr{D}\beta)$ *respectively*, $\alpha T^{-1} = \beta T^{-1}$, $\alpha V^{-1} = \beta V^{-1}$, $f \in \mathbf{m}^*_+ \mathscr{D}\beta$ *but* g non $\in e_{\beta,V,\mathbf{V}} f$ *although obviously* $g(\omega) = \int h(., \omega)\, \mathrm{d}\overline{\beta T^{-1}}\ \ [\overline{\beta V^{-1}}]$.

Proof. If g non $\in e_{\alpha,V,\mathbf{V}} f$, then the Theorem holds. We shall consider the case $g \in e_{\alpha,V,\mathbf{V}} f$. Since it is assumed h non $\in \mathbf{i}(p_{\alpha,V,\mathbf{V}} T^{-1}, \mathbf{V}) = \mathbf{m}^*_+ \mathscr{D}\overline{\nu}_T = \mathbf{m}^*_+ \mathscr{D}\overline{\alpha[T, V]^{-1}}$, $h = c_A$, we have A non $\in \mathscr{D}\overline{\alpha[T, V]^{-1}}$. Put

$$\mathbf{S} = \{\tilde{B};\ \tilde{B} = [T, V]^{-1}(B),\ B \in \mathbf{T} \times \mathbf{V}\}$$

and denote $\alpha_0 = \alpha_{\mathbf{S}}$. Then α_0 is a probability, too, and obviously $\tilde{A}$ non $\in \mathscr{D}\overline{\alpha}_0$, where $\tilde{A} = [T, V]^{-1}(A)$. Hence if follows that there exist infinitely many measures β defined on $\mathbf{s}\{\mathbf{S} \cup \{\tilde{A}\}\}$ and such that $\beta \succ \alpha_0$. (See for example [4], Sec. 16, Ex. 2.) Choose β in such a way that $\beta(\tilde{A}) \neq \alpha(\tilde{A})$ (we note that $\tilde{A} \in \mathscr{D}\alpha$ since $f \in \mathbf{m}\mathscr{D}\alpha$). Since[12]) $\beta[T, V]^{-1} = \alpha[T, V]^{-1}$ all the assertions of the Theorem are obvious, except possibly the assertion g non $\in e_{\beta,V,\mathbf{V}} f$.

We have $g \in e_{\alpha,V,\mathbf{V}} f$ and thus, since $\alpha V^{-1} = \beta V^{-1}$, $\int e_{\alpha,V,\mathbf{V}} f\, \mathrm{d}\alpha V^{-1} = \int f\, \mathrm{d}\alpha$, we obtain $\int g\, \mathrm{d}\beta V^{-1} = \int g\, \mathrm{d}\alpha V^{-1} = \int f\, \mathrm{d}\alpha = \alpha(\tilde{A}) \neq \beta(\tilde{A}) = \int f\, \mathrm{d}\beta$. Thus $\int g\, \mathrm{d}\beta V^{-1} \neq \int f\, \mathrm{d}\beta$, which gives g non $\in e_{\beta,V,\mathbf{V}} f$.

10.12. Examples of non regular conditional probabilities. Theorem 10.11 together with Theorem III of [3] enables us to construct many examples of non regular conditional probabilities. (A conditional probability $p_{\alpha,V,\mathbf{V}} = p$ is regular, if there exists a function P defined on $\mathscr{D}p \times \Omega$ such that $P(., \omega)$ is a probability for every $\omega \in \Omega$ and $P(A, .) \in p(A)$ for every $A \in \mathscr{D}p$.)

Suppose that the conditions of the preceding Theorem are satisfied and that in addition the σ-algebras $\mathbf{V}$ and $\mathbf{T}$ possess countable bases, i. e. that there exist two countable systems $\mathbf{S}_1$, $\mathbf{S}_2$ such that $\mathbf{s}\mathbf{S}_1 = \mathbf{V}$ and $\mathbf{s}\mathbf{S}_2 = \mathbf{T}$. Finally let $\mathbf{V}$ contain every set $\{\omega\}$, where $\omega \in \bigcup \mathbf{V}$.

Now if β satisfies the assertions of the Theorem, in particular if g non $\in e_{\beta,V,\mathbf{V}} f$, then $p_{\beta,V,\mathbf{V}}$ cannot be regular. Indeed, if $p_{\beta,V,\mathbf{V}}$ is regular, then according to Theorem III of [3] $g \in e_{\beta,V,\mathbf{V}} f$, which is impossible.

The following example shows that the assumptions can be satisfied:

Let $\mathscr{B}_1$ be the system of all Borel subsets of $\langle 0, 1\rangle$, let Λ be the Lebesgue measure on $X = \langle 0, 1\rangle \times \langle 0, 1\rangle$. Choose a set $A \subset X$ in such a way, that A non $\in \mathscr{D}\Lambda$ but that for every $\omega \in \langle 0, 1\rangle$ the set $\{t;\ (t, \omega) \in A\}$ belongs to $\mathscr{B}_1$.

