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CHARACTERIZATION OF FOURIER-STIELTJES TRANSFORMS OF VECTOR AND OPERATOR VALUED MEASURES

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Let S be a locally compact Abelian group, Σ its character group, $\mathscr{B} = \mathscr{B}(\Sigma)$ the system of Borel sets in Σ and X a Banach space. A function $\varphi : S \rightarrow X$ is called the Fourier-Stieltjes transform of a vector measure $m : \mathscr{B} \rightarrow X$, if $\varphi(s) = \int \langle \overline{s, \sigma} \rangle \, \mathrm{d}m(\sigma)$ for $s \in S$. It is proved in Section 3 that a function φ is the Fourier-Stieltjes transform of a vector measure if and only if the set $\{\int f(s) \, \varphi(s) \, \mathrm{d}s \mid \|\hat{f}\|_\infty \leqq 1, \, f \in L^1(S)\}$ is relatively weakly compact in X ($\hat{f}$ denotes the Fourier transform of f). Another necessary and sufficient condition for φ to be a Fourier-Stieltjes transform is the relatively weak compactness of the set $\{\int \varphi(s) \, \mathrm{d}\mu(s) \mid \|\hat{\mu}\|_\infty \leqq 1, \, \mu \in M_d\}$, where M_d is the algebra of all discrete measures on S. The results just mentioned constitute "vector" generalizations of analogous theorems of W. F. EBERLEIN [8].

The results of Section 3 are used in Section 4 to express a representation $U : S \rightarrow L(X)$ of S in the form $U(s) = \int \langle \overline{s, \sigma} \rangle \, \mathrm{d}P(\sigma)$, where $P : \mathscr{B} \rightarrow L(X)$ is a spectral measure and $L(X)$ denotes the algebra of bounded linear operators on X. Such an expressing is possible if and only if, for every $x \in X$, the set $\{\int f(s) \, U(s) \, x \, \mathrm{d}s \mid \|\hat{f}\|_\infty \leqq 1, \, f \in L^1(S)\}$ or, equivalently, the set $\{\int U(s) \, x \, \mathrm{d}\mu(s) \mid \|\hat{\mu}\|_\infty \leqq 1, \, \mu \in M_d(S)\}$ is relatively weakly compact in X. This is an extension onto an arbitrary Banach space of Ambrose's [1] generalization of a classical theorem of M. H. STONE (see e.g. [18; 36E]).

Sections 1 and 2 are introductory. Section 1 contains some results concerning vector and operator valued measures and representation of some transformations in form of integral with respect to such measures. In Section 2 some sufficient conditions are given for a vector-valued function to be integrable and for a representation of a group to be equivalent to the representation of its group-algebra.

In Section 5 the results of Section 4 are applied to give some criteria for an operator to be spectral in the sense of DUNFORD ([5] or [6]).

1. VECTOR AND OPERATOR MEASURES

Let S be a locally compact Hausdorff space. Let $\mathscr{B}_0(S)$, $\mathscr{B}_1(S)$ and $\mathscr{B}(S)$ (in short $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}$) be the σ-ring generated by all compact G_δ sets, all compact sets and all open sets, respectively. We call $\mathscr{B}_0$ the system of Baire sets, $\mathscr{B}_1$ the system of Borel sets in a restricted sense and $\mathscr{B}$ the system of Borel sets.

This terminology is slightly different from that of [9]. But for some purposes of the Harmonic Analysis it is very useful if every continuous function is Borel measurable. For instance we wish to integrate characters with respect to some finite measure on $\mathscr{B}$ (see also [10]).

Let X be a Banach space, X' its dual. $L(X)$ will stand for the algebra of all continuous linear operators on X.

Let $\mathscr{R}$ be a ring of sets. A function $m : \mathscr{R} \to X$ will be called a vector measure if it is σ-additive. A function $P : \mathscr{R} \to L(X)$ will be called an operator measure if it is σ-additive in the strong operator topology, i.e. if, for every $x \in X$, $P(\,)\,x$ is a vector measure. It is known that if $\mathscr{R}$ is a σ-ring then every weakly σ-additive function on $\mathscr{R}$ is a vector or operator measure. An operator measure is called multiplicative if, for $E, F \in \mathscr{R}$, $P(E \cap F) = P(E)\,P(F)$. A spectral measure is a multiplicative operator measure whose domain $\mathscr{R}$ is an algebra, i.e. $S \in \mathscr{R}$, and $P(S) = I$, where I is the identity operator on X.

A vector measure $m : \mathscr{B}_\iota \to X$ is said to be regular (ι denotes 0, 1 or is omitted) if, for every $E \in \mathscr{B}_\iota$ and $\varepsilon > 0$, there exists a compact set $C \in \mathscr{B}_\iota$ and an open set $U \in \mathscr{B}_\iota$ such that $C \subset E \subset U$ and $\|m(F)\| < \varepsilon$ for every $F \subset U - C$, $F \in \mathscr{B}_\iota$ (see [7; III.5.11] or [3]).

An operator measure $P : \mathscr{B} \to L(X)$ is said to be regular if, for every $x \in X$, the vector measure $P(\,)\,x$ is regular.

Lemma 1. *Let $m_0 : \mathscr{B}_0 \to X$ be a Baire measure. Then m_0 is regular and there exists a unique regular vector measure $m : \mathscr{B} \to X$ such that $m(E) = m_0(E)$ for $E \in \mathscr{B}_0$.*

Proof. It is proved in [3; Theorem 4] that m_0 is regular. It is proved further (Theorem 5) that there exists a unique regular vector measure $m_1 : \mathscr{B}_1 \to X$ coinciding with m_0 on $\mathscr{B}_0$. Following [13; Theorem 3.1], there exists a set $S_0 \in \mathscr{B}_1$ such that $m_1(E) = m_1(S_0 \cap E)$ for every $E \in \mathscr{B}_1$. As for every $S_1 \supset S_0$, $S_1 \in \mathscr{B}_1$, the relation $m_1(E) = m_1(S_1 \cap E)$ holds also for every $E \in \mathscr{B}_1$, and every set of $\mathscr{B}_1$ is contained in an open set of $\mathscr{B}_0$, we can choose a Baire set for S_0.

Define the vector measure $m : \mathscr{B} \to X$ by $m(E) = m_1(S_0 \cap E)$ for $E \in \mathscr{B}$. It is easy to see that m is a regular vector measure coinciding with m_0 on $\mathscr{B}_0$.

