Czechoslovak Mathematical Journal

Otomar Hájek

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Czechoslovak Mathematical Journal, Vol. 17 (1967), No. 3, 372–381,382–398

Persistent URL: `http://dml.cz/dmlcz/100784`

Czechoslovak Mathematical Journal, 17 (92) 1967, Praha

THEORY OF PROCESSES, II.

OTOMAR HÁJEK, Praha

(Received March 3, 1966)

The present paper is a direct continuation of [I], and the conventions and notation introduced there – in particular those of the Appendix – and also the mode of reference will be preserved. The results of [I] will be referred to by prefixing I, so that I,4.9 refers to [I], section 4, item 9. Occasionally it will be useful to abbreviate the notation

$$r : (P, R, p) \to (P', R', p') \quad \text{in } \mathsf{Proc}\,,$$

exhibiting a morphism and objects in the category Proc (I,3.1), simply to $r : p \to p'$ in Proc, if the P, R, etc. are fixed previously or immaterial.

1. LATTICE-THEORETIC PROPERTIES

1. Consider the set of all processes in P over R, for fixed but arbitrary P and R. Under the natural relation-inclusion, this constitutes a partially ordered set (cf. I,1.20); its elementary properties, in particular the construction of joins and meets, will be studied in this section.

2. Merely for the internal needs of the present paper, it will be useful to introduce the term pre-process for processes without the compositivity property. More precisely, p is a *pre-process* in P over R iff P is a set, $R \subset \mathsf{R}^1$, and p is a relation on $P \times R$ such that

1° $(x, \alpha)\, p\, (y, \beta)$ implies $\alpha \geqq \beta$,
2° $(x, \alpha)\, p\, (y, \alpha)$ implies $x = y$.

In analogy with I,1.1 one defines the individual relations of such a pre-process as the relations ${}_\alpha p_\beta$ on P, for all $\alpha \geqq \beta$ in R, with

$$x\; {}_\alpha p_\beta\; y \quad \text{iff} \quad (x, \alpha)\, p\, (y, \beta)\,.$$

Observe again that p is completely determined by the system $\{{}_\alpha p_\beta \mid \alpha \geqq \beta \text{ in } R\}$; property 2° is then equivalent with ${}_\alpha p_\alpha \subset 1$ for all $\alpha \in R$.

3. The set of all pre-processes in P over R constitutes a principal ideal in the Boolean algebra of all relations on $P \times R$. In greater detail, let $p_{\max}$ be the maximal process in P over R, defined thus: $(x, \alpha)\, p_{\max}\, (y, \beta)$ iff either $\alpha > \beta$ and x, y are arbitrary, or $(x, \alpha) = (y, \beta)$. Then a relation p on $P \times R$ is a pre-process in P over R iff $p \subset p_{\max}$. In particular, if p_i are pre-processes in P over R, then so are $\bigcup p_i$ and $\bigcap p_i$.

4. Two special types of pre-process will be particularly important: the *partializable* pre-processes (in P over R), characterised by

$$ {}_\alpha p_\gamma \subset {}_\alpha p_\beta \circ {}_\beta p_\gamma \quad \text{for all} \quad \alpha \geqq \beta \geqq \gamma \quad \text{in} \quad R\,; $$

and the *transitive* pre-processes p characterised simply by transitivity of p, i.e. by $p \circ p \subset p$, or, equivalently, by

$$ {}_\alpha p_\gamma \supset {}_\alpha p_\beta \circ {}_\beta p_\gamma \quad \text{for all} \quad \alpha \geqq \beta \geqq \gamma \quad \text{in} \quad R\,. $$

The definition I,1.2 of a process may now be formulated thus: a pre-process is a process iff it is both partializable and transitive; this trivial but important result will often be used, without explicit reference.

5. Let p_i be pre-processes in P over R; it is easily shown that $\bigcup p_i$ is partializable if all p_i are such, and that $\bigcap p_i$ is transitive if all p_i are such.

Now, the union $\bigcup p_i$ of processes p_i may well not be a process; the preceding assertion traces this to $\bigcup p_i$ failing to be transitive. However, a simple remedy suggests itself (from I, Appendix, recall the notation p^{T} for the transitivization of a relation p):

6. Lemma (and definition). *Let p be a partializable pre-process in P over R. Then the transitivization p^{T} of p is a process, indeed it is the least process q in P over R with $q \supset p$; it will be termed the upper modification of p.*

Proof. The individual relations of p^{T} are easily described:

$$ (1) \quad {}_\alpha p^{\mathsf{T}}_\beta = \bigcup \{ {}_{\theta_1} p_{\theta_2} \circ {}_{\theta_2} p_{\theta_3} \circ \dots \circ {}_{\theta_{n-1}} p_{\theta_n} : \alpha = \theta_1 \geqq \dots \geqq \theta_n = \beta \text{ in } R,\ n \in \mathrm{C}^1 \}\,; $$

it follows immediately that p^{T} is indeed a partializable pre-process in P over R. The remaining assertions follow from the fact that p^{T} is the transitivization of p.

7. Now consider a system of processes p_i, all in P over R. From 5, $\bigcup p_i$ is a partializable pre-process, and hence one may form its upper modification say p according to 6. Thus p is the least process in P over R with $p \supset p_i$ for all the p_i, i.e. their least upper bound (in the partially ordered set of all processes in P over R). In the customary manner we denote p by $\bigvee p_i$, etc.

Evidently $\bigvee p_i$ may be described by

$$ (2) \qquad \bigvee p_i = \bigcup_i p_i \cup \bigcup_{i,j} (p_i \circ p_j) \cup \bigcup_{i,j,k} (p_i \circ p_j \circ p_k) \cup \dots, $$

or in terms of the individual relations,

$$_\alpha(\bigvee p_i)_\beta = \bigcup\{_{\theta_1}(p_{i_1})_{\theta_2} \circ \dots \circ _{\theta_{n-1}}(p_{i_{n-1}})_{\theta_n} : \alpha = \theta_1 \geqq \dots \geqq \theta_n = \beta,\ n \in \mathrm{C}^1\}$$

(and similar formulae hold if the p_i's are merely partializable pre-processes, and $\bigvee p_i$ the upper modification of $\bigcup p_i$). In particular, the least upper bound (l. u. b.) of two processes p, q in P over R is

$$p \vee q = p \cup q \cup (p \circ q) \cup (q \circ p) \cup (p \circ q \circ p) \cup \dots;$$

in the very special case that p *commutes* with q, $p \circ q = q \circ p$, one has that

$$p \vee q = p \cup q \cup (p \circ q), \tag{3}$$

whereupon $p \vee q = p \circ q$ if also their solution-spaces coincide.

As an elementary example, consider the two differential processes in R^1 over R^1 associated with the differential equations $\mathrm{d}x/\mathrm{d}\theta = 1$ and $\mathrm{d}x/\mathrm{d}\theta = -1$. Then their l. u. b. is the process p described by $x\ {}_\alpha p_\beta\ y$ iff $|x - y| \leqq \alpha - \beta$.

To summarize the main result (cf. [1], chap. IV, § 1, theorem 2),

8. Theorem. *The set of all processes in P over R, partially ordered by relation-inclusion, is a complete lattice.*

A like result holds for bi-processes. (The assertion concerning bi-processes follows from twofold application of the otherwise obvious result that $p_1 \subset p_2$ for bi-processes iff $p_1' \subset p_2'$ for the associated processes; cf. I,3.9). The extremal elements of the lattice described are, of course, the process $p_{\max}$ from 3, and the least relation O. More generally, given an arbitrary subset $D \subset P \times R$, the greatest process in P over R with D as domain may be described thus: $(x, \alpha)\ p\ (y, \beta)$ iff both $(x, \alpha), (y, \beta) \in D$, and either $(x, \alpha) = (y, \beta)$ or $\alpha > \beta$ and x, y are unrestricted. Similarly, the least process p in P over R with domain D has $(x, \alpha)\ p\ (y, \beta)$ iff $(x, \alpha) = (y, \beta) \in D$; it is trivial (in the sense of I,1.12) and hence a bi-process.

9. It is now in place to describe, as far as reasonable, the connection between various properties of processes p_i and those of $\bigvee p_i$. Thus, let p_i be processes in P over R, set $p = \bigvee p_i$, and write D for the domain of p, D_i for that of p_i, etc. (refer to I,1 throughout). Then

$$D = \bigcup D_i, \tag{4}$$

since D = domain p, and hence from (2)

$$D \supset \bigcup \text{domain } p_i = \bigcup D_i$$

and also

$$D = \bigcup \text{domain } (p_{i_1} \circ p_{i_2} \circ \dots \circ p_{i_n}) \subset \bigcup \text{domain } p_{i_n} = \bigcup D_i .$$

For the carriers and parameter-domains there follows from (4) directly

$$C = \bigcup C_i\,, \quad B = \bigcup B_i\,.$$

The description of the interval-components of p is easily obtained, but slightly more involved: α and β are in the same interval-component of p iff there is a chain $I_1, \ldots, I_n$ such that each I_k is the interval component of some p_{i_k}, $\alpha \in I_1$, $\beta \in I_n$, and I_k intersects I_{k-1} for $1 < k \leqq n$. In particular, if all p_i are extensive, then p is extensive iff B is an interval in R.

As concerns the solutions of the processes involved, one has the following easily established assertion (cf. I,2.7).

10. Lemma. *Let p_i be a process and S_i a solution system base with $S_i \subset sol\ p_i$ (cf.* I,2), *all in P over R; then $\bigcup S_i$ is a solution system base and* $\bigcup S_i \subset \operatorname{sol} \bigvee p_i$.

In particular, one may take for S_i the complete solution system of p_i. Hence, if each s_k is a solution of some p_{i_k}, if domain $s_k = [\theta_{k+1}, \theta_k]$ for $1 \leqq k < n \in \mathrm{C}^1$ and

$$s_k\theta_k = s_{k-1}\theta_k \quad \text{for} \quad 1 < k \leqq n\,,$$

then $\bigcup_{k=1}^{n} s_k$ is a solution of $\bigvee p_i$. In particular, $\bigvee p_i$ is solution-complete if each p_i is such.