Let now α be a probability such that $\alpha \succ \Lambda$ and $\mathscr{D}\alpha = \mathbf{s}(\mathscr{D}\Lambda \cup \{A\})$. Put $\mathbf{T} = \mathbf{V} = \mathscr{B}_1$, $T(t, \omega) = t$, $V(t, \omega) = \omega$ for every $(t, \omega) \in X$. It is easy to see

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[12]) Again we put $\mathscr{D}\beta(T, V)^{-1} = \mathbf{T} \times \mathbf{V}$.

that all required conditions are satisfied (see Theorem 10.11 and assumptions 10.2). In particular $h(.,\omega) \in \mathbf{m}^*_+\mathscr{B}_1 \subset \mathbf{m}^*_+\overline{\mathscr{D}\alpha T^{-1}}$, $h(T(.), V(.)) = h = c_A \in \mathbf{m}^*_+\mathscr{D}\alpha$. Since $\overline{\alpha[T, V]^{-1}} = A$, we have also h non $\in \mathbf{m}^*_+\mathscr{D}\alpha[T, V]^{-1}$. Finally T and V are α-independent and $\mathbf{T} = \mathbf{V} = \mathscr{B}_1$ has a countable basis and contains every set $\{\omega\} \subset \bigcup \mathbf{V} = \langle 0, 1\rangle$.

Some symbols

$\mathscr{A}_\vee, \mathscr{A}_\wedge, \mathscr{A}_-(\mathscr{B}), \mathscr{A}_-, \mathscr{A}_{\sigma\vee}$	(1.5)	$\mathbf{S}_1 \times \mathbf{S}_2$	(7.1)
$\mathscr{A}_+, \mathscr{A}_{\sigma+}, \mathscr{A}_\times$	(4.1)	$\sup_Y A$	(1.7)
$<<$	(2.1)	$\subset\cdot, \cdot\supset, \doteq$	(1.1)
$\mathfrak{B}$	(1.2)	$\inf_Y A$	(1.7)
$\succ$	(1.3)	$\inf A$	(3.6)
c_A	(1.5)	$\int . \, d\mu$	(1.6), (6.3)
$\mathbf{c}$	(1.5)	$\mathbf{W} \int . \, d\mu$	(8.1)
χ_A	(1.3)	$\mathbf{WI} \int . \, d\mu$	(8.4)
$\mathscr{D}$	(1.3)	$\bar{J}$	(5.6)
$\mathscr{D}^2$	(5.1)	$\mathbf{k}$	(1.5)
E, E^*, E_+, E^*_+	(1.1)	$\mathbf{m}, \mathbf{m}^*, \mathbf{m}_+, \mathbf{m}^*_+$	(1.5)
$e_{\alpha,z,\mathbf{V}}$	(9.3)	$\bar{\mu}$	(5.14)
$f^\Omega, {}^\Omega f$	(7.2)	μZ^{-1}	(8.11)
$\bigvee_{i=1}^{k} f_i = f_1 \vee \ldots \vee f_k$	(1.4)	$\mathbf{n}, \mathbf{n}^*, \mathbf{n}_+, \mathbf{n}^*_+$	(1.6)
$\bigwedge_{i=1}^{k} f_i = f_1 \wedge \ldots \wedge f_k$	(1.4)	$\frac{d\nu}{d\mu}$	(2.3)
f_+, f_-	(1.4)	$\mathbf{P}_A$	(3.14)
$f = g\ [\mu, \mathbf{W}]$	(8.2)	$p_{\alpha,z,\mathbf{V}}$	(9.2)
f_A	(1.3)	$\mathbf{q}, \mathbf{q}_0$	(1.8)
$\mathbf{f}, \mathbf{f}^*, \mathbf{f}_+, \mathbf{f}^*_+$	(1.4)	$\mathbf{r}$	(1.2)
$\mathbf{i}$	(8.2)	$\mathscr{R}$	(1.3)
$\mathfrak{s}$	(1.2)	T_A	(1.3)
$\mathbf{S}_\cup, \mathbf{S}_{\sigma\cup}, \mathbf{S}_\cap, \mathbf{S}_-$	(1.2)	VT	(1.3)
$\mathbf{S}_1 \circ \mathbf{S}_2$	(7.1)	$\mathbf{V}_\mu$	(7.1)
		$\tilde{Y}, -\tilde{\infty}, +\tilde{\infty}$	(3.7)

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Резюме

О МЕРАХ, ЗНАЧЕНИЯ КОТОРЫХ — КЛАССЫ ЭКВИВАЛЕНТНЫХ ИЗМЕРИМЫХ ФУНКЦИЙ

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Пусть ξ — вероятность определенная на σ-алгебре $\mathbf{V}$ подмножеств множества Ω. Пространство всех ($\mathbf{V}$) измеримых функций разбито на классы функций, почти всюду взаимно равных. Эти классы называем случайными величинами. Пространство всех неотрицательных (не обязательно конечных) случайных величин обозначим через $\mathbf{n}_+^* \xi$.