Let m' and m'' be two regular measures on $\mathscr{B}$ such that $m'(E) = m''(E)$ for $E \in \mathscr{B}_0$. Put $m'_\iota(E) = m'(E)$ and $m''_\iota(E) = m''(E)$, $E \in \mathscr{B}_\iota$, $\iota = 0, 1$. Then m'_ι and m''_ι are regular measures on B_ι. As $m'_0 = m''_0$, [3; Theorem 5] implies that $m'_1 = m''_1$, i.e. $m'(E) = m''(E)$ for all $E \in \mathscr{B}_1$.

Let us suppose now that $E \in \mathscr{B}$ is a set such that $m'(E) \neq m''(E)$. Denote $\varepsilon = \|m'(E) - m''(E)\|$. It follows from the regularity that there exist compact sets C' and C'' such that $C' \subset E$, $C'' \subset E$ and for every F, $C' \subset F \subset E$ or $C'' \subset F \subset E$ we have either $\|m'(E) - m'(F)\| < \frac{1}{2}\varepsilon$ or $\|m''(E) - m''(F)\| < \frac{1}{2}\varepsilon$. Then by putting $C = C' \cup C''$ with regard to $\varepsilon \leqq \|m'(E) - m'(C)\| + \|m''(E) - m''(C)\| < \varepsilon$ we obtain a contradiction.

Corollary 1. *If $m : \mathscr{B} \to X$ is a regular vector measure, then there exists a σ-compact (even Baire) set S_0 such that $m(E) = m(E \cap S_0)$ for $E \in \mathscr{B}$.*

Corollary 2. *Let m be a function on $\mathscr{B}_1$ or $\mathscr{B}$ with values in X such that $\langle m(\ ), x' \rangle$ is a regular scalar measure for every $x' \in X'$; then m is a regular vector measure.*

Proof. It is well known [7; IV.10.1] that m is a vector measure. Let m_0 be the restriction of m to $\mathscr{B}_0$. According to Lemma 1 there exists a unique regular vector measure m_1 on $\mathscr{B}_1$ or $\mathscr{B}$ coinciding with m_0 on $\mathscr{B}_0$. It is evident that the measures $\langle m_1(\ ), x' \rangle$, $x' \in X'$, are regular and coincide with $\langle m_0(\ ), x' \rangle$ on $\mathscr{B}_0$. The measures $\langle m(\ ), x' \rangle$, $x' \in X'$, also possess the same property. It follows that $\langle m(\ ), x' \rangle = \langle m_1(\ ), x' \rangle$ for $x' \in X'$. Therefore $m = m_1$.

As for integration with respect to vector measures we refer to [7] and [14] (in [7] the domain of the vector measure is supposed to be a σ-algebra, in [14] it may be an arbitrary δ-ring).

Integral $\int f \,\mathrm{d}m$ of a scalar-valued function f with respect to a vector measure m is the element of X such that

$$\left\langle \int f \,\mathrm{d}m, x' \right\rangle = \int f(s) \,\mathrm{d}\langle m(s), x' \rangle$$

for every $x' \in X'$.

The semi-variation $\|m\|$ of a vector measure $m : \mathscr{R} \to X$ is defined [7; IV.10.3] by

$$\|m\| (E) = \sup \left\| \sum_{i=1}^{k} \alpha_i \, m(E_i) \right\|$$

for every $E \in \mathscr{R}$, where the least upper bound is taken over all finite systems $E_1, \ldots, E_k$ of disjoint subsets of E belonging to $\mathscr{R}$ and complex numbers $|\alpha_i| \leqq 1$.

If $\mathscr{R}$ is a σ-ring, then $\|m\|$ is finite on $\mathscr{R}$.

If P is an operator measure and f a scalar function, then $\int f \,\mathrm{d}P$ is defined as the element of $L(X)$ for which

$$\left(\int f \,\mathrm{d}P \right) x = \int f(s) \,\mathrm{d}P(s)\, x$$

for $x \in X$.

Now let $C_0(S)$ stand for the set (Banach algebra) of all continuous functions vanishing at infinity.

Lemma 2. *Let D be a dense linear subspace of $C_0(S)$. Let $\Phi : D \to X$ be a linear mapping and let the set*

$$(1)\qquad \{\Phi(f) \mid \|f\|_\infty \leqq 1, f \in D\}$$

be relatively weakly compact in X.

Then Φ is a bounded operator and there exists a unique regular vector measure $m : \mathscr{B} \to X$ such that

$$(2)\qquad \Phi(f) = \int f \,\mathrm{d}m$$

for every $f \in D$. Moreover, $\|\Phi\| = \|m\|(S)$.

Conversely, if there exists a vector measure $m : B \to X$ such that (2) *holds, then* (1) *is a relatively weakly compact set.*

Proof. If the set (1) is relatively weakly compact, it is bounded and, consequently Φ is a bounded operator. D being dense, Φ can be extended onto all $C_0(S)$ without change of norm. The extension of Φ to $C_0(S)$ will be also denoted by Φ.

Being a subset of the closure of (1) and convex, the set $\{\Phi(f) \mid \|f\|_\infty \leqq 1, f \in C_0(S)\}$ is also relatively weakly compact in X [7; V.6.1].

For the case that S is a compact space the lemma is proved in [7; VI.7.3].

Suppose S is not compact. Denote $\tilde{S} = S \cup \{\infty\}$ the one-point compactification of S. $C(\tilde{S})$ stands for the space of all continuous complex functions on $\tilde{S}$. For every $\tilde{f} \in C(\tilde{S})$ there exists a unique function $f \in C_0(S)$ such that $\tilde{f}(s) = f(s) + \tilde{f}(\infty)$ for $s \in S$. Put $\tilde{\Phi}(\tilde{f}) = \Phi(f)$. It is easy to see that $\{\tilde{\Phi}(\tilde{f}) \mid \|\tilde{f}\|_\infty \leqq 1, \tilde{f} \in C(\tilde{S})\} \subset \{\Phi(f) \mid \|f\|_\infty \leqq 2, f \in C_0(S)\}$; hence $\tilde{\Phi}$ is a weakly compact operator. According to [7; VI.7.3] there exists a unique regular vector measure $\tilde{m} : \mathscr{B}_1(\tilde{S}) \to X$ such that $\tilde{\Phi}(\tilde{f}) = \int \tilde{f} \,\mathrm{d}\tilde{m}$. Define $m_1(E) = \tilde{m}(E)$ for every $E \in \mathscr{B}_1(S)$. Since every function $f \in C_0(S)$ is $\mathscr{B}_1(S)$-measurable, $\Phi(f) = \int f \,\mathrm{d}m_1$. Since for every $x' \in X'$,

$$(3)\qquad \langle \Phi(f), x' \rangle = \int f(s) \,\mathrm{d}\langle m_1(s), x' \rangle ,$$

it follows from classical theorems on the uniqueness of scalar measure that m_1 is the unique regular vector measure on $\mathscr{B}_1$ for which (3) holds.