11. Preserve the assumptions and notation of item 9. For the escape times one has, evidently,

$$\varepsilon(x, \alpha) \geqq \sup_i \varepsilon_i(x, \alpha)$$

(with the supremum taken over those indices i which satisfy $(x, \alpha) \in D_i$, cf. (4)); hence p has globality at (x, α) if some p_i does. Moreover, it is also immediate that $(x, \alpha) \in D$ has local existence relative to p iff it does relative to some p_i; thus (x, α) is an end-pair (or start-pair) relative to p iff it is such relative to all p_i with $(x, \alpha) \in D_i$.

Quite obviously, unicity is usually not preserved on passing to l. u. bounds (possibly this is best seen from lemma 10 or the example in item 9); however, in a significant special case, one does have a positive result. The processes of a system $\{p_i \mid i \in I\}$ may be termed *consecutive* iff, for all $i \neq j$ in I, the intersection $B_i \cap B_j$ of their parameter-domains contains one point at most. If this is so, then

$$\delta(x, \alpha) = \min_i \delta_i(x, \alpha)\,,$$

so that local or global unicity obtains at $(x, \alpha) \in D$ relative to p iff it obtains relative to all p_i with $(x, \alpha) \in D_i$.

12. Lemma. *Let* $r : (P, R, p_i) \to (P', R', p'_i)$ *in* Proc. *If* domain $r \supset \bigcup$ domain p_i, *then also* $r : \bigvee p_i \to \bigvee p'_i$ *in* Proc.

Proof. Set $p = \bigvee p_i$, $p' = \bigvee p'_i$, and take any finite subset $\{1, 2, \ldots, n\}$ of the index set. Then

$$r \circ p_1 \circ p_2 \circ \ldots \circ p_n \circ r^{-1} \subset (r \circ p_1 \circ r^{-1}) \circ (r \circ p_2 \circ r^{-1}) \circ$$
$$\circ \ldots \circ (r \circ p_n \circ r^{-1}) \subset p'_1 \circ p'_2 \circ \ldots \circ p'_n \subset p'$$

since domain $p_i \subset$ domain r by assumption. Hence and from (2), $r \circ p \circ r^{-1} \subset p'$ as asserted.

Thus, in the style of I,4, $p = \bigvee p_i$ admits period τ if all the p_i do; p is stationary or r-symmetric if all the p_i are such. Obviously the procedures of taking l. u. bounds and of changing orientation commute. However, additivity is usually not preserved, see 18.7°.

13. According to 8, there exists a greatest lower bound (g. l. b.) $\bigwedge p_i$ of arbitrarily given processes p_i in P over R; it may be obtained as the l. u. b. of all processes p' such that $p' \subset p_i$ for all indices i. However a more constructive description is possible in terms of transitive pre-processes, on the lines of item 7.

Assume given a transitive pre-process p in P over R. For $\alpha \geqq \beta$ in R define relations ${}_\alpha p'_\beta$ on P by

$${}_\alpha p'_\beta = \bigcap \{{}_\alpha p_\theta \circ {}_\theta p_\beta : \alpha \geqq \theta \geqq \beta \text{ in } R\}\,.$$

Obviously these are the individual relations of a pre-process p' in P over R. It is easily verified that p' is again transitive with $p' \subset p$, and also that

(5) $$q \subset p' \subset p \quad \text{for all partializable} \quad q \subset p$$

(q a pre-process in P over R); in particular,

(6) $$p' = p \quad \text{iff } p \text{ is partializable}\,.$$

Next define, by transfinite induction, transitive pre-processes p_α as follows: $p_0 = p$, $p_{\alpha+1} = p'_\alpha$, $p_\omega = \bigcap_{\alpha<\omega} p_\alpha$ (for the moment, the prime denotes the operation described above). The sequence $\{p_\alpha\}_{\alpha \geqq 0}$ is then constant starting at least from some ordinal connected with the cardinality of $P \times R$; thus $p'_\alpha = p_\alpha$ for some ordinal α. (6) then yields that p_α is partializable (and transitive), and thus a process. Hence and from (5), it is the greatest process q in P over R with $q \subset p$, and will therefore be termed the *lower modification* of p.

14. Using this and item 5, one may also describe the g. l. b. $\bigwedge p_i$ of arbitrary processes p_i in P over R as the lower modification of $\bigcap p_i$. Of course, the construction is considerably more involved than that of the upper modification; thus there is no simple formula analogous to (2) or (3).

15. As in item 9, we shall now examine the effect of taking g. l. bounds on the concepts usually associated with processes. Thus, let p_i be processes in P over R, and set $p = \bigwedge p_i$ (then $p \subset p_i$, and I,1.20 yields the trivial parts of the assertions to follow).

First

$$D = \bigcap D_i\,; \tag{7}$$

here $D \subset \bigcap D_i$ is trivial, and the opposite inclusion is obtained from the definition I,1.6 of the domain of a process by following through construction 13 (or by observing that the minimal process with domain $\bigcap D_i$ is a common lower bound to all p_i). Hence p is cartesian if all p_i are such, whereupon also $C = \bigcap C_i$, $D = \bigcap D_i$.

The description of solutions of p is particularly simple:

16. Lemma. *Let S_i be the solution system of a process p_i in P over R. Then $\bigcap S_i$ is the solution system of $\bigwedge p_i$; i.e., s is a solution of $\bigwedge p_i$ iff it is a solution of all p_i.*

Proof. The non-trivial part is to show that a common solution s of all p_i is also a solution of $\bigwedge p_i$. This follows from the construction of 13 and the following observation: if

$$s\alpha \; {}_\alpha p_\beta \; s\beta \quad (\text{for } \alpha \geqq \beta \text{ in domain } s)$$

holds for some pre-processes p, then it also holds for their set-intersection, and also for the pre-process p' as in 13.

For the two numerical characteristics ε and δ we have, in general, only the obvious relations

$$\varepsilon(x, \alpha) \leqq \inf_i \varepsilon_i(x, \alpha)\,, \quad \delta(x, \alpha) \geqq \sup_i \delta_i(x, \alpha)\,.$$

Thus an end-pair (or start-pair) of some p_i is also such relative to $p = \bigwedge p_i$; similarly, local or global unicity obtains at (x, α) relative to p if it does relative to some p_i.

17. Lemma. *Let $r : (P, R, p_i) \to (P', R', p_i')$ in* Proc, *and assume that $r = (r', r'')$ is a partial map with $r''(x, \alpha)$ independent of x and mapping R onto an interval of R'. Then also $r : \bigwedge p_i \to \bigwedge p_i'$ in* Proc.

Proof. Set $p = \bigwedge p_i$, $q = \bigcap p_i'$. Then p is a process, and hence, by the assumptions on r, $r \circ p \circ r^{-1}$ is a partializable pre-process in P' over R'. Secondly, q is a transitive pre-process with $r \circ p \circ r^{-1} \subset q$, since

$$r \circ p \circ r^{-1} \subset r \circ p_i \circ r^{-1} \subset p_i'$$

for all i's. The construction of 13 (together with (5)) applied to q then yields $r \circ p \circ r^{-1} \subset p'$ for the lower modification $p' = \bigwedge p_i'$ of q, as asserted.

Thus, as in 12, $p = \bigwedge p_i$ admits period τ if all the p_i do; p is stationary, or r-symmetric, if all the p_i are such. Obviously the procedures of taking g. l. bounds and of changing orientation commute.

18. This item contains suggestions for minor extensions of the preceding results.

1° Specify the connection between pre-processes (possibly partializable or transitive) and solution system sub-bases in the manner of I,2.

2° Define *pre-bi-processes* (pardoning the terminological atrocity) as symmetric relations with property 1.2°, and formulate appropriate versions of partializability and transitivity. Specify the connection between these and pre-processes in analogy with that obtaining between bi-processes and processes (I,3.8 − 9).

3° Prove the results of items 3, 5, 6, 8, 12, 13, 17 for pre-bi-processes or bi-processes, respectively. (However note that the upper modification of a pre-bi-process is not its transitivization.)

4° Let p be a process in P over R, and R' a subset of R; obtain a necessary and sufficient condition for the partialization $p \mid (P \times R')$ to be a process in P over R (hints: the condition involves only R and the interval-components of p; apply item 3). In the positive case describe the domain, extents of unicity, etc.

5° Let $\{p_i \mid i \in I\}$ be a system of processes in P over R; prove that $\bigwedge p_i$ has unicity if ${}_\alpha(p_i)_\beta \circ {}_\alpha(p_j)_\beta^{-1} \subset 1$ for all i, j in I, $\alpha \geqq \beta$ in R.

6° Obtain results stronger than 9 to 11 and 15 to 16 for monotone systems of processes. Hence show that to every process p with unicity there exists a process p' (both in P over R) maximal relative to the following properties: $p' \supset p$, p' has unicity.

7° Prove that the g. l. b. of additive processes is additive (hint: make appropriate modifications in the construction of item 13). Show that the l. u. b. of additive processes need not be additive (suggestion: $dx/d\theta = x$ and $dx/d\theta = -x$ in R^1; however, in R^1_+ this l. u. b. is again additive).

8° Let p be a process in P over R, and let $\tau \in \mathsf{R}^1$ be such that $\alpha \in R$ implies $\alpha \pm \tau \in R$. Now define, for all $\alpha \geqq \beta$ in R, relations ${}_\alpha(p_\tau)_\beta$ on P by

$$ {}_\alpha(p_\tau)_\beta = {}_{\alpha+\tau}p_{\beta+\tau} \, . $$

Verify that these are the individual relations of a process p_τ in P over R (also observe that then also $p_{n\tau}$ is defined for all $n \in \mathsf{C}^1$). Prove that

$$ p' = \bigvee_{n \in \mathsf{C}^1} p_{n\tau} \quad \text{and} \quad p'' = \bigwedge_{n \in \mathsf{C}^1} p_{n\tau} $$

are processes admitting period τ, and show that they are, in a sense, upper and lower modifications of p.