В работе рассматриваются понятия меры, интеграла и функционала, значения которых не вещественные числа, но случайные величины. Доказана теорема о распространении меры с кольца на σ-кольцо.

Пусть мера μ определена на σ-кольце $\mathbf{S}$ подмножеств множества X. Слабый интеграл $\int . \mathrm{d}\mu$ определен для неотрицательных ($\mathbf{S}$) измеримых функций так, что он аддитивен и что $f_n \nearrow f \Rightarrow \int f_n \,\mathrm{d}\mu \nearrow \int f \,\mathrm{d}\mu$. Доказана теорема о представлении линейного функционала в виде слабого интеграла.

Для σ-алгебры $\mathbf{W} \subset \mathbf{V}$ определяется понятие $\mathbf{W}$-интеграла $\mathbf{W}\int . \mathrm{d}\mu$ для абстрактных функций, определенных на множестве X, значения которых ($\mathbf{W}$) измеримые вещественные функции. Очевидно, такие абстрактные функции можно рассматривать как вещественные функции на $X \times \Omega$.

Итак, $\mathbf{W}\int . \mathrm{d}\mu$ определяется для неотрицательных ($\mathbf{S} \times \mathbf{W}$) измеримых функций. При этом ($\mathbf{S} \times \mathbf{W}$) — σ-кольцо, порожденное классом всех множеств вида $A \times B$, $A \in \mathbf{S}$, $B \in \mathbf{W}$.

$\mathbf{W}\int . \mathrm{d}\mu$ (a) аддитивен, (b) $f_n \nearrow f \Rightarrow \mathbf{W}\int f_n \, \mathrm{d}\mu \nearrow \mathbf{W}\int f \, \mathrm{d}\mu$ и (c), если $f(x, \omega) = \tilde{f}(\omega)$ для ($\mathbf{S} \times \mathbf{W}$) измеримой неотрицательной функции f, то $\mathbf{W}\int f \, \mathrm{d}\mu = \int \tilde{f} \, \mathrm{d}\mu$. Наконец, $\mathbf{W}$-интеграл однороден в следующем смысле: (d) если f и g — ($\mathbf{S} \times \mathbf{W}$) измеримые неотрицательные функции, $g(x, \omega) = \tilde{g}(\omega)$, то $\mathbf{W}\int g . f \, \mathrm{d}\mu = \bar{g} . \mathbf{W}\int f \, \mathrm{d}\mu$, где $\bar{g}$ означает случайную величину, содержающаю функцию $\tilde{g}$. Нетрудно показать, что, если μ σ-конечна, то $\mathbf{W}$-интеграл определяется условиями (a)—(d) однозначно, но мы не знаем, существует-ли он всегда. Если $\mathbf{W}_1$, $\mathbf{W}_2$ — σ-алгебры, $\mathbf{W}_1 \subset \mathbf{W}_2 \subset \mathbf{V}$ и если $\mathbf{W}_2 \int . \mathrm{d}\mu$ существует, то существует и $\mathbf{W}_1 \int . \mathrm{d}\mu$. Если существует $\mathbf{V}\int . \mathrm{d}\mu$, то мы скажем, что μ — сильная мера. Приведены общие достаточные условия для того, чтобы μ была сильной мерой. Например, если $\mathbf{S}$ — σ-кольцо всех борелевских или бэровских множеств локально компактного пространства, или если $\mathbf{V}$ — σ-алгебра всех борелевских множеств локально компактного хаусдорфова пространства, то μ — сильная мера. Также, если μ — условная вероятность, то μ является сильной мерой. Наоборот, если μ — сильная мера, и если $\mu(X) = 1$, то существует условная вероятность p на σ-кольце $\mathbf{Q}$ и отображение T множества $\bigcup \mathbf{Q}$ на множество X так, что $\mu(A) = p(T^{-1}(A))$ для всякого множества $A \in \mathbf{S}$.

Для $\mathbf{W}$-интеграла доказаны теоремы, аналогичные теоремам Фубини и Радона-Никодима. Между $\mathbf{W}$-интегралом и обыкновенным интегралом Лебега такая связь: Пусть $\nu(A, \omega)$ для каждого фиксированного $\omega \in \Omega$ является вещественной мерой на $\mathbf{S}$, пусть для каждого $A \in \mathbf{S}$ $\nu(A, \omega)$, как функция переменной $\omega \in \Omega$, является элементом случайной величины $\mu(A)$. Пусть существует интеграл $\mathbf{W}\int . \mathrm{d}\mu$ и пусть f — неотрицательная ($\mathbf{S} \times \mathbf{W}$) измеримая функция. В таком случае функция $h(\omega) = \int f(., \omega) \, \mathrm{d}\nu(., \omega)$ является элементом случайной величины $\mathbf{W}\int f \, \mathrm{d}\mu$.

В конце работы рассматриваются приложения, касающиеся условных вероятностей.