According to Lemma 1, m_1 can be extended uniquely to a regular measure $m : \mathscr{B} \to X$. For this measure m the relation (2) will be true. It follows from (2) that $\|\Phi\| = \|m\|(S)$.

If there exists a vector measure $m : \mathscr{B} \to X$ such that (2) holds, then according to [14] the set $\{\int f \,\mathrm{d}m \mid \|f\|_\infty \leqq 1, f \text{ is } m\text{-integrable}\}$ is relatively weakly compact in X. The set (1) is its part and, therefore, it is also relatively weakly compact.

Corollary. *Let X be a weakly complete Banach space. Let D be a dense subspace of $C_0(S)$ and $\Phi : D \to X$ a bounded linear mapping. Then* (1) *is a relatively weakly compact set in X.*

Proof. According to [7; VI.7.6], $\{\tilde{\Phi}(\tilde{f}) \mid \|\tilde{f}\|_\infty \leqq 1, \tilde{f} \in (\tilde{S})\}$ is a relatively weakly compact set (notations as in the proof of Lemma 2).

Lemma 3. *Let $P : \mathscr{B} \to L(X)$ be a regular operator measure and D be a dense subalgebra of $C_0(S)$. If*

$$\int fg \,\mathrm{d}P = \int f \,\mathrm{d}P \int g \,\mathrm{d}P \tag{4}$$

for $f, g \in D$, then P is multiplicative.

Proof. It follows from density of D and continuity of integral that (4) holds for every $f, g \in C_0(S)$.

If C_1, C_2 are compact G_δ sets, then there exist decreasing sequences $\{f_n\}$, $\{g_n\}$ of functions in $C_0(S)$ such that $0 \leqq f_n(s) \leqq 1$, $0 \leqq g_n(s) \leqq 1$ and $\chi_{C_1}(s) = \lim f_n(s)$, $\chi_{C_2}(s) = \lim g_n(s)$ for $s \in S$. By passing to limits in scalar integrals using (4) we obtain $\langle P(C_1 \cap C_2)\, x, x' \rangle = \langle P(C_1)\, P(C_2)\, x, x' \rangle$ for every $x \in X$, $x' \in X'$, i.e. P is multiplicative on the system of all compact G_δ sets. By a direct computation it can be seen that P is multiplicative on the system of all sets of the form $C_1 - C_2$, where C_1 and C_2 are compact G_δ. As every set of the ring $\mathscr{R}$ generated by compact G_δ sets is a finite union of disjoint sets of the form $C_1 - C_2$, the multiplicativity of P on $\mathscr{R}$ can be also easily deduced.

For $E \in \mathscr{R}$, let $\mathscr{M}_E$ be the system of all sets $F \in \mathscr{B}$ such that $P(E \cap F) = P(E)\, P(F) = P(F)\, P(E)$. The system $\mathscr{M}_E$ is monotone and contains $\mathscr{R}$, hence $\mathscr{B}_0 \subset \mathscr{M}_E$ (see [9; Theorem 6B]). For a fixed $F \in \mathscr{B}_0$ let $\mathscr{M}_F$ be the system of all sets $E \in \mathscr{B}$ such that $P(E \cap F) = P(E)\, P(F) = P(F)\, P(E)$. The system $\mathscr{M}_E$ is also monotone and contains $\mathscr{R}$, i.e. $\mathscr{B}_0 \subset \mathscr{M}_E$. We have proved that P is multiplicative on $\mathscr{B}_0$.

From the multiplicativity of P on $\mathscr{B}_0$ there follows the multiplicativity on the whole $\mathscr{B}$ according to [15; Theorem 5]. (In [15] the space S is supposed to be σ-compact, but because of Corollary 1 of Lemma 1 this supposition presents no loss of generality.)

2. REPRESENTATIONS

We use the symbol $M(S)$ for the set of all finite regular complex measures on $\mathscr{B}(S)$. $M(S)$ may be identified with $C_0(S)'$, the dual space of $C_0(S)$.

A function $f : S \to X$ is called weakly μ-measurable for $\mu \in M(S)$ if, for every compact set $K \subset S$ and $\varepsilon > 0$, there exists a compact set $K_1 \subset K$ such that $|\mu|\,(K - K_1) < \varepsilon$ and f is weakly continuous on K_1 ($|\mu|$ is the variation of μ). The strong measurability is defined analogously.

A function $f : S \to X$ is said to be scalar μ-measurable if, for every $x' \in X'$, the function $s \to \langle f(s), x' \rangle$ is μ-measurable. All these definitions agree with [2].

Lemma 4. *Let $\mu \in M(S)$ and $f : S \to X$ be a bounded function, $\|f(s)\| \leqq k$ for $s \in S$. Let one of the following conditions be satisfied:*

(a) *f is weakly μ-measurable.*

(b) *f is scalar μ-measurable and X is a reflexive space.*

(c) *f is scalar μ-measurable and, for every compact set $K \subset S$, $f(K)$ is contained in a separable subspace of X.*

Then f is μ-integrable, i.e. there exists an element $\int f \,\mathrm{d}\mu \in X$ such that, for every $x' \in X'$, the equality

$$\left\langle \int f \,\mathrm{d}\mu, x' \right\rangle = \int \langle f(s), x' \rangle \,\mathrm{d}\mu(s) \tag{5}$$

holds. Moreover $\|\int f \,\mathrm{d}\mu\| \leqq k|\mu|(S)$.

Proof. (a) We proceed similarly as in the proof of Proposition 8 in [2; VI.1.2]. Let S_μ be a σ-compact set such that $|\mu|(E) = 0$ for $E \in \mathscr{B}$, $E \cap S_\mu = \emptyset$. S_μ being σ-compact and f weakly measurable, there exist disjoint compact sets K_n, $n = 1, 2, \ldots$, and a $|\mu|$-null set N such that $S_\mu - N = \bigcup_{n=1}^{\infty} K_n$ and f is weakly continuous on every K_n. Hence the sets $f(K_n)$ are weakly compact. According to the Krein-Šmulian theorem [7; V.6.4] the closed convex envelope B_n of $f(K_n)$ is also weakly compact. From the Corollary of Proposition 5 in [2; VI.1.2] there follows the existence of an element $z_n \in |\mu|(K_n) B_n$ for which

$$\langle z_n, x' \rangle = \int_{K_n} \langle f(s), x' \rangle \,\mathrm{d}\mu(s)$$

for every $x' \in X'$. Hence $\|z_n\| \leqq k|\mu|(K_n)$. The convergence of the series $\sum_{n=1}^{\infty} |\mu|(K_n)$ implies that of $\sum_{n=1}^{\infty} z_n$. Denote $z = \sum_{n=1}^{\infty} z_n$; hence $\|z\| \leqq k|\mu|(S)$. From the well-known theorems on integration it follows that $z = \int f \,\mathrm{d}\mu$, i.e. (5) holds.