9° Define, prove existence and describe the upper and lower stationary modifications of a process. (Hint: 8°.)

10° Let p be a process in P over R, q its stationarization as in I,4.11 (hence a process in $P \times R$), $\mathrm{proj}_1 : P \times R \to P$ the natural projection, and finally p' the upper stationary modification of p as in 9°. Prove that

$$ \mathrm{proj}_1 \times 1 : q \to p' \quad \text{in} \quad \mathsf{Proc} \, . $$

11° Let p be a process in P over R, and $r : P \to P$ one-to-one onto. Show that

$$p_r = (r \times 1) \circ p \circ (r \times 1)^{-1}$$

is a process p_r in P over R (also observe that then also p_{r^n} are defined for all $n \in \mathsf{C}^1$). Prove that $\bigvee_{n \in \mathsf{C}^1} p_{r^n}$ and $\bigwedge_{n \in \mathsf{C}^1} p_{r^n}$ are the upper and lower r-symmetric modifications of p.

12° Let p be a partializable pre-process in a group P over R. For each pair $\alpha \geqq \beta$ in R define a relation ${}_\alpha p'_\beta$ in P by letting $x \, {}_\alpha p'_\beta \, y$ iff there are $x_i, y_i \in P$ $(i = 1, 2)$ with

$$x_1 - x_2 = x\,, \quad y_1 - y_2 = y\,, \quad x_i \, {}_\alpha p_\beta \, y_i\,.$$

Show that the ${}_\alpha p'_\beta$ are the individual relations of a partializable pre-process p' in P over R. By alternating this procedure with transitivization sufficiently often, construct the (appropriately defined) upper linear modification of p.

13° Let $p = \bigvee p_i$ with the p_i consecutive processes (cf. item 11) in P over R. Collect appropriate results to obtain the following assertions: If each p_i is a local (global) semi-flow (flow, semi-dynamical system, dynamical system), then so is p.

19. The remaining items are concerned with interpretations of l. u. b. formation outside process-theory proper.

1° Describe the effect of formation of l. u. b. and g. l. b. in graphs, interpreting these as stationary processes over C^1; and similarly for partially ordered sets (cf. I, 4.19.22°).

2° For $i \in I$ let v_i be an orientor field in a normed linear space L; denote by p_i the corresponding processes (defined via classical or other solutions, cf. I,2.10.12°). Show that the process p similarly associated with any orientor field v, $v(x, \theta) \supset \supset \bigcup_{i \in I} v_i(x, \theta)$, has $p \supset \bigvee p_i$; also consider equality instead of inclusion, under further appropriate conditions. (Suggestion: try $v_i(x, \theta)$ and $\bigcup v_i(x, \theta)$ convex and compact.)

Finally, we shall be concerned with *mixed-type regulated systems*, usually denoted by

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x, u, \theta)\,, \quad u \in U\,; \tag{8}$$

here f and u are partial maps $L \times Q \times \mathsf{R}^1 \to L$ and $L \times \mathsf{R}^1 \to Q$ respectively, L a normed space, Q a set (this description comprises not only the pure forcing terms $u : \mathsf{R}^1 \to L = Q$ as in I,2.10.14°, but also autonomous regulators $u : L \to Q$, and regulators of mixed type). Concerning the set U of regulators we shall assume only that proj_2 [domain u] is an interval in R^1 for each $u \in U$.

3° Using I,1.3 (or rather I,2.10.5°) interpret two-position regulation as a system $\{p_i \mid i = 1, 2\}$ consisting of two processes only (in L over R^1). Interpret (8) similarly; also show that finite switching regimes of (8) are adequately represented by formation of the l. u. b. of the processes p_u corresponding to (8), even up to behaviour at switch points. (Hint: use U as the index set; apply lemma 10.)

4° Define *regulated-processes* in P over R with regulators from U (P a set, $R \subset \mathsf{R}^1$ U a partially ordered set) as maps p of U into the set of processes in P over R, having the following three properties:

1′ If $\bigvee u_i$ exists in U, then $p(\bigvee u_i) = \bigvee pu_i$,
2′ If $\bigwedge u_i$ exists in U, then $p(\bigwedge u_i) = \bigwedge pu_i$,
3′ If O exists in U, then $pO = O$.

(Here the notation on the left sides of the formulae concerns the partial order in U, that on the right the partial order as in item 8.) Show that $u \leqq u'$ in U implies $pu \subset \subset pu'$.

5° Show that each regulated system (8) defines, in a natural manner, an associated regulated-process; also show that then

$$\text{domain } pu = \text{domain } u \cap \text{domain} f(., u, .) .$$

6° Define *functional-differential regulated systems*, as suggested by the symbolic form

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x_\theta \mid J, u(x_\theta \mid J, \theta), \theta), \quad u \in U$$

(with $J = [-\tau, 0]$ or $J = (-\infty, 0]$, cf. I,1.21.9°), and construct the corresponding regulated-process.

2. TRANSFORMATION THEORY

1. The basic results of this section are items 2 and 9 on inverse and direct generation of processes. The role of these in the present paper is similar to that of the *Deux méthodes générales de définition d'une topologie*, familiar from [2, § 7]; namely, as fundamental apparatus in the elementary categorial constructions of processes (indeed, this was the motivation). They may also be interpreted as the abstract counterpart of classical transformation theory for differential equations; see item 12.

The problem to be considered may be formulated, loosely but succinctly, thus: given the relation r in a morphism condition $r \circ p \circ r^{-1} \subset p'$, i.e.

(1) $$r : p \to p' \quad \text{in} \quad \mathsf{Bipr} ,$$

find p to given p', or conversely, possibly with various further properties.

2. Transformation theorem (and definition). *For $i \in I$ let p'_i be a process in P'_i over R'_i, and r_i a relation between $P'_i \times R'_i$ and $P \times R$. Then there exists a unique process p, in P over R, maximal relative to the following properties*:

$$(2) \qquad r_i : p \to p'_i\,, \quad \text{domain } p \subset \text{domain } r_i \quad (\textit{for all } i \in I)\,.$$

p is then said to be inversely generated by the p'_i and r'_i.

Proof. It suffices to show that the l. u. b. of all processes p satisfying (2) also does so. For convenience of notation, let p_j vary over all processes in P over R which satisfy (2), and set $p = \bigvee p_j$. According to 1.6, p may also be obtained thus: take the partializable pre-process $q = \bigcup p_j$; then p is the transitivization of q, i.e. $p = \bigcup q^n$.

Now proceed to verify (2) for p. Fix an arbitrary $i \in I$; then

$$r_i \circ q \circ r_i^{-1} = \bigcup_j (r_i \circ p_j \circ r_i^{-1}) \subset p'_i\,,$$

$$(3) \qquad \text{domain } q = \text{range } q = \bigcup_j \text{domain } p_j \subset \text{domain } r_i\,.$$

Hence

$$r_i \circ q^2 \circ r_i^{-1} \subset r_i \circ q \circ r_i^{-1} \circ r_i \circ q \circ r_i^{-1} \subset p'_i \circ p'_i = p'_i\,,$$

etc., i.e. $r_i \circ q^n \circ r_i^{-1} \subset p'_i$; thus

$$r_i \circ p \circ r_i^{-1} = \bigcup_n (r_i \circ q^n \circ r_i^{-1}) \subset p'_i\,.$$

Thus p indeed satisfies the first condition from (2); the second follows similarly from (3). This concludes the proof.

3. Lemma. *In the situation of* 2, *let $s : R \to P$ be partial map such that domain s is an interval in R and* $\text{proj}_2\, r_i[s]$ *an interval in R_i for each $i \in I$. Then s is a solution of p iff $s \subset \bigcap_i$* domain r_i *and $r_i[s]$ is a solution of p'_i for each $i \in I$.*

(Note that $r_i[s]$ is the image of the set s under r_i; cf. I,3.4.)

Proof. One part of the assertion follows directly from (2) (and I,3.4). Conversely, let s have the indicated properties. Then it is easily shown that $s' = \{(s\alpha, \alpha, s\beta, \beta) : : \alpha \geqq \beta \text{ in domain } s\}$ is a process in P over R and satisfies (2); from maximality, $s' \subset p$, and hence s is indeed a solution of p.

4. In the diagram

$$\begin{array}{ccc} p'_i & \xrightarrow{u_i} & q'_i \\ {\scriptstyle r_i}\uparrow & & \uparrow{\scriptstyle u_i \circ r_i} \\ p & \xrightarrow[u]{} & q \end{array}$$

let the processes p'_i and q'_i be given, and also the morphisms u_i and relations r_i as indicated. Also let p be inversely generated by the p'_i and r_i, and similarly q be inversely generated by the q'_i and $u_i \circ r_i$. Then u is an inclusion; thus $p \subset q$, and hence commutativity obtains in the diagram. (This is an immediate consequence of the definition and maximality of q.)

By suitable choice of the u_i's we obtain the following assertion: In the situation of 2 let $r_i = r'_i \times 1$ (thus r'_i is a relation between P'_i and P); if all the p'_i admit period τ or all are stationary then p has the same property.

5. The transformation theorem seems satisfactorily general; indeed, it may even be presented as evidence of the appropriateness of the process concept as given in I,3. The inner mechanism of its proof may be seen to depend on the existence and construction of l. u. bounds.

On the other hand, the transformation theorem is, possibly, too general, in that some of most elementary properties of p are not particularly simply connected with those of the p'_i and r_i. Thus, the domain D of p satisfies $D \subset \bigcap_i r_i^{-1}[D_i]$, a simple consequence of (2) only; however, a complete description of D in terms of p'_i and r_i is rather involved (cf. 13.2°). For the purpose of the following section a rather special case will be more than sufficient. It may be noticed that the conditions to be imposed on the relations r_i are similar to those appearing in 1.17 (and I,3.4).

6. Proposition. *For $i \in I$ let p'_i be a process in P'_i over R'_i, and $r_i = (r'_i, r''_i)$ a partial map $P \times R \to P'_i \times R'_i$ such that $r''_i(x, \alpha)$ is independent of $x \in P$ and*

$$\bigcap_i (r_i^{-1} \circ r_i) \subset 1 . \tag{4}$$

Then the process inversely generated by the p'_i and r_i coincides with the lower modification of $\bigcap_i (r_i^{-1} \circ p'_i \circ r_i)$.

Proof. 1° Set $q_i = r_i^{-1} \circ p'_i \circ r_i$ for $i \in I$, and $q = \bigcap_i q_i$. First we shall need to prove that q is indeed a transitive pre-process. Each q_i is a symmetric and transitive relation, since p'_i is such and $r_i \circ r_i^{-1} \subset 1$; therefore q is symmetric and transitive, and it only remains to verify the initial-value condition 1.2.2° for q.

Let $(x, \alpha)\, q\, (y, \beta)$. Then for each $i \in I$ there exist pairs in $P'_i \times R'_i$ such that

$$(x'_i, \alpha'_i)\, r_i\, (x, \alpha)\,, \quad (y'_i, \beta'_i)\, r_i\, (y, \beta)\,, \quad (x'_i, \alpha'_i)\, p'_i\, (y'_i, \beta'_i)\,;$$

in particular, the assumption on r''_i yields that

$$\alpha'_i = r''_i \alpha\,, \quad \beta'_i = r''_i \beta\,. \tag{5}$$

It is required to show that $\alpha = \beta$ implies $x = y$; thus, let $\alpha = \beta$. Then (5) yields $\alpha'_i = \beta'_i$, and then $x'_i = y'_i$ since p'_i is a process. Hence $(x'_i, \alpha'_i) = (y'_i, \beta'_i)$, implying $(x, \alpha)\, r_i^{-1} \circ r_i (y, \alpha)$; and since this holds for all $i \in I$, (4) yields $x = y$ as required.

2° Thus q is indeed a transitive pre-process, and one may form its lower modification, say p. The assertion is then that p satisfies (2) and is maximal with this property. Now,

$$r_i \circ p \circ r_i^{-1} \subset r_i \circ q_i \circ r_i^{-1} = r_i \circ r_i^{-1} \circ p_i' \circ r_i \circ r_i^{-1} \subset p_i'$$

since $r_i \circ r_i^{-1} \subset 1$; and obviously

$$\text{domain } p \subset \text{domain } q_i \subset \text{domain } r_i .$$

To prove maximality, take any other process $\bar{p}$ in P over R satisfying a condition as (2); then

$$\bar{p} \subset r_i^{-1} \circ r_i \circ \bar{p} \circ r_i^{-1} \circ r_i \subset r_i^{-1} \circ p_i' \circ r_i = q_i$$

since domain $\bar{p}$ = range $\bar{p} \subset$ domain r_i; and hence, in turn, $\bar{p} \subset \bigcap q_i = q$, $\bar{p} \subset p$ (cf. 1.13 (5) with a different notation). Therefore p is indeed the greatest process with (2) as asserted; this completes the proof.

7. Some remarks to proposition 6 are appropriate; preserve the assumptions and notation. Condition (4) is satisfied trivially if some r_i is one-to-one, i.e. if $r_i^{-1} \circ r_i \subset 1$. A partial converse to this is that if the index set is a singleton, then (4) is precisely the requirement that r be one-to-one.

Of course, the results of items 3 and 4 apply a fortiori in the situation of 6. In 3 the requirement on $\text{proj}_2\, r_i[s]$ now reduces to that $r_i''[\text{domain } s]$ is to be an interval in R_i; this is satisfied automatically if r_i'' maps R onto an interval in R_i. For the domains we now have that

$$(6) \qquad D = \bigcap_i r_i^{-1}[D_i]$$

(to prove the non-trivial inclusion show that the maximal process with domain $\bigcap_i r_i^{-1} . [D_i]$ satisfies (2)).

Now assume that, in addition, all the r_i'' increase; then, from I,3.6

$$\varepsilon \leqq \inf_i r_i''^{-1} \circ \varepsilon_i' \circ r_i .$$

In particular, $(x, \alpha) \in D$ is an end- or start-pair relative to p if some $r_i(x, \alpha)$ is such relative to p_i'. We shall also show that, in the same situation,

$$(7) \qquad \delta \geqq \inf_i r_i''^{-1} \circ \delta_i' \circ r_i .$$

This will imply that p has positive or negative local or global unicity at $(x, \alpha) \in D$ if some p_i' has the corresponding property at $r_i(x, \alpha)$.

The proof of (7) is a variation on that used in I,3.7, and exploits (4). It suffices to consider the case $\delta(x, \alpha) < +\infty$. Then there exist $\theta \in R$ arbitrarily near $\delta(x, \alpha)$ and $u \neq v$ in R with $u\ {}_\theta p_\alpha\ x$, $v\ {}_\theta p_\alpha\ x$ (cf. I,1.17). Since $D \subset r_i^{-1}[D_i]$, one also has

$$r_i(u, \theta)\ p_i'\ r_i(x, \alpha) \quad \text{with} \quad r_i(u, \theta) = (r_i'(u, \theta), r_i''\theta)$$

for all $i \in I$, and similarly for (v, θ). Now, $r_i''\theta < \delta_i' r_i(x, \alpha)$ would imply $r_i(u, \theta) = r_i(v, \theta)$ by definition of δ_i'; however $u \neq v$ and (4) yield that this cannot occur for all $i \in I$. Thus $r_i''\theta \geqq \delta_i' r_i(x, \alpha)$ for some i, implying (7) on taking $\theta \searrow \delta(x, \alpha)$.

8. Now consider the second problem suggested in 1 by (1), that of finding p' to given p and r. This case is considerably more complicated; e.g., such processes p' need not exist at all. To see this, observe that (1) implies that $r \circ p \circ r^{-1}$ is a pre-process, and hence $r \circ p \circ r^{-1} \subset p'_{\max}$ for the maximal process $p'_{\max}$ in P' over R' (cf. 1.3); however, arbitrarily given p and r may well violate this necessary condition. On the other hand, in a reasonable situation related to that of item 6, one can obtain positive results. First let us describe the situation in the general case.

For $i \in I$ let p_i be a process in P_i over R_i, and r_i a relation between $P' \times R'$ and $P_i \times R_i$. Iff there exists a minimal process p' in P' over R' with

$$r_i : p_i \to p' \quad \text{for all} \quad i \in I\,,$$

then p will be said to be *directly generated* by the p_i and r_i.

9. Proposition. *For each $i \in I$ let $r_i = (r_i', r_i'')$ be a partial map $P_i \times R_i \to P' \times R'$ such that $r_i''(x, \alpha)$ is independent of $x \in P_i$, one-to-one, and maps R_i onto an interval in R'. Then, for any processes p_i in P_i over R_i, the upper modification of $\bigcup(r_i \circ p_i \circ r_i^{-1})$ is the process p' directly generated by the p_i and r_i.*

Proof. From the assumptions on r_i, each $r_i \circ p_i \circ r_i^{-1}$ is a partializable pre-process on P' over R'; hence so is their set-union (cf. 1.5), and

$$r_i \circ p_i \circ r_i^{-1} \subset \bigcup_i (r_i \circ p_i \circ r_i^{-1}) \quad \text{for all} \quad i \in I\,.$$

Now merely apply 1.7 to conclude the proof.

10. In the situation of 9, for the corresponding domains and parameter-domains one has

$$D' = \bigcup_i r_i[D_i]\,, \quad B' = \bigcup_i r_i''[B_i].$$

The description of some solutions of p' follows from 1.10. To take at least one case in greater detail, assume that all r_i'' increase; also let s_k be a solution of p_{i_k} with domain $s_k = [\beta_k, \alpha_k]$ (for $1 \leqq k \leqq n \in \mathrm{C}^+$, $i_k \in I$) and assume that

$$r_{i_k}(s_k\beta_k, \beta_k) = r_{i_{k+1}}(s_{k+1}\alpha_{k+1}, \alpha_{k+1}) \quad \text{for} \quad k < n\,;$$

in particular this implies $r''_{i_k}\beta_k = r''_{i_{k+1}}\alpha_{k+1}$, i.e. the $\mathrm{proj}_2\, r_{i_k}[s_k]$ are contiguous intervals. Then

$$s = \bigcup_{k=1}^{n} r_{i_k}[s_k]$$

is a solution of p. (The modifications necessary to cover the case of s with non-closed domain or with not all r_i increasing are perhaps obvious.)

As concerns the relation between the escape times of p_i and p', from I,3.6 it follows that

$$\varepsilon'(x', \alpha') \geqq \sup \{r''_i(\varepsilon_i(x, \alpha) - 0) : (x', \alpha')\, r_i\, (x, \alpha),\ i \in I\} ;$$

in general little more can be asserted even if I is a singleton. At least, then, local existence relative to p' obtains at $(x', \alpha') \in D'$ if $(x', \alpha')\, r_i\, (x, \alpha)$ and local existence relative to p_i obtains at (x, α) for some $i \in I$.

11. In the diagram

$$\begin{array}{ccc} p_i & \xleftarrow{\;u_i\;} & q_i \\ {\scriptstyle r_i}\big\downarrow & & \big\downarrow{\scriptstyle r_i \circ u_i} \\ p' & \xleftarrow[\;u\;]{} & q' \end{array}$$

let the processes p_i and q_i be given, and also the morphisms u_i and relations r_i as indicated. Assume that the r_i satisfy the assumptions of 9, and the u_i satisfy similar assumptions. Let p' be directly generated by p_i and r_i, and q' be directly generated by q_i and $r_i \circ u_i$. Then u is an inclusion; thus $q' \subset p'$, and commutativity obtains in the diagram. (This is, of course, a restricted case of the dual to item 4.)