(b) If f is scalar μ-measurable and X is a reflexive space, the integral on the right hand of (5) exists for every $x' \in X'$ and depends linearly and continuously on x'. The existence of $\int f \,\mathrm{d}\mu \in X$ such that (5) holds, follows readily.

(c) If condition (c) is satisfied, then f is μ-measurable according to [2; IV.5.5] and hence also weakly μ-measurable. Now we can use the proved part (a).

A function $F : S \to L(X)$ is called weakly, strongly or scalar μ-measurable if, for every $x \in X$, the function $s \to F(s)\,x$ possesses the corresponding property.

$\int F \,\mathrm{d}\mu$ stands for the operator in $L(X)$ such that

$$\left\langle \left(\int F \,\mathrm{d}\mu \right) x, x' \right\rangle = \int \langle F(s)\,x, x' \rangle \,\mathrm{d}\mu(s)$$

for $x \in X$ and $x' \in X'$. Evidently, $\int F \,\mathrm{d}\mu$ is determined uniquely by F and μ provided it exists.

From now on let S be a locally compact group. In this case $M(S)$ is a Banach algebra with respect to usual linear operations and with respect to convolution as a multiplication in $M(S)$. The convolution of elements $\mu, \nu \in M(S)$ is the unique element $\mu * \nu \in M(S)$ satisfying the condition

$$\int f \, \mathrm{d}\mu * \nu = \iint f(st) \, \mathrm{d}\mu(s) \, \mathrm{d}\nu(t) = \iint f(st) \, \mathrm{d}\nu(t) \, \mathrm{d}\mu(s) \tag{6}$$

for every $f \in C_0(S)$ (see [10; 19.10]).

The norm in $M(S)$ is defined by $\|\mu\| = |\mu| (S)$.

A representation of the group S in X is a mapping $U : S \to L(X)$ such that $U(st) = U(s) \, U(t)$ for $s, t \in S$ and $U(e) = I$, where e is the identity element in S (see e.g. [10]).

Lemma 5. *Let A be a subalgebra of $M(S)$. Let $U : S \to L(X)$ be a representation. Suppose that the operator $T(\mu) = \int U \, \mathrm{d}\mu$ exists for every $\mu \in A$.*

Then $T : \mu \to T(\mu)$ is a representation of the algebra A, i.e. a homomorphism of A into $L(X)$.

Proof. The operator $T(\mu)$ depends linearly on μ. Hence it suffices to prove that $T(\mu * \nu) = T(\mu) \, T(\nu)$ or, equivalently, $\langle T(\mu * \nu) \, x, x' \rangle = \langle T(\mu) \, T(\nu) \, x, x' \rangle$ for $x \in X$, $x' \in X'$.

Let $U'(s)$ be the adjoint operator to $U(s)$. Hence $U'(s) \in L(X')$, $s \in S$.

Let $\mu, \nu \in A$, $x \in X$, $x' \in X'$ be arbitrary. Then according to (6),

$$\langle T(\mu * \nu) \, x, x' \rangle = \int \langle U(s) \, x, x' \rangle \, \mathrm{d}\mu * \nu(s) =$$

$$= \iint \langle U(st) \, x, x' \rangle \, \mathrm{d}\nu(t) \, \mathrm{d}\mu(s) = \iint \langle U(s) \, U(t) \, x, x' \rangle \, \mathrm{d}\nu(t) \, \mathrm{d}\mu(s) =$$

$$= \iint \langle U(t) \, x, U'(s) \, x' \rangle \, \mathrm{d}\nu(t) \, \mathrm{d}\mu(s) = \int \langle T(\nu) \, x, U'(s) \, x' \rangle \, \mathrm{d}\mu(s) =$$

$$= \int \langle U(s) \, T(\nu) \, x, x' \rangle \, \mathrm{d}\mu(s) = \langle T(\mu) \, T(\nu) \, x, x' \rangle \, .$$

Remark. The proof of Lemma 5 is almost the same as in [10; 22.3], where it has been used for the case that X is a reflexive space. It is explicitly mentioned in [10] (p. 336 foot-note 1) that the proof is based on reflexivity of X. In [10] a representation $U : S \to L(X')$ is considered (for technical reasons) and if X were not reflexive, it could not be guaranteed that $U'(s) \in L(X)$.

The following theorem is a consequence of Lemma 4 and Lemma 5.

Theorem 1. *Let $U : S \to L(X)$ be a bounded representation of a locally compact group S. Suppose $\|U(s)\| \leqq k$ for $s \in S$. Let A be a subalgebra of $M(S)$. Suppose one of the following conditions be fulfilled.*

(a) *U is weakly μ-measurable for every $\mu \in A$.*

(b) *X is reflexive space and U is scalar μ-measurable for every $\mu \in A$.*

(c) *U is scalar μ-measurable for every $\mu \in A$ and, for every $x \in X$ and every compact set $K \subset S$, $U(K)\,x$ is a subset of a separable subspace of X.*

Then for every $\mu \in A$ there exists a unique operator $T(\mu) = \int U(s)\,\mathrm{d}\mu(s) \in L(X)$. The mapping $T: \mu \to T(\mu)$ is a bounded representation of A in X with $\|T\| \leqq k$.

The case (b) of this theorem is the same as Theorem 22.3 in [10].

3. FOURIER TRANSFORMS OF VECTOR MEASURES

Let S be a locally compact Abelian group and Σ its character group. (For terminology and basic facts of Harmonic Analysis we refer to [18].) The value of a character σ at a point $s \in S$ will be written as $\langle s, \sigma \rangle$.

For $\mu \in M(S)$ we put

$$\hat{\mu}(\sigma) = \int \overline{\langle s, \sigma \rangle}\,\mathrm{d}\mu(s)\,.$$

For a subset $A \subset M(S)$ we use the symbol $\hat{A}$ for the set $\{\hat{\mu} \mid \mu \in A\}$.

We consider $L^1(S)$ (the measure omitted in the notation is a fixed Haar measure on S) as a subalgebra of $M(S)$. We do not distinguish between a function $f \in L^1(S)$ and the measure $\mu \in M(S)$ for which $\mathrm{d}\mu(s) = f(s)\,\mathrm{d}s$, i.e. for which $\int \varphi(s)\,\mathrm{d}\mu(s) = \int \varphi(s) f(s)\,\mathrm{d}s$, $\varphi \in C_0(S)$, where $\int \ldots \mathrm{d}s$ denotes integration with respect to the fixed Haar measure.

A vector-valued function $\varphi : S \to X$ is called a Fourier-Stieltjes transform if there exists a regular vector measure $m : \mathscr{B}(\Sigma) \to X$ such that

$$\varphi(s) = \int \overline{\langle s, \sigma \rangle}\,\mathrm{d}m\,, \quad s \in S\,. \tag{7}$$

More precisely, φ is called the Fourier-Stieltjes transform of m.

If there exists a regular vector measure m such that (7) holds, then it is unique. This follows from the uniqueness theorem for Fourier-Stieltjes transforms of scalar measures. Namely, it follows from (7) that

$$\langle \varphi(s), x' \rangle = \int \overline{\langle s, \sigma \rangle}\,\mathrm{d}\langle m(\sigma), x' \rangle \tag{8}$$

for $x' \in X'$.

Theorem 2. *A function $\varphi : S \to X$ is a Fourier-Stieltjes transform if and only if φ is weakly continuous and*

$$\left\{\int f(s)\,\varphi(s)\,\mathrm{d}s \,\Big|\, \|\hat{f}\|_\infty \leqq 1,\ f \in L^1(S)\right\} \tag{9}$$

is a relatively weakly compact subset of X.

Proof. Define the transformation $\Phi : \hat{L}^1(S) \to X$ by

$$\Phi(\hat{f}) = \int f(s)\,\varphi(s)\,\mathrm{d}s\,, \quad f \in L^1(S)\,. \tag{10}$$

Since $f = g$ is implied by $\hat{f} = \hat{g}$, (10) defines the transformation Φ unambiguously.

By Lemma 2 it follows from the relatively weak compactness of the set (9) that there exists a regular vector measure $m : \mathscr{B}(\Sigma) \to X$ such that

$$\int f(s)\,\varphi(s)\,\mathrm{d}s = \Phi(\hat{f}) = \int \hat{f}(\sigma)\,\mathrm{d}m(\sigma)$$

for every $f \in L^1(S)$.

For the integral $\int \overline{\langle s, \sigma\rangle}\,\mathrm{d}m(\sigma)$ existing for every $s \in S$, we can write

$$\left\langle \int f(s)\,\varphi(s)\,\mathrm{d}s, x' \right\rangle = \int \hat{f}(\sigma)\,\mathrm{d}\langle m(\sigma), x'\rangle =$$

$$= \int\left(\int f(s)\,\overline{\langle s, \sigma\rangle}\,\mathrm{d}s\right)\mathrm{d}\langle m(\sigma), x'\rangle = \int f(s)\left(\int \overline{\langle s, \sigma\rangle}\,\mathrm{d}\langle m(\sigma), x'\rangle\right)\mathrm{d}s$$

for every $f \in L^1(S)$ and $x' \in X'$. Hence

$$\langle \varphi(s), x'\rangle = \int \overline{\langle s, \sigma\rangle}\,\mathrm{d}\langle m(\sigma), x'\rangle \tag{11}$$

almost everywhere on S. From the weak continuity of φ we deduce that (11) holds everywhere and, hence, (7) is valid.

Conversely, let (7) hold. Evidently, φ is a bounded function. We prove first that φ is (strongly) continuous. Let $\{s_\alpha\}$ be a net converging to s_0. Let $\varepsilon > 0$ be fixed. There exists a compact set $K \subset \Sigma$ such that $\|m(E)\| < \frac{1}{4}\varepsilon$ for $E \in B(\Sigma)$, $E \cap K = \emptyset$. Since $\langle s_\alpha, \sigma\rangle \to \langle s_0, \sigma\rangle$ uniformly for $\sigma \in K$, there exists α_0 such that $|\langle s_\alpha, \sigma\rangle - \langle s_0, \sigma\rangle| < \varepsilon/(2\|m\|\,(K))$ for $\alpha \geqq \alpha_0$ and $\sigma \in K$. Then

$$\|\varphi(s_\alpha) - \varphi(s_0)\| = \left\|\int(\langle s_\alpha, \sigma\rangle - \langle s_0, \sigma\rangle)\,\mathrm{d}m(\sigma)\right\| \leqq$$

$$\leqq \left\|\int_K (\langle s_\alpha, \sigma\rangle - \langle s_0, \sigma\rangle)\,\mathrm{d}m(\sigma)\right\| + \left\|\int_{\Sigma - K} (\langle s_\alpha, \sigma\rangle - \langle s_0, \sigma\rangle)\,\mathrm{d}m(\sigma)\right\| \leqq$$

$$\leqq \frac{\varepsilon}{2\|m\|\,(K)}\,\|m\|\,(K) + 2\,\frac{\varepsilon}{4} = \varepsilon$$

for $\alpha \geqq \alpha_0$.

Due to continuity and boundedness of φ the integral $\int f(s)\,\varphi(s)\,\mathrm{d}s$ exists for every $f \in L^1(S)$. We have further

$$\int f(s)\,\langle \varphi(s), x' \rangle\,\mathrm{d}s = \int f(s) \left(\int \overline{\langle s, \sigma \rangle}\,\mathrm{d}\langle m(\sigma), x' \rangle \right) \mathrm{d}s =$$

$$= \int \left(\int f(s)\,\overline{\langle s, \sigma \rangle}\,\mathrm{d}s \right) \mathrm{d}\langle m(\sigma), x' \rangle = \int \hat{f}(\sigma)\,\mathrm{d}\langle m(\sigma), x' \rangle$$

for every $x' \in X'$. Using Lemma 2 we obtain the relatively weak compactness of the set (9).

Corollary 1. *If φ is the Fourier-Stieltjes transform of a vector measure m, then the mapping $\Phi : \hat{L}^1(S) \to X$ defined by* (10) *is bounded and* $\|\Phi\| = \|m\|\,(S)$.

Corollary 2. *If X is a weakly complete space and the mapping Φ defined by* (10) *is bounded on $\hat{L}^1(S)$, then φ is a Fourier-Stieltjes transform.*

Proof. We use the Corollary of Lemma 2.

Corollary 3. *If φ is a Fourier-Stieltjes transform, then it is bounded and strongly continuous.*

Remark. If we replace the weak continuity of φ by scalar measurability, then from the relatively weak compactness of (9) we can deduce the existence of a regular vector measure m such that (11) holds for every $x' \in X'$ almost everywhere on S.

Theorem 2 gives a characterization of Fourier-Stieltjes transforms in terms of the algebra $L^1(S)$ of absolutely continuous measures on S. The following theorem presents such a characterization in terms of the algebra of discrete measures. Moreover, the system of measures with finite supports will suffice for a characterization.