Hence, by suitable choice of the u_i's, we obtain the following assertion: In the situation of item 9 let $r_i = r'_i \times 1$; if all the p_i admit period τ or all are stationary, then p has the same property.

12. The reader familiar with differential equation theory may share the present author's opinion that classical transformation theory is rather unsatisfactory in one respect. To be sure, one cannot object e.g. to linear transformation theory for linear equations; however, in the more general case over-strict assumptions are necessary to obtain well-founded results by classical methods. Thus, it seems regrettable to be forced to exclude transformations such as $y = x^2$ or $\sigma = \theta^3$ in $\mathrm{d}x/\mathrm{d}\theta = f(x, \theta)$, even though perfectly good equations may well result by a formal procedure, and regularity of the transformation may be quite immaterial for the problem considered.

The treatment suggested by the results of this section is to interpret the given equation as a process p in the usual manner, to describe the transformation by a suitable relation r, and then to construct the process (inversely or directly) generated by p and r or r^{-1}. This may then be followed by a separate inquiry as to differentiability of the so-obtained process.

13. (Suggestions for further results.) 1° Show that inverse generation as in 2 can be composed from the formation of g. l. bounds and the inverse generation by a single bi-process and relation.

2° Let the process p be inversely generated by p' and $r:(P, R, p) \to (P', R', p')$. Prove that the domain of p is the intersection of the following two subsets of $P \times R$:

$$\text{domain } r - r^{-1}[P' \times R' - D'], \quad \bigcup\{X \subset P \times R : (r \mid X) \circ (r \mid X)^{-1} \subset p'\}.$$

In particular, the second set may be omitted if $r \circ r^{-1} \subset p'$. (Hint: for the more difficult inclusion consider the minimal process with given domain, 1.8).

3° Let $r = (r', r'')$ be a one-to-one partial map $P \times R \to P' \times R'$ with $r''(x, \alpha)$ independent of $x \in P$, one-to-one, and mapping R onto an interval of R'. Show that the process p, inversely generated by a bi-process p' in P' over R' and the relation r, coincides with the lower modification of the transitive pre-process $r^{-1} \circ p' \circ r$. Also give reasonable conditions on r for p to be directly generated by p' and r^{-1}.

4° Prove the following complements to proposition 6: If $r_i = r_i' \times r_i''$, and to any $x_i' \in P_i'$ there exists an $x \in P$ with $x_i' \, r_i \, x$ for all $i \in I$, then $\bigcap r_i^{-1} \circ p_i' \circ r_i$ is partializable, and hence coincides with the process inversely generated by the p_i' and r_i.

5° In the situation of (7) show that $\delta \geqq r_i''^{-1} \circ \delta_i' \circ r_i''$ if r_i'' is one-to-one; more generally, if $J \subset I$ is such that $\bigcap_{i \in J} (r_i^{-1} \circ r_i) \subset 1$, then $\delta = \inf_{i \in J} r_i''^{-1} \circ \delta_i' \circ r_i''$.

6° Formulate the necessary condition spoken of in item 8 in terms of p and r only.

7° Treat r-symmetry and additivity under inverse and direct generation (by a single relation).

8° Prove the existence of the process directly generated by a process p in P over R and the collapsion of P to a singleton (multiplied by the identity map of R). Describe its domain, and characterize local existence in terms of p.

3. CATEGORIAL CONSTRUCTIONS

1. In this section there are performed the elementary constructions of processes from given ones: relativization, direct product, factorization, direct sum (for inverse and direct limits see item 19). The unifying concept is that of process generation as described in the preceding section.

2. Let p be a process in P over R, and D' an arbitrary subset of its domain D; let $i : D' \subset D$ be the inclusion map, interpreted as a relation between $P \times R$ and $P \times R$. Then the process p' inversely generated by p and i will be termed the *relativization* of p to D'.

Thus p' is the maximal process in $P \times R$ with $p' \subset p$ and domain $p' \subset D'$. A direct description may be obtained from 2.6: p' is the lower modification of the transitive pre-process $p \mid D'$ (this leads to a construction which is a special case of 1.13).

3. It follows immediately that $p' \subset p$, that the domain of p' is precisely D', that s is a solution of p' iff it is a solution of p with $s \subset D'$, that

$$\varepsilon'(x, \alpha) \leqq \varepsilon(x, \alpha)\,, \quad \delta'(x, \alpha) \geqq \delta(x, \alpha)$$

for $(x, \alpha) \in D'$, etc. In particular each end-pair of p in D' is also an end-pair of p'; however p' may have further end-pairs.

Preserving the notation, if p_1' is the relativization of a second process p_1 to D_1', and $r : p \to p_1$ a morphism in Proc with $r[D'] \subset D_1'$, then commutativity obtains in the diagram

$$\begin{array}{ccc} p & \xrightarrow{\;r\;} & p_1 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i_1} \\ p' & \xrightarrow[r/D']{} & p_1' \end{array}$$

It follows that if p admits period τ and D' is such that $(x, \alpha) \in D'$ implies $(x, \alpha \pm \tau) \in D$, then p' also admits period τ. If p is stationary and $D' = P' \times R'$ with $P' \subset P$ and R' a subgroup of R, then p' is also stationary. If p is additive and $(x, \alpha) \in D'$, $(y, \alpha) \in D'$ imply $(x - y, \alpha) \in D$, then p' is also additive. If p is r-symmetric and $r[D'] = D'$, then p' is also r-symmetric.

4. For $i \in I$ let p_i be a process in P_i over R. Then the process p in ΠP_i over R inversely generated by the p_i and $\text{proj}_i \times 1$ will be termed the *direct product* of the p_i (and occasionally may be denoted by $\dot{\Pi} p_i$, or as in $p_1 \dot{\times} p_2$, etc.). For the moment it will also be convenient to set

$$P = \Pi P_i\,, \quad r_i = \text{proj}_i \times 1\,.$$

Thus p is the maximal process on P such that all the natural projections are admissible relative to $p \to p_i$ in Proc (cf. 2.2 with $\bigcap$ domain $r_i = P$). However, both 2.6 and 2.13.4° are also applicable. Hence p may also be described thus: denoting elements of $P = \Pi P_i$ by $[x_i]$, etc., one has

$$(1) \qquad [x_i]\ {}_\alpha p_\beta\ [y_i] \quad \text{iff} \quad x_i\ {}_\alpha(p_i)_\beta\ y_i \quad \text{for all} \quad i \in I\,.$$

5. The notation of 4 is to apply. From 2.3, a partial map $s : R \to P$ is a solution of p iff all $\text{proj}_i \circ s$ are solutions of p_i. From 2.7 one has that $([x_i], \alpha) \in D$ iff $(x_i, \alpha) \in \in D_i$ for all $i \in I$; hence, easily

$$C \subset \Pi C_i\,, \quad B = \bigcap B_i\,,$$

and in the special case that all p_i are cartesian (i.e. $D_i = C_i \times B_i$) p is also cartesian and $D = (\Pi C_i) \times (\bigcap B_i)$.

Now set $x = [x_i]$; then directly from (1),

$$(2) \qquad \varepsilon(x, \alpha) = \inf_i \varepsilon_i(x_i, \alpha)\,;$$

and from 2.7 (7),

$$(3) \qquad \delta(x, \alpha) \geqq \inf_i \delta_i(x_i, \alpha)\,.$$

Hence local existence obtains at all (x_i, α) if it obtains at (x, α), and global existence obtains at all (x_i, α) iff it does at (x, α); thus (x, α) is an end-pair if some (x_i, α) is such (but p may have other end-pairs than these). In (3) inequality may actually occur, cf. 20.5°; at least, then, global unicity obtains at (x, α) if it does at all (x_i, α).

According to 2.4, p admits period τ or is stationary if all p_i are such. Evidently, if all p_i are additive (in groups P_i) and P is endowed with the so-called complete direct product group-structure [4, V, § 17], then p is additive.

6. In the situation of item 4, several slightly stronger results may be had in the case that the index set I is finite. Assume this. Then, of course, (2) above reduces to $\varepsilon = \min \varepsilon_i$, so that all end-pairs of p are as described in 5; explicitly, $([x_i], \alpha)$ is an end- or start-pair iff some (x_i, α) is such.

A less trivial consequence is that if I is finite then (3) may be replaced by

$$(4) \qquad \delta(x, \alpha) = \min_i \delta_i(x_i, \alpha)$$

(with $x = [x_i]$). Indeed, assume the contrary; then $\delta_i < +\infty$, so that $\delta_i \leqq \varepsilon_i$ for all $i \in I$ (cf. I,1.17) and

$$\min \delta_i(x_i, \alpha) \leqq \min \varepsilon_i(x_i, \alpha)\,.$$

Now fix a $j \in I$ and choose $\theta \in R$ such that

$$(5) \qquad (u_j, \theta)\ p_j\ (x_j, \alpha)\,, \quad (v_j, \theta)\ p_j\ (x_j, \alpha)$$

for some $u_j \neq v_j$ in P_j, and that $\alpha \leqq \theta < \min \varepsilon_i$ and $\theta < \delta$ (this is the contradictory assumption; note that if there were $\alpha = \min \varepsilon_i$, then some $\delta_j = +\infty$ by I,1.17, contradicting an assumption). Since $\theta < \min \varepsilon_i(x_i, \alpha)$, for all $i \neq j$ in I, there exist u_i, v_i in P_i with (5) (and j replaced by i). Then $[u_i] \neq [v_i]$ since $u_j \neq v_j$, and from (1)

$$[u_i]\,{}_\theta p_\alpha\,[x_i]\,, \quad [v_i]\,{}_\theta p_\alpha\,[x_i]\,,$$

contradicting $\theta < \delta(x, \alpha)$. This concludes the proof of (4).

Thus for finite index sets, local or global unicity obtains at $([x_i], \alpha)$ iff it obtains at all (x_i, α).

7. Let p be a process in P over R, and $\sim$ an equivalence relation on P; denote by $e : P \to P/\sim$ the canonical surjection associated with $\sim$ (i.e. ex is the equivalence class modulo $\sim$ of $x \in P$). Then the process p' in $P/\sim$ over R directly generated by p and $e \times 1$ will be termed the *factor process* of p modulo $\sim$, and may be denoted by $p/\sim$.

According to 2.9 and 1.6, p' is the transitivization of the partializable pre-process $(e \times 1) \circ p \circ (e \times 1)^{-1}$; the individual relations of p' may then be described as (cf. 1.6 (1), observe that $e^{-1} \circ e = \sim$)

(6)
$$\begin{aligned} {}_\alpha p'_\beta = e \circ \bigcap \{ & {}_{\theta_1}p_{\theta_2} \circ \sim \circ {}_{\theta_2}p_{\theta_3} \circ \sim \circ \ldots \circ \sim \circ {}_{\theta_{n-1}}p_{\theta_n} : \\ & \alpha = \theta_1 \geqq \ldots \geqq \theta_n = \beta,\ n \in \mathrm{C}^1 \} \circ e^{-1} \end{aligned}$$

8. According to 2.10 (primes refer to concepts associated with p'),

$$D' = (e \times 1)[D]\,, \quad B' = B\,;$$

obviously even the interval-components of p coincide with those of p', and $C' = e[C]$.

If s_k are solutions of p with domain $s_k = [\theta_{k+1}, \theta_k]$ for $1 \leqq k \leqq n \in \mathrm{C}^1$, and with

$$s_k\theta_k \sim s_{k-1}\theta_k \quad \text{for} \quad 1 < k \leqq n\,,$$

then $\bigcup_{k=1}^{n} e \circ s_k$ is a solution of p' (there are obvious modifications for non-closed domains).

If p admits period τ or is stationary, then p' has the corresponding property; however, p' may well be stationary without p being such. For the escape times one has

$$\varepsilon'(x', \alpha) \geqq \sup \{\varepsilon(x, \alpha) : x' = ex\}\,;$$

in general little more can be asserted concerning ε' (and nothing concerning δ'). Items 9 to 12 concern some special situations in which, loosely speaking, unicity is preserved under factorization.

9. Thus, let p be a process in P over R, and $\sim$ an equivalence relation on P. There will be presented two of several possible variants of a compatibility relation between p and $\sim$. First, p will be termed *compatible* with $\sim$ (and vice versa) iff

$$(7) \qquad {}_\alpha p_\beta \circ \sim \circ ({}_\alpha p_\beta)^{-1} \subset \sim, \quad {}_\alpha p_\alpha \circ \sim \subset \sim \circ {}_\alpha p_\alpha$$

for all $\alpha \geqq \beta$ in R; and p will be termed *strictly compatible* with $\sim$ (and vice versa) iff

$$(8) \qquad {}_\alpha p_\beta \circ \sim \subset \sim \circ {}_\alpha p_\beta$$

for all $\alpha \geqq \beta$ in R (to prevent misunderstanding, in general there are no implications between these two concepts).

Also let q be the bi-process associated with p, and p' the orientation-changed process (cf. I,3.12). Iff p is compatible with $\sim$, then we will also say that p and q are positively compatible with $\sim$, and p' negatively compatible with $\sim$. Iff p is both positively and negatively compatible with $\sim$, then the bi-process q will be termed compatible with $\sim$, and p, q, p' bilaterally compatible with $\sim$; obviously this obtains iff (6) holds for unrestricted α, β in R. Similarly for strict compatibility.

10. For processes with global unicity, strict compatibility implies compatibility. Indeed, ${}_\alpha p_\beta \circ ({}_\alpha p_\beta)^{-1} \subset 1$ and (8) for all $\alpha \geqq \beta$ in R yield

$${}_\alpha p_\beta \circ \sim \circ ({}_\alpha p_\beta)^{-1} \subset \sim \circ {}_\alpha p_\beta \circ ({}_\alpha p_\beta)^{-1} \subset \sim .$$

For processes with global existence, compatibility implies strict compatibility. Indeed, global existence for p implies $({}_\alpha p_\beta)^{-1} \circ {}_\alpha p_\beta \supset {}_\beta p_\beta$, whereupon (7) yields

$$\sim \circ {}_\alpha p_\beta \supset ({}_\alpha p_\beta \circ \sim \circ ({}_\alpha p_\beta)^{-1}) \circ {}_\alpha p_\beta \supset {}_\alpha p_\beta \circ \sim \circ {}_\beta p_\beta \supset$$
$$\supset {}_\alpha p_\beta \circ {}_\beta p_\beta \circ \sim = {}_\alpha p_\beta \circ \sim$$

Thus if p has global existence and global unicity (i.e. if p defines a global semi-flow, I,4.18.3°), then compatibility and strict compatibility are equivalent.

Some further properties are exhibited in the following items.

11. Let p be a process in P over R and $\sim$ an equivalence relation on P. If p is compatible or strictly compatible with $\sim$, then the domain D is invariant under $\sim$ in the sense that $(x, \alpha) \in D$, $x \sim y$ imply $(y, \alpha) \in D$. Indeed, in either case there is ${}_\alpha p_\alpha \circ \sim \subset \sim \circ {}_\alpha p_\alpha$ for $\alpha \in R$, so that

$$x \, {}_\alpha p_\alpha \, x, \quad x \sim y \quad \text{imply} \quad x \sim y', \quad y' \, {}_\alpha p_\alpha \, y$$

for some y', and hence $(y, \alpha) \in D$.

In the case that p is strictly compatible with $\sim$, the individual relations of the corresponding factor process p' are simply (cf. 7)

$$(9) \qquad {}_\alpha p'_\beta = e \circ {}_\alpha p_\beta \circ e^{-1} .$$

The factor process p' may then be described rather concisely: there is $x' \, {}_\alpha p'_\beta \, y'$ for x', y' in $P/\sim$ and $\alpha \geqq \beta$ in R iff there exist $x \in x'$ and $y \in y'$ with $x \, {}_\alpha p_\beta \, y$. (Another formulation is that the partializable pre-process $(e \times 1) \circ p \circ (e \times 1)^{-1}$ is transitive and hence coincides with p'.) The following are two results on unicity under factorization.

12. Lemma. *Let p' be the factor process of a process p modulo an equivalence relation $\sim$. Then*

1° *p' has unicity iff $\sim$ is compatible with p,*

2° *If $\sim$ is strictly compatible with p, then, for all $(x, \alpha) \in D$,*

$$\varepsilon'(ex, \alpha) = \varepsilon(x, \alpha), \quad \delta'(ex, \alpha) \geqq \delta(x, \alpha).$$

Proof. First recall that $e \circ e^{-1} = 1$, $e^{-1} \circ e = \sim$. If $\sim$ is compatible with p, then (7) judiciously applied in (6) yields ${}_\alpha p'_\beta \circ ({}_\alpha p'_\beta)^{-1} \subset 1$ as asserted. Conversely, if a map $r : P \to P'$ is admissible relative to $p \to p'$ in Proc (e.g. $r = e$), then it is easily verified that the equivalence relation $\sim$ with $x \sim y$ iff $rx = ry$ is strictly compatible with p. This proves 1°.

In the second assertion, the part concerning the escape times follows quite obviously from (9); as for the extents of unicity, assume the contrary. Then there exist $x \sim y$ and $u \nsim v$ in P, $\theta \geqq \alpha$ in R such that

$$(10) \qquad u \, {}_\theta p_\alpha \, x, \quad v \, {}_\theta p_\alpha \, y, \quad \theta < \delta(x, \alpha);$$

but then (8) yields $v \sim w$, $w \, {}_\theta p_\alpha \, x$ for some w, and necessarily $w = u$ (from (10), using the definition of δ). But this contradicts $u \nsim v \sim w$.

13. Let $\{P_i \mid i \in I\}$ be a disjoint system of sets, and for $i \in I$ let p_i be a process in P_i over R. Denote the disjoint sum of the P_i by ΣP_i, and the natural injection $P_i \to \Sigma P_i$ by inj_i (since the P_i's are disjoint, we may even assume $\Sigma P_i = \bigcup P_i$, $\mathrm{inj}_i : P_i \subset \bigcup P_i$). Then the process p in ΣP_i over R directly generated by the p_i and $\mathrm{inj}_i \times 1$ will be termed the *direct sum* of the p_i, and may be denoted by $\dot{\Sigma} p_i$, etc.

In the slightly more general situation when disjointness is not required one may put $P_i^* = P_i \times (i)$, and define p_i^* as the direct product of p_i with the maximal process on the singleton (i); thus one reverts to a situation as above. However, for the sake of simplicity, the original assumption on disjointness will be preserved.

The properties of the direct sum are particularly simple, and follow from the obvious relations

$$p = \bigvee_{i \in I} p_i = \bigcup_{i \in I} p_i, \quad p_i \wedge p_j = p_i \cap p_j = O \quad \text{for} \quad i \neq j \quad \text{in} \quad I.$$

Hence (also see 1.9 to 1.11)

$$D = \bigcup D_i\,,\quad C = \bigcup C_i\,,\quad B = \bigcup B_i\,.$$

The solution system of p is directly the set-union of those of the p_i;

$$\varepsilon(x, \alpha) = \varepsilon_i(x, \alpha)\,,\quad \delta(x, \alpha) = \delta_i(x, \alpha)$$

for $x \in P_i$. The process p admits period τ or is stationary iff all the p_i are such.

14. The reader with a weakness for formally balanced exposition may have noticed that the four elementary constructions, as presented above, do not attain the same level of generality. Thus, even informally, the (category-theoretic) dual of the factorization procedure of item 7 is not the relativization described in 2, but rather relativization only to those subsets of $P \times R$ which have the special form $D' = P' \times R$ with $P' \subset P$. (On the other hand, the formation of direct products and sums does seem satisfactorily general, see 20.7°–8°.)

This indicates that processes on P over R should be factorised modulo an equivalence on $P \times R$ rather than on P only. However, the set of the resulting equivalence classes should have a canonic cartesian product structure. Thus the first task is to select a suitable type of equivalence relations; and since in any case we shall have to employ direct generation of processes, one may be guided by the maps treated in 2.9. This will be sketched in the following item.

15. An equivalence relation $\sim$ on $P \times R$ may be termed *time-dependent* iff

$$(x, \alpha) \sim (y, \beta) \quad \text{implies} \quad \alpha = \beta\,; \tag{11}$$

iff, furthermore, $(x, \alpha) \sim (y, \alpha)$ implies $(x, \theta) \sim (y, \theta)$ for all $\theta \in R$, then $\sim$ may be termed *stationary*. This latter case occurs iff $\sim$ is of the form $\sim\, = \,\simeq \times 1$ with $\simeq$ an equivalence relation on P (i.e. the case considered in item 7). Obviously (11) implies that $\sim$ determines (and is determined by) an otherwise arbitrary system $\{\sim_\alpha \mid \alpha \in R\}$ of equivalence relations $\sim_\alpha$ on P.

Now let $e : P \times R \to (P \times R)/\!\sim$ be the canonic surjection; according to (11), e is of the form $e = (e', 1)$ (and $e = e' \times 1$ for stationary $\sim$). To introduce a cartesian product structure for the image-set $(P \times R)/\!\sim$, it may be considered as a subset of $(\exp P) \times R$; it suffices to identify $e(x, \alpha)$ with (x', α), where

$$x' = \{y \in P : (y, \alpha) \sim (x, \alpha)\} \in \exp P$$

($\exp P$ denotes the set of all subsets of P).

Finally, let there also be given a process p in P over R; then the process p' in $\exp P$ over R directly generated by p and

$$e : P \times R \to (P \times R)/\!\sim\, \subset (\exp P) \times R$$

may again be termed the factor process of p modulo $\sim$. For time-independent $\sim \, = \, \simeq \times 1$, p' may also be considered as a process in $P/\simeq \, \subset \exp P$ over R; since the corresponding solution-space lies within $(P/\simeq) \times R$, this cannot lead to ambiguities.

The compatibility relations are easily extended to time-dependent equivalence relations: $p \circ \sim \, \subset \, \sim \circ \, p$ for strict compatibility, and

$$p \circ \sim \circ \, p^{-1} \subset \, \sim , \quad \sim [D] \subset D$$

for compatibility. (A natural example of time-dependent equivalences appears in 18.13°.)

16. (Suggested further results on the relativization of processes.)

1° Exhibit an example showing that in the situation of item 2, $p \mid D'$ need not be partializable. Formulate partializability of $p \mid D'$ in terms of time-convexity (cf. I.4.10).

2° Show that the relativized process may possess end-pairs other than those originally present in its domain. (Suggestion: relativize the differential process in R^1 associated with $\mathrm{d}x/\mathrm{d}\theta = 1$ to some closed interval.)

3° Let p be a process in P over R with local existence, and $D' \subset P \times R$. Describe a necessary and sufficient condition on D' (and p) for the corresponding relativized process to have local existence.

4° Assume that a process p defines a local semi-dynamical system in P over R, let $P' \subset P$ be given, and let p' be the relativization of p to $P' \times R$. Using the preceding subitem, obtain necessary and sufficient conditions for p' to define a local semi-dynamical system.

5° For $k = 1, 2, 3$ let p_k be a process in P over R such that p_2 is a relativization of p_1 and $p_3 \subset p_2$. Show that p_3 is a relativization of p_2 iff it is a relativization of p_1; in particular, then, the relativization procedure is compositive.

6° Let p be a process in P over R, and J the system of its interval components; for each $i \in J$ set $p_i = p \mid (P \times i)$. Show that each p_i is a process (directly or using 1.18.4°), and that

$$p = \bigvee p_i = \bigcup p_i \,, \quad p_i \wedge p_j = p_i \cap p_j = O \quad \text{for} \quad i \neq j \quad \text{in} \quad J \,.$$

Prove that each p_i is extensive (i.e. has a unique interval-component, cf. I,1.8), and that, for any extensive process $q \subset p$, there is $q \subset p_i$ for some $i \in J$. The system $\{p_i \mid i \in J\}$ may be termed the canonic resolution of p into extensive processes.

7° Show that additivity (and also r-symmetry) carry over from a process to all members of its canonic resolution, and conversely.

8° Let p be a process with unicity in P over R, $D' \subset D$, and p' the relativization of p to D'. Show that the escape times ε' of p' satisfy

$$\varepsilon'(x, \alpha) = \sup \{\lambda : \lambda \geqq \theta \text{ and } u\ {}_\theta p_\alpha\ x \text{ imply } (u, \theta) \in D'\} ;$$

obtain hence a direct description of p'.

9° Let p be the process defined by the Carathéodory solutions of a differential equation in a normed linear space P, determined by a partial map $f : P \times \mathsf{R}^1 \to P$ as in I,2.11.4°, let $D' \subset \text{domain} f$ and let p' be the process associated similarly with $f \mid D'$. Show that p' is the relativization of p to D'.

17. (Suggested further results on direct products and sums of processes.)

1° Obtain an example of processes p, p' with $p \neq O \neq p'$ but $p \dot{\times} p' = O$. (Hint: utilize $B = \bigcap B_i$ in item 5.)

2° Show that each component J of $p_1 \dot{\times} p_2$ has the form $J = J_1 \cap J_2$ for appropriate interval-components J_i of p_i; also extend to finitely many factors. (Hint: use the proof of (4).)

3° In the situation of item 4 show that $\delta(x, \alpha) > \inf \delta_i(x_i, \alpha)$ is possible only if, simultaneously, $\inf \varepsilon_i(x_i, \alpha) = \inf \delta_i(x_i, \alpha)$, $\delta(x, \alpha) = +\infty$, I is infinite. (Hint: $\inf \varepsilon_i >$ $> \inf \delta_i < \delta$ implies existence of a θ with $\delta_i < \theta < \varepsilon_i$, $\theta < \delta$, and one can show that this leads to a contradiction; then prove that $\delta < +\infty$ or I finite imply $\inf \varepsilon_i >$ $> \inf \delta_i$.)

4° Show that, nevertheless, $\delta(x, \alpha) > \inf \delta_i(x_i, \alpha)$ is indeed possible. (Hint: choose p_i and (x_i, α) so that $\alpha < \delta_i(x_i, \alpha) < \varepsilon_i(x_i, \alpha) \to \alpha$ for $i \to \infty$; then $x = [x_i]$ has $\varepsilon(x, \alpha) = \alpha = \inf \delta_i(x_i, \alpha)$, hence it is an end-pair, whereupon $\delta(x, \alpha) = +\infty$.)

5° Let $p = \dot{\Pi} p_i$ for stationary processes p_i with local existence. Prove that p has local existence iff all the p_i, with only finitely many exceptions, have global existence. (Hint: one part follows from (2) in 5; for the other choose suitable $x_i \in P_i$ with $\varepsilon_i(x_i, 0) \to 0$.) (This was first obtained for local dynamical systems in [3], II,3.2.)

Let $R \subset \mathsf{R}^1$ be fixed, and consider the following category Proc_R: its objects are processes over R, and its morphisms $r : p \to p'$ in Proc_R are those set-theoretical partial maps r : carrier $p \to$ carrier p' which have

$$(r \times 1) \circ p \circ (r \times 1)^{-1} \subset p' ,$$

i.e. are admissible relative to $p \to p'$ in Proc. (Hence Proc_R is a non-full subcategory of Proc.)

6° Prove that direct sums in the category Proc_R coincide (up to isomorphism, of course) with the direct products of processes as defined in item 4. (Hint: Essentially, only two further facts need be established: that if $p = \dot{\Pi} p_i$, then the natural projections proj_i partialize on carrier p to morphisms in Proc_R; and that if also $r_i : p' \to p_i$ in Proc_R, then $[r_i\, x] \in$ carrier p for each $x \in$ carrier p'.)

7° Prove that free sums in Proc_R coincide (up to isomorphism) with the direct sums as defined in item 13. Also show that the canonic resolution into extensive processes (cf. 16.6°) describes a free sum in Proc, but usually not in Proc_R.

8° Characterize direct and free sums in the full subcategory of Proc_R consisting of the full processes. (Hint: relativize direct products so as to obtain a full process.)

9° Describe the behaviour of processes which define local or global (semi-) flows or (semi-) dynamical systems under formation of direct products and sums. (Hint: for the stationary situation 6° yields necessary and sufficient conditions.)

10° Describe direct products and sums of differentiable processes.

18. (Suggested further results on compatibility of equivalence relations and factorization of processes.)

1° Let p be a stationary process on P over R with global existence and global unicity; show that $\bigcup_{\alpha \geqq 0} ({}_{\alpha}p_0)^{-1} \circ {}_{\alpha}p_0$ is an equivalence relation on P strictly compatible with p. Describe the corresponding factor-process in the category Proc_R.

2° Obtain necessary and sufficient conditions on p and $\sim$ for (8) to hold.

3° Verify that a process p in P is compatible with the identity on P iff p has global unicity. (Suggestion: direct verification, but a satisfactory indirect proof utilizes 12.1°.)

4° Obtain an example showing that compatibility and strict compatibility need not be preserved under orientation-change of the process concerned. Then define the appropriate bilateral versions of (strict) compatibility, and apply to bi-processes.

5° Collect results to prove the following assertions: The property of processes of defining (in the sense of I,4.18) a local or global semi-flow or semi-dynamical system is preserved under factorization modulo a compatible equivalence relation; similarly for flows of dynamical systems, modulo bilaterally compatible equivalence relations.