For $s \in S$, the symbol δ_s will stand for the measure on $\mathscr{B}(S)$ defined by $\delta_s(E) = c_E(s)$, where c_E is the characteristic function of E.

Let $M_d(S)$ be the set of all measures $\mu \in M(S)$ which can be written in the form

$$\mu = \sum_{i=1}^{\infty} a_i \delta_{s_i}\,, \tag{12}$$

where $s_i \in S$ and a_i are complex numbers such that $\sum_{i=1}^{\infty} |a_i| < \infty$. $M_d(S)$ is a subalgebra of $M(S)$.

The set of elements $\mu \in M_d(S)$ such that only a finite number of a_i's are different from zero will be denoted by $M_{dd}(S)$.

Theorem 3. *A function $\varphi : S \to X$ is a Fourier-Stieltjes transform if and only if it is weakly continuous and*

$$\left\{\int \varphi(s)\,\mathrm{d}\mu(s) \mid \|\hat{\mu}\|_\infty \leqq 1\,,\quad \mu \in M_{dd}(S)\right\} \tag{13}$$

is a relatively weakly compact subset of X.

Proof. The relatively weak compactness of the set (13) implies its boundedness and the boundedness of φ. Hence $\int \varphi(s)\,\mathrm{d}\mu(s)$ exists for every $\mu \in M(S)$, in particular, $\int f(s)\,\varphi(s)\,\mathrm{d}s$ exists for $f \in L^1(S)$.

Define the transformation $\Phi : \hat{M}(S) \to X$ by $\Phi(\hat{\mu}) = \int \varphi(s)\,\mathrm{d}\mu(s)$ for $\mu \in M(S)$. Φ is defined unambiguously since μ is determined uniquely by $\hat{\mu}$. It follows from the supposition on weak compactness of (13) that the restriction of Φ to $\hat{M}_{dd}(S)$ is continuous, i.e. there exists a constant k such that

$$\left\|\int \varphi(s)\,\mathrm{d}\mu(s)\right\| \leqq \|\Phi(\hat{\mu})\| \leqq k\|\hat{\mu}\|_\infty\,,\quad \mu \in M_{dd}(S)\,.$$

Hence,

$$\left|\int \langle \varphi(s), x' \rangle\,\mathrm{d}\mu(s)\right| = \langle \Phi(\hat{\mu}), x' \rangle \leqq k\|x'\|\,\|\hat{\mu}\|_\infty$$

for every $x' \in X'$.

According to [8; Theorem 1] the function $\langle \varphi(\;), x' \rangle$ is a Fourier-Stieltjes transform, i.e. there exists a measure $m_{x'} \in M(\Sigma)$ such that

$$\langle \varphi(s), x' \rangle = \int \overline{\langle s, \sigma \rangle}\,\mathrm{d}m_{x'}(\sigma)\,,\quad s \in S\,,\quad x' \in X'\,.$$

Consequently

$$\langle \Phi(\hat{\mu}), x' \rangle = \int \langle \varphi(s), x' \rangle\,\mathrm{d}\mu(s) = \int \hat{\mu}(\sigma)\,\mathrm{d}m_{x'}(\sigma)$$

for $\mu \in M(S)$, $x' \in X'$.

Let $AP(\Sigma)$ denote the uniform closure of $\hat{M}_{dd}(S)$. Φ being continuous on $\hat{M}_{dd}$, it can be extended uniquely to a continuous transformation on $AP(\Sigma)$ denoted also by Φ.

Evidently,

$$\langle \Phi(g), x' \rangle = \int g(\sigma)\,\mathrm{d}m_{x'}(\sigma)$$

for every $g \in AP(\Sigma)$. Moreover, the set

$$\{\Phi(g) \mid \|g\|_\infty \leqq 1,\; g \in AP(\Sigma)\} \tag{14}$$

is relatively weakly compact being a subset of the closure of (13).

We wish to prove that the set (9) is relatively weakly compact. By the Eberlein-

Šmulian theorem and the relatively weak compactness of (14) it suffices to prove that (9) is a subset of the weak closure of the set $\{\Phi(g) \mid \|g\|_\infty \leqq 2, g \in AP(\Sigma)\}$.

Let $f \in L^1(S)$, $\|\hat{f}\|_\infty \leqq 1$. Let, further, $x_i' \in X'$, $i = 1, 2, \ldots, j$, and $\varepsilon > 0$ be arbitrary. There exists a compact set $K_i \subset \Sigma$ such that $|m_{x_i'}| (\Sigma - K_i) < \frac{1}{3}\varepsilon$, $i = 1, 2, \ldots, j$ Put $K = \bigcup_{i=1}^{j} K_i$. $AP(\Sigma)$ being a uniformly closed self-adjoint algebra which separates the points of Σ, by the Stone-Weierstrass theorem there exists a function $g \in AP(\Sigma)$ such that $\hat{f}(\sigma) = g(\sigma)$ for $\sigma \in K$. Moreover, the set of real-valued functions belonging to $AP(\Sigma)$ is a lattice (see e.g. [18; Lemma 4D]); hence, g may be chosen so that $\|\operatorname{Re} g\|_\infty \leqq 1$, $\|\operatorname{Im} g\|_\infty \leqq 1$, i.e. $\|g\|_\infty \leqq 2$.

For $i = 1, 2, \ldots, j$, we have

$$|\langle \Phi(\hat{f}), x_i' \rangle - \langle \Phi(g), x_i' \rangle| = \left| \int (\hat{f}(\sigma)\, g(\sigma))\, \mathrm{d}m_{x_i'}(\sigma) \right| \leqq$$

$$\leqq \left| \int_K (\hat{f}(\sigma) - g(\sigma))\, \mathrm{d}m_{x_i'}(\sigma) \right| + \left| \int_{\Sigma - K} (\hat{f}(\sigma) - g(\sigma))\, \mathrm{d}m_{x_i'}(\sigma) \right| < \varepsilon\,.$$

Hence, $\Phi(\hat{f})$ belongs to the weak closure of $\{\Phi(g) \mid \|g\|_\infty \leqq 2, g \in AP(\Sigma)\}$. It follows that (9) is a relatively weakly compact set and, by Theorem 2, that φ is a Fourier-Stieltjes transform.

Conversely, let φ be the Fourier-Stieltjes transform of a vector measure $m : \mathscr{B}(\Sigma) \to$ $\to X$. Then $\Phi(\hat{\mu}) = \int \hat{\mu}(\sigma)\, \mathrm{d}m(\sigma)$, $\mu \in M(S)$ and, by [14], the set $|\int g(\sigma)\, \mathrm{d}m(\sigma)| \; \|g\|_\infty \leqq$ $\leqq 1$, g m-integrable$\}$ is relatively weakly compact, hence the set (13), being its subset, is relatively weakly compact, too.

Remark. Theorem 3 may be stated also in the following form. A necessary and sufficient condition for a function $\varphi : S \to X$ to be a Fourier-Stieltjes transform is the relatively weak compactness of the set of all vectors $\sum_{i=1}^{k} a_i\, \varphi(s_i)$, where a_i are complex numbers, $s_i \in S$, $i = 1, 2, \ldots, k$; $k = 1, 2, \ldots$, such that $\sup_{\sigma \in \Sigma} \left| \sum_{i=1}^{k} a_i \langle s_i, \sigma \rangle \right| \leqq$ $\leqq 1$. In this form it is closer to the formulation given in [8] for the case when X is the complex number field.

Corollary 1. *If X is a weakly complete space and if there exists a constant k such that $\|\int \varphi(s)\, \mathrm{d}\mu(s)\| \leqq k \|\hat{\mu}\|_\infty$ for $\mu \in M_{dd}(S)$, then φ is a Fourier-Stieltjes transform.*

Proof. If there exists such a constant, then the set (13) is bounded. Similarly as in the proof of Theorem 3 we show that the set (9) is a part of the weak closure of a multiple of the set (13), hence it is also bounded. By Corollary 2 of Theorem 2, φ is a Fourier-Stieltjes transform.

4. FOURIER-STIELTJES TRANSFORMS OF GROUP-REPRESENTATIONS

In this section we use Theorem 2 and Theorem 3 to obtain conditions for the existence of spectral resolution for a given group representation. We use the notations of Section 3.

If a representation $U : S \to L(X)$ can be written in the form

$$U(s) = \int \langle \overline{s, \sigma} \rangle \, \mathrm{d}P(\sigma) \,, \quad s \in S \,, \tag{15}$$

where $P : \mathscr{B}(\Sigma) \to L(X)$ is a regular spectral measure, then the measure P is determined uniquely by U. Indeed, for every $x \in X$ and $x' \in X'$, we have

$$\langle U(s) \, x, x' \rangle = \int \langle \overline{s, \sigma} \rangle \, \mathrm{d}\langle P(\sigma) \, x, x' \rangle$$

and the scalar measures $\langle P(\ ) \, x, x' \rangle$ are determined uniquely by their Fourier-Stieltjes transforms.

Theorem 4. *Let $U : S \to L(X)$ be a weakly continuous representation of the group S. Suppose, for every $x \in X$, the set*

$$\left\{ \int f(s) \, U(s) \, x \, \mathrm{d}s \mid \|\hat{f}\|_\infty \leqq 1, \, f \in L^1(S) \right\} \tag{16}$$

be relatively weakly compact.

Then there exists a regular spectral measure $P : \mathscr{B}(\Sigma) \to L(X)$ such that (15) *holds.*

Conversely, if there exists an operator measure $P : \mathscr{B}(\Sigma) \to L(X)$ such that (15) *holds, then, for every $x \in X$, the set* (16) *is relatively weakly compact and U is strongly continuous and bounded.*

Proof. The weak compactness of (16) implies the existence of a constant k such that

$$\left\| \int f(s) \, U(s) \, x \, \mathrm{d}s \right\| \leqq k \|\hat{f}\|_\infty \, \|x\| \,. \tag{17}$$

By Theorem 2 and its Corollary 1 there exists a regular vector measure $m_x : \mathscr{B}(\Sigma) \to X$ such that

$$U(s) \, x = \int \langle \overline{s, \sigma} \rangle \, \mathrm{d}m_x(\sigma) \,, \quad x \in X \,,$$

and $\|m_x(E)\| \leqq k \|x\|$ for every $E \in \mathscr{B}(\Sigma)$. The function $P : \mathscr{B}(\Sigma) \to L(X)$ defined by $P(E) \, x = m_x(E)$, $E \in \mathscr{B}(\Sigma)$, $x \in X$, is a regular operator measure such that (15) holds.

Since by Lemma 5

$$\int (f * g)(s)\, U(s)\, \mathrm{d}s = \int f(s)\, U(s)\, \mathrm{d}s \int g(s)\, U(s)\, \mathrm{d}s$$

and $\int f(s)\, U(s)\, \mathrm{d}s = \int \hat{f}(\sigma)\, \mathrm{d}P(\sigma)$, the transformation $\hat{f} \to \int \hat{f}(\sigma)\, \mathrm{d}P(\sigma)$ is multiplicative on $\hat{L}^1(S)$. It follows by Lemma 3 that P is a multiplicative measure.

If we substitute the neutral element of S for s in (15), we obtain that $P(\Sigma) = I$. Hence P is a spectral measure.

The converse part of the theorem is a consequence of Theorem 2.

Corollary. *If X is a weakly complete space and $U : S \to L(X)$ a weakly continuous representation of the group S, then there exists a spectral measure $P : \mathscr{B}(\Sigma) \to L(X)$ such that* (15) *holds if and only if there exists a constant k such that* (17) *holds.*

Remarks 1. If the assumption of weak continuity of the representation U in Theorem 4 is replaced by the measurability of $\langle U(\ )\, x, x' \rangle$ for every $x \in X$, $x' \in X'$, then there exists a regular spectral measure $P : \mathscr{B}(\Sigma) \to L(X)$ such that

$$\langle U(s)\, x, x' \rangle = \int \langle \overline{s, \sigma} \rangle\, d\langle P(\sigma)\, x, x' \rangle$$

for almost each $s \in S$.

2. If X is a Hilbert space, the assumptions of Theorem 4 are fulfilled for every bounded weakly continuous representation $U : S \to L(X)$. In fact, for every bounded representation U in a Hilbert space X there exists by [4] an operator $A \in L(X)$ such that $A^{-1} \in L(X)$ and the representation $s \to V(s) = A^{-1}\, U(s)\, A$ is unitary. However, if V is a unitary representation, then, by [18; 26F and 32B], $\| \int f(s)\, V(s)\, \mathrm{d}s \| \leqq \| \hat{f} \|_\infty$. Hence, (17) is valid for $k = 1 = \|A\|\, \|A^{-1}\|$.

Theorem 5. *A weakly continuous representation $U : S \to L(X)$ can be written in the form* (15), *where $P : \mathscr{B}(\Sigma) \to L(X)$ is a regular spectral measure if and only if the set*

$$\left\{ \int U(s)\, x\, \mathrm{d}\mu(s) \,\middle|\, \|\hat{\mu}\|_\infty \leqq 1,\ \mu \in M_{dd}(S) \right\} \tag{18}$$

is relatively weakly compact for every $x \in X$.