6° For processes p in P over R admitting a period, the stationarization procedure of I,4 may be modified to obtain a considerably simpler phase-space. Let q be the stationarization of p as in I,4.11, so that q is a stationary process on $P \times R$ over the subgroup R' generated by R in R^1; also assume, for simplicity, that the period admitted by p is precisely 2π. Now define an equivalence relation $\sim$ on $P \times R$ by letting $(x, \xi) \sim (y, \eta)$ iff $x = y$ and $\xi \equiv \eta \pmod{2\pi}$. Then the factor process $q/\sim$ may be termed the *cylindrical stationarization* of p. Describe $q/\sim$ and its carrier directly, using the 1-sphere $\mathsf{S}^1 = \{e^{i\xi} : \xi \in R\}$ as a representation of $R \pmod{2\pi}$.

7° In the preceding situation show that $\sim$ is strictly compatible with q; that it is compatible iff p has global unicity; and that $(x, e^{i\xi})$ is a τ-periodic pair (cf. I,4.3) iff $x \;{}_{\xi+\tau}p_{\xi}\; x$ and $\tau = 2k\pi$ for some $k \in \mathsf{C}^1$.

8° Let $r : P \to P$ be a symmetry of P (i.e. $r \circ r = 1$, cf. I,4.15). Show that the relation $\sim$ on P defined by letting $x = y$ iff either $x = y$ or $x = ry$ is an equivalence relation on P; and that $\sim$ is strictly compatible with each r-symmetric process. Describe the corresponding factor process and its carrier.

9° For a time-dependent equivalence relation $\sim$ on $P \times R$ define $\sim_\alpha$ for $\alpha \in R$ as in item 15: $x \sim_\alpha y$ iff $(x, \alpha) \sim (y, \alpha)$. Describe (strict) compatibility between $\sim$ and a process p in P over R in terms of the $\sim_\alpha$ and $_\alpha p_\beta$.

10° Obtain results similar to those in 5° for time-dependent equivalence relations.

11° Consider a functional-differential equation

$$\frac{dx}{d\theta} = f(x_\theta \mid J, \theta) \tag{12}$$

with $J = (-\infty, 0]$ and partial $f : C(-\infty, 0] \times R^1 \to R^1$ (x_θ is the θ-translate of x as in I,1.22); and assume that f depends on a bounded time-lag only, in the sense that for some interval $I = [-\tau, 0] \subset J$,

$$x \mid I = y \mid I \quad \text{implies} \quad f(x, \theta) = f(y, \theta) \quad \text{for all} \quad \theta\,. \tag{13}$$

Secondly, consider the process p in $C(-\infty, 0]$ over R^1, associated with (12) as in I,2.11.8°; and also the equivalence relation $\sim$ on $C(-\infty, 0]$ defined by

$$x \sim y \quad \text{iff} \quad x \mid I = y \mid I\,. \tag{14}$$

Prove that $\sim$ is strictly compatible with p, and that it is compatible iff p has unicity (i.e. (12) has positive unicity in the customary sense); also describe the corresponding factor process and its carrier.

12° Describe the modification of 11° appropriate to a difference-differential equation

$$\frac{dx}{d\theta} = f(x\theta, x(\theta - \tau), \theta)$$

with given time-lag τ, $\theta \leqq \tau \in R^1$. (Replace (14) by $x \sim y$ iff $x0 = y0$ and $x(-\tau) = y(-\tau)$.)

13° In (12) allow time-dependent lag bounds, in the sense that (13) is to be replaced by

$$x \mid [-\tau\theta, 0] = y \mid [-\tau\theta, 0] \quad \text{implies} \quad f(x, \theta) = f(y, \theta) \tag{15}$$

for each $\theta \in R^1$, where $\tau : R^1 \to [0, +\infty)$ is a given map. Then define a time-dependent equivalence relation $\sim$ on $C(-\infty, 0] \times R^1$ by letting $(x, \theta) \sim (y, \theta)$ iff x, y, θ satisfy the premiss of (15). Prove that $\sim$ is strictly compatible with the corresponding process p if

$$\frac{\tau\alpha - \tau\beta}{\alpha - \beta} \leqq 1 \quad \text{for all} \quad \alpha \neq \beta \quad \text{in} \quad R^1\,, \tag{16}$$

and in the positive case that $\sim$ is compatible with p iff p has unicity. Describe directly the solution-space and carrier of the factor process, and also in the case that (16) is not satisfied. (Hint: if (16) is not assumed, also consider $\tau^*\theta = \theta - \inf(\lambda - \tau\lambda)$.)

14° Paralleling 12°, modify 13° to apply to a difference-differential equation $dx/d\theta = f(x\theta, x(\theta - \tau\theta), \theta)$ with given time-dependent lag $\tau : \mathrm{R}^1 \to [0, +\infty)$.

15° Prove that solution-completeness carries over from a process to its factorization modulo a time-dependent equivalence relation.

19. This item concerns the construction of direct and inverse limits of processes. To fix terminology and notation, the corresponding construction for abstract sets will first be recalled.

Let I be a partially ordered (index-) set; and consider a system $\{r_j^i \mid i \geqq j \text{ in } I\}$ of maps with the property that

$$(17) \qquad r_j^i \circ r_k^j = r_k^i \quad \text{for} \quad i \geqq j \geqq k \quad \text{in} \quad I$$

and that each r_i^i is an identity map. For convenience set $P_i = \text{domain } r_i^i$, so that (17) yields $r_j^i : P_j \to P_i$.

Form, first, the set ΠP_i; its subset $P_{-\infty}$, defined as consisting of all $[x_i] \in \Pi P_i$ with $x_i = r_j^i x_j$ whenever $i \geqq j$ in I; and then maps $r_{-\infty}^i : P_{-\infty} \to P_i$ for $i \in I$, defined as the composition of the maps in the diagram

$$P_{-\infty} \subset \Pi P_i \xrightarrow{\text{proj}_i} P_i \,.$$

Then obviously $r_j^i \circ r_{-\infty}^j = r_{-\infty}^i$. The set $P_{-\infty}$ together with the system $\{r_{-\infty}^i \mid i \in I\}$ is called the *inverse limit* of the r_j^i (or of the P_i relative to the r_j^i), and occasionally one writes

$$P_{-\infty} = \lim\text{inv}\, P_i \,, \quad r_{-\infty}^i = \lim_{j \to -\infty} \text{inv}\ r_j^i \,.$$

(There is an appropriate and obvious extremal property of $\{r_{-\infty}^i \mid i \in I\}$ in the category of all sets and mappings.)

For the second construction assume that I is directed (upward directed, i.e. to any i, j in I there is a common "roof" $k \in I$ with $i \leqq k \geqq j$), and, for convenience, that the P_i's are disjoint. Now form the set ΣP_i, and also

$$P_{+\infty} = (\Sigma P_i)/\sim \,,$$

with $\sim$ the following equivalence relation: $x_i \sim x_j$ with $x_i \in P_i$ and $x_j \in P_j$ iff $r_i^k x_i = r_j^k x_j$ for some $k \geqq i, j$ in I; and finally – maps $r_i^{+\infty} : P_i \to P_{+\infty}$ for some $i \in I$, defined as the composition of the maps in the diagram

$$P_i \subset \Sigma P_i \xrightarrow{e} (\Sigma P_i)/\sim \; = P_{+\infty}$$

with e the natural surjection. Then obviously $r_i^{+\infty} \circ r_j^i = r_j^{+\infty}$. The set $P_{+\infty}$ together with the system $\{r_i^{+\infty} \mid i \in I\}$ is called the inductive or *direct limit* of the (P_i relative to the) r_j^i, and occasionally one writes

$$P_{+\infty} = \lim \operatorname{dir} P_i\,, \quad r_j^{+\infty} = \lim_{i \to +\infty} \operatorname{dir} r_j^i\,.$$

(There is an appropriate and obvious extremal property of $\{r_i^{+\infty} \mid i \in I\}$ in the category of all sets and mappings.)

1° In the situation described above assume that each r_j^i is admissible in Proc relative to $p_i \to p_j$, where p_i is a process in P_i over R. Define the inverse limit $p_{-\infty}$ of the p_i relative to r_j^i as the relativization of $\dot{\Pi} p_i$ (in ΠP_i over R) to $P_{-\infty} \times R$. Show that each $r_{-\infty}^i$ is admissible relative to $p_{-\infty} \to p_i$ in Proc. Describe $p_{-\infty}$ directly.

2° With the preceding notation, show that

$$\varepsilon_{-\infty}([x_i], \alpha) \leqq \inf \varepsilon_i(x_i, \alpha) = \limsup_{i \to -\infty} \varepsilon_i(x_i, \alpha)\,, \tag{18}$$

interpreting $\limsup\limits_{i \to -\infty}$ as $\inf\limits_{i} \sup\limits_{j \leqq i}$. Also prove that if all p_i have global unicity then so does $p_{-\infty}$ and equality obtains in (18). (Hint: in ΠP_i, $x \in P_{-\infty}$ and $(x', \theta)\ \dot{\Pi} p_i\ (x, \alpha)$ imply $x' \in P_{-\infty}$.)

3° In 1° define the direct limit $p_{+\infty}$ of the p_i relative to r_j^i as the factor process of $\dot{\sum} p_i$ (in $\sum P_i$ over R) modulo the equivalence relation $\sim$ described in item 19. Show that each $r_i^{+\infty}$ is admissible relative to $p_i \to p_{+\infty}$ in Proc. Describe $p_{+\infty}$ directly.

4° Verify that $\sim$ is compatible with $\dot{\sum} p_i$ if all p_i have global unicity, whereupon $p_{+\infty}$ also has global unicity.

5° Let p be a process in P over R; denote the set of the positive periods which p admits by I; for $\lambda \in I$ let q_λ be the cylindrical stationarization of p corresponding to λ, cf. 18.6°; finally define a partial order in I by letting $\lambda \leqq \mu$ iff λ/μ is an integer. Describe the direct and inverse limits of the q_λ relative to the appropriately defined maps r_λ^μ.

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Author's address: Sokolovská 83, Praha 8 - Karlín, ČSSR (Matematicko-fyzikální fakulta KU).