Proof. From the weak compactness of the set (18) we deduce the boundedness of the representation U. Then we can proceed analoguously as in the proof of Theorem 4, only we use Theorem 3 instead of Theorem 2.

Corollary. *If X is a weakly complete space, then a weakly continuous representation $U : S \to L(X)$ can be written in the form* (15) *if and only if there exists a constant k such that $\| \int U(s)\, \mathrm{d}\mu(s) \| \leqq k \|\hat{\mu}\|_\infty$ for every $\mu \in M_{dd}(S)$.*

Theorems of Section 4 shall now be used to obtain some results in the Spectral Theory of operators, namely to obtain criteria for an operator to be spectral in the Dunford sense ([5] or [6]).

For example we use Theorem 5 for the case that $S = N$, where N is the additive group of integers with the discrete topology. Then $\Sigma = K$, where K is the multiplicative group of all complex numbers of magnitude 1 topologized as a subset of the plane.

Denote $\mathcal{N}$ the system of finite subsets of N.

The following theorem is a direct consequence of Theorem 5 (in its formulation a_i denote arbitrary complex numbers).

Theorem 6. *Let $T \in L(X)$ and let the set*

$$\{\sum_{n\in\pi} a_n T^n x \mid \sup_{|z|=1} |\sum_{n\in\pi} a_n z^n| \leqq 1,\ \pi \in \mathcal{N}\} \tag{19}$$

be relatively weakly compact for every $x \in X$. Then the spectrum $\sigma(T)$ of T is a subset of K and there is a spectral measure $P : \mathscr{B}(K) \to L(X)$ such that

$$T^n = \int z^n \,\mathrm{d}P(z)\,, \quad n \in N\,. \tag{20}$$

Conversely, if $\sigma(T) \subset K$ and if (20) holds for $n = 1$, then (19) is a relatively weakly compact set for every $x \in X$.

If, for $T \in L(X)$, the assumptions of Theorem 6 are satisfied, i.e. (19) is relatively weakly compact, then $\|T^n\| = O(1)$, $n \to \pm\infty$. The class of operators T, called weakly almost periodic, such that T^n is bounded has been introduced by E. R. Lorch [19]. It has been investigated also by G. K. Leaf [17] together with some generalizations. Operators for which there exists a spectral measure $P : \mathscr{B}(K) \to L(X)$ such that (20) holds are called pseudounitary. They were introduced in [16].

For operators in weakly complete spaces the criterium of Theorem 6 can be simplified.

Corollary. *If X is a weakly complete Banach space, then $T \in L(X)$ is a pseudounitary operator if and only if there exists a constant k such that*

$$\|\sum_{n\in\pi} a_n T^n\| \leqq k \sup_{|z|=1} |\sum_{n\in\pi} a_n z^n|$$

for arbitrary complex numbers a_n and $\pi \in \mathcal{N}$.

An operator $T \in L(X)$ is said after S. Kantorovitz [11] to be pseudohermitian if there exists a spectral measure $P : \mathscr{B}(-\infty, \infty) \to L(X)$ such that $T = \int s \,\mathrm{d}P(s)$.

Applying Theorem 4 to the additive group $S = (-\infty, \infty)$ we obtain the following theorem.

Theorem 7. *An operator $T \in L(X)$ is pseudohermitian if and only if the set*

$$\left\{\int f(s)\, e^{isT} x \,\mathrm{d}s \,\Big|\, \|\hat{f}\|_\infty \leqq 1,\ f \in L^1(-\infty, \infty)\right\} \tag{21}$$

is relatively weakly compact for every $x \in X$.

Proof. In the investigated case $\Sigma = (-\infty, \infty)$. By Theorem 4 the relatively weak compactness of the set (21) is the necessary and sufficient condition for the existence of a spectral measure $P : \mathscr{B}(-\infty, \infty) \to L(X)$ such that

$$e^{isT} = \int e^{is\sigma}\, \mathrm{d}P(\sigma), \quad s \in (-\infty, \infty). \tag{22}$$

It is proved in [11] (p. 170, from (2) to Remarks on p. 171) that from (22) the pseudohermiticity of T follows.

Corollary. *If X is a weakly complete Banach space, then an operator $T \in L(X)$ is pseudohermitian if and only if there exists a constant k such that*

$$\left\| \int f(s)\, e^{isT}\, \mathrm{d}s \right\| \leqq k \|\hat{f}\|_\infty, \quad f \in L^1(-\infty, \infty).$$

The result of this Corollary has been obtained using other methods by S. Kantorovitz in [12], and for reflexive spaces in [11].

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Резюме

ХАРАКТЕРИЗАЦИЯ ПРЕОБРАЗОВАНИЙ ФУРЬЕ-СТИЛТЬЕСА ВЕКТОРНЫХ И ОПЕРАТОРНЫХ МЕР

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Пусть S — локально компактная абелева гриппа, Σ — дуальная группа группы S, $\mathscr{B}$ — система борелевских множеств в Σ и X — пространство Банаха. Функция $\varphi : S \to X$ называется преобразованием Фурье-Стилтьеса, если существует векторная мера $m : \mathscr{B} \to X$ такая, что $\varphi(s) = \int\langle s, \sigma\rangle \, \mathrm{d}m(\sigma)$, $s \in S$. Доказывается, что φ является преобразованием Фурье-Стилтьеса тогда и только тогда, если множество $\{\int f(s)\, \varphi(s)\, \mathrm{d}s \mid \|\hat{f}\|_\infty \leqq 1, f \in L^1(S)\}$ относительно слабо компактно в X. Другим необходимым и достаточным условием является относительная слабая компактность множества $\{\int \varphi(s)\, \mathrm{d}\mu(s) \mid \|\hat{\mu}\|_\infty \leqq 1, \mu \in M_d\}$, где M_d — множество всех дискретных мер на S.

Пусть $L(X)$ — алгебра всех линейных непрерывных операторов на X и пусть $U : S \to L(X)$ — представление группы S. Потом $U(s) = \int\langle s, \sigma\rangle \, \mathrm{d}P(\sigma)$, $s \in S$, где P некоторая спектральная мера на $\mathscr{B}$ с значениями в $L(X)$ тогда и топлко тогда, если для всякого $x \in X$ множество $\{\int f(s)\, U(s)\, x \, \mathrm{d}s \mid \|\hat{f}\|_\infty \leqq 1, f \in L^1(S)\}$ или множество $\{\int U(s)\, x\, \mathrm{d}\mu(s) \mid \|\hat{\mu}\|_\infty \leqq 1, \mu \in M_d\}$ относительно слабо компактно в